Justification of the KP-II approximation in dynamics of two-dimensional FPU systems

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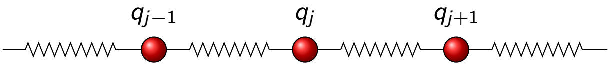
The Fermi-Pasta-Ulam problem



- System of particles on the line
- Nearest neighbour interactions with Hamiltonian given by $H = \sum_j \frac{1}{2}\dot{q}_j^2 + V(q_{j+1} - q_j)$
- Equations of motion are given by $\ddot{q}_j = V'(q_{j+1} - q_j) - V'(q_j - q_{j-1})$
- Potential $V(q) = \frac{1}{2}q^2 + \frac{1}{3}\alpha q^3$
- Numerical experiments showed recurrent formation of solitons for long time scales

A. Vainchtein, "Solitary waves in FPU-type lattices", Physica D 434 (2022) 133252 (22 pages)

KdV limit for small-amplitude, long-scale waves

- Ansatz in the strain variables:

$$r_j(t) = q_{j+1}(t) - q_j(t) := \varepsilon^2 R\left(\varepsilon\left(j - t\right), \varepsilon^3 t\right) + \text{error}$$

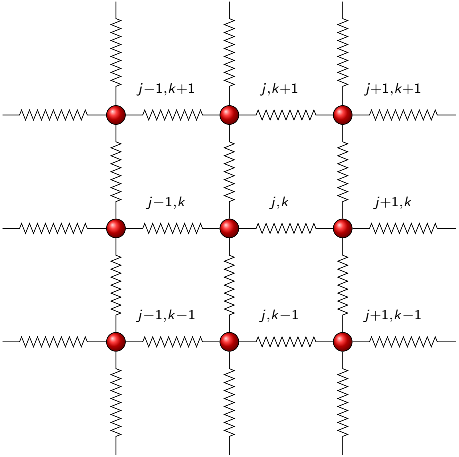
- Approximation satisfies the FPU system to $O(\varepsilon^6)$ if R satisfies the KdV equation:

$$\partial_\tau R + \alpha R \partial_\xi R + \frac{1}{24}\partial_\xi^3 R = 0$$

- Rigorous justification:
  Schneider–Wayne (1999), Friesecke–Pego (1999-2004), Bambusi–Ponno (2005-2006)
- Follow-up work: log-KdV in Hertzian potential (Games–P, 2014; Dumas–P., 2014), generalized KdV on extended time intervals (Khan–P, 2017), polyatomic case (Gaison–Moskow–Wright–Zhang, 2014), nonlocal interaction (Herrmann–Mikikits–Leitner, 2016), and many more.

KdV is an attractive model due to integrability and asymptotic stability of solitary waves.

2D Square Lattice



KP-II limit for small-amplitude, long-scale, transversely modulated waves

There exist two versions of the two-dimensional generalization of the KdV equation:

$$\text{(KP-I)} \quad \partial_\xi(\partial_\tau R + \alpha R \partial_\xi R + \frac{1}{24}\partial_\xi^3 R) - \partial_\eta^2 R = 0$$

and

$$\text{(KP-II)} \quad \partial_\xi(\partial_\tau R + \alpha R \partial_\xi R + \frac{1}{24}\partial_\xi^3 R) + \partial_\eta^2 R = 0$$

For water waves, (KP-I) arises for problems with surface tension and (KP-II) arises for gravity waves.

For the defocusing Gross–Pitaevskii equation,

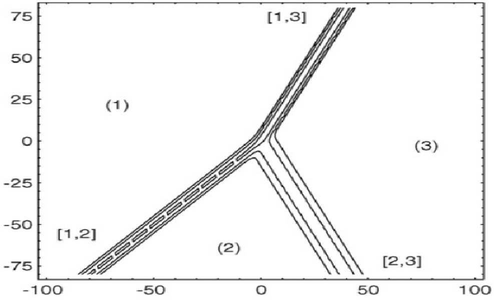
$$i\psi_t + \psi_{xx} + \psi_{yy} - |\psi|^2\psi = 0,$$

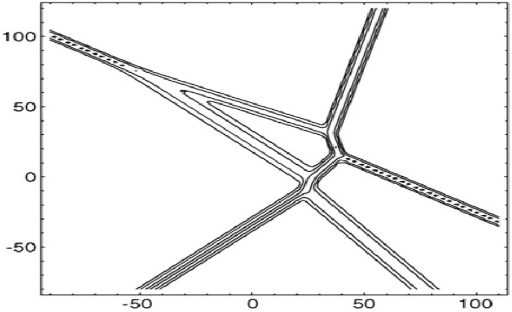
only (KP-I) arises in the asymptotic reduction on the nonzero background.

For the FPU lattice on the square lattice, only (KP-II) arises in the asymptotic reduction.

KP-II limit for small-amplitude, long-scale, transversely modulated waves

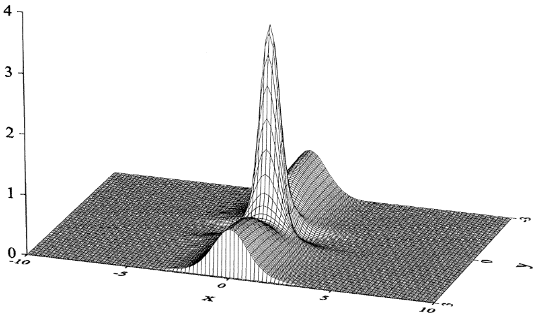
KP-II is equally attractive due to asymptotic stability of line solitary waves (Mizumachi, 2015), transverse stability of line periodic waves (Haragus, Li, P, 2017), and the web patterns of line solitons.

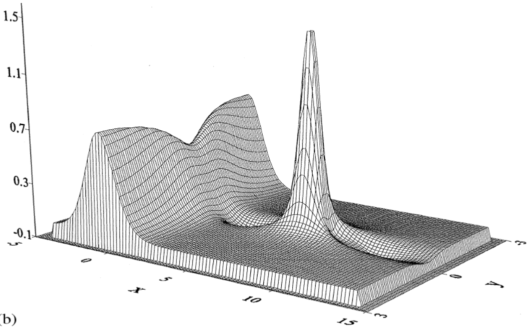




On comparison with KP-I

Line solitary and periodic waves are unstable for KP-I and the perturbations evolve into two-dimensional solitons called lumps.





(b)

Justification of KP-II limit

- By using the scalar model on 2D square lattice,

$$H = \sum_{(j,k)} \frac{1}{2}\dot{q}_{j,k}^2 + \frac{1}{2}(q_{j+1,k} - q_{j,k})^2 + \frac{1}{3}\alpha(q_{j+1,k} - q_{j,k})^3 + \frac{1}{2}\varepsilon^2(q_{j,k+1} - q_{j,k})^2$$

Duncan–Eilbeck–Zakharov (1991) formally derived KP-II equation

$$\partial_\xi(\partial_\tau R + \alpha R \partial_\xi R + \frac{1}{24}\partial_\xi^3 R) + \partial_\eta^2 R = 0$$

- Rigorous justification of the KP-II limit has been an open problem for 20 years!
- It was only justified recently: Gallone–Pasquali (2021) on $\mathbb{T}^2$ and Hristov–P (2021) on $\mathbb{R}^2$

Vector FPU model on square lattice

- We look at the following Hamiltonian

$$H = \frac{1}{2}\sum_{(j,k)\in\mathbb{Z}^2} \dot{x}_{j,k}^2 + \dot{y}_{j,k}^2 + V(x_{j+1,k}-x_{j,k}, y_{j+1,k}-y_{j,k}) + \sum_{(j,k)\in\mathbb{Z}^2} V(y_{j,k+1}-y_{j,k}, x_{j,k+1}-x_{j,k}),$$

  where

$$V(r,s) = \frac{1}{2}(c_1^2 r^2 + c_2^2 s^2) + \frac{1}{3}\alpha_1 r^3 + \frac{1}{2}\alpha_2 r s^2.$$

- Aproximating function for the horizontal propagation

$$x_{j+1,k} - x_{j,k} = \varepsilon^2 A(\varepsilon\,(j - c_1 t)\,, \varepsilon^2 k, \varepsilon^3 t) + error$$

  with $\xi = \varepsilon(j - c_1 t)$, $\eta = \varepsilon^2 k$, and $\tau = \varepsilon^3 t$

- The approximation satisfies FPU up to a small error if A solves the KP-II equation given by

$$2c_1 \partial_\xi \partial_\tau A + \frac{c_1^2}{12}\partial_\xi^4 A + 2\alpha_1 \partial_\xi\,(A\partial_\xi A) + c_2^2 \partial_\eta^2 A = 0 \qquad \text{(KP)}$$

Strain variables

- For the justification analysis it is more convenient to introduce the following strain variables:

$$\left\{\begin{array}{ll} u^{(1)}_{j,k} := x_{j+1,k} - x_{j,k}, & u^{(2)}_{j,k} := x_{j,k+1} - x_{j,k}, \\ v^{(1)}_{j,k} := y_{j+1,k} - y_{j,k}, & v^{(2)}_{j,k} := y_{j,k+1} - y_{j,k}, \\ w_{j,k} := \dot{x}_{j,k}, & \\ z_{j,k} := \dot{y}_{j,k}. & \end{array}\right.$$

- This allows us to rewrite two second-order equations for $\ddot{x}_{j,k}$ and $\ddot{y}_{j,k}$ as six first-order equations with two compatibility conditions:

$$\begin{array}{ll} \dot{u}^{(1)}_{j,k} = w_{j+1,k} - w_{j,k}, & \dot{u}^{(2)}_{j,k} = w_{j,k+1} - w_{j,k}, \\ \dot{v}^{(1)}_{j,k} = z_{j+1,k} - z_{j,k}, & \dot{v}^{(2)}_{j,k} = z_{j,k+1} - z_{j,k}, \end{array}$$

$$\dot{w}_{j,k} = c_1^2 \left(u^{(1)}_{j,k} - u^{(1)}_{j-1,k} \right) + c_2^2 \left(u^{(2)}_{j,k} - u^{(2)}_{j,k-1} \right) + \cdots,$$

$$\dot{z}_{j,k} = c_1^2 \left(v^{(2)}_{j,k} - v^{(2)}_{j,k-1} \right) + c_2^2 \left(v^{(1)}_{j,k} - v^{(1)}_{j-1,k} \right) + \cdots.$$

Theorem (Horizontal propagation)

Theorem (Hristov–P, 2021)

Let $A \in C^0\left([-\tau_0,\tau_0], H^{s+9}\left(\mathbb{R}^2\right)\right)$ *be a solution to the KP-II equation with fixed* $s \geq 0$*, whose initial data* $A(\xi,\eta,0) = A_0$ *satisfies* $A_0 \in H^{s+9}\left(\mathbb{R}^2\right)$*,* $\partial_\xi^{-2}\partial_\eta^2 A_0 \in H^{s+9}\left(\mathbb{R}^2\right)$*, and*

$$\partial_\xi^{-1}\partial_\eta^2\left[\partial_\xi^{-2}\partial_\eta^2 A_0 + A_0^2\right] \in H^{s+3}\left(\mathbb{R}^2\right).$$

Then there are constants $C_0, C_1, \varepsilon_0 > 0$ *such that for* $\varepsilon \in (0,\varepsilon_0)$ *if the initial conditions of the two-dimensional FPU system satisfies*

$$\left\|u_{in}^{(1)} - \varepsilon^2 A_0\right\|_{\ell^2} + \left\|u_{in}^{(2)}\right\|_{\ell^2} + \left\|w_{in} + \varepsilon^2 c_1 A_0\right\|_{\ell^2} + \left\|v_{in}^{(1)}\right\|_{\ell^2} + \left\|v_{in}^{(2)}\right\|_{\ell^2} + \|z_{in}\|_{\ell^2} \leq C_0 \varepsilon^{\frac{5}{2}}$$

then the solution to the two-dimensional FPU system satisfies for $t \in [-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}]$

$$\left\|u^{(1)}(t) - \varepsilon^2 A\right\|_{\ell^2} + \left\|u^{(2)}(t)\right\|_{\ell^2} + \left\|w(t) + \varepsilon^2 c_1 A\right\|_{\ell^2} + \left\|v^{(1)}(t)\right\|_{\ell^2} + \left\|v^{(2)}(t)\right\|_{\ell^2} + \|z(t)\|_{\ell^2} \leq C_1 \varepsilon^{\frac{5}{2}}.$$

Outline of Proof

1. Well-posedness of KP-II equation in Sobolev spaces of high regularity
2. Approximation bound between sequences on $\mathbb{Z}^2$ and smooth functions on $\mathbb{R}^2$ with slow scaling
3. Expansions to satisfy the compatibility conditions
4. Control of residual terms
5. Energy estimates
6. Bounds on the approximation error in the time evolution

Vertical propagation follows by symmetry of the square lattice.
Diagonal or any oblique propagation leads to further problems and open questions.

Time evolution of the KP-II equation

The KP-II equation

$$2c_1\partial_\xi\partial_\tau A + \frac{c_1^2}{12}\partial_\xi^4 A + 2\alpha\partial_\xi\left(A\partial_\xi A\right) + c_2^2\partial_\eta^2 A = 0$$

can be written in the evolution form

$$\partial_\tau A + \frac{c_1}{24}\partial_\xi^3 A + \frac{\alpha}{c_1}\left(A\partial_\xi A\right) + \frac{c_2^2}{2c_1}\partial_\xi^{-1}\partial_\eta^2 A = 0,$$

where $\partial_\xi^{-1} A = \int_{-\infty}^{\xi} A(\xi')d\xi'$.

- In the residual terms, we need a bound in Sobolev norm for terms of the form $\partial_\xi^{-1}\partial_\tau^2 A$. These terms are related to $\partial_\xi^{-2}\partial_\eta^2\partial_\tau A$ or $\partial_\tau^3 A$.
- No such terms arise in the justification of KP-II from 2D Boussinesq [Gallay–Schneider, 2001]

$$\partial_t^2 u - \Delta u + \Delta^2 u + \Delta(u^2) = 0.$$

Time evolution of the KP-II equation

Lemma (Hristov–P, 2021)

For any $A_0 \in H^{s+9}\left(\mathbb{R}^2\right)$ *such that* $\partial_\xi^{-2}\partial_\eta^2 A_0 \in H^{s+9}\left(\mathbb{R}^2\right)$ *and*

$$\partial_\xi^{-1}\partial_\eta^2\left[\partial_\xi^{-2}\partial_\eta^2 A_0 + A_0^2\right] \in H^{s+3}\left(\mathbb{R}^2\right)$$

with fixed $s \geq 0$, *there exists* $\tau_0 > 0$ *such that the Cauchy problem admits a unique solution*

$$A \in C^0\left(\left[-\tau_0, \tau_0\right], H^{s+9}\right) \cap C^1\left(\left[-\tau_0, \tau_0\right], H^{s+6}\right) \cap C^2\left(\left[-\tau_0, \tau_0\right], H^{s+3}\right) \cap C^3\left(\left[-\tau_0, \tau_0\right], H^{s}\right).$$

Analysis is based on writing the evolution problm for $D := \partial_\xi^{-2}\partial_\eta^2 A$,

$$\partial_\tau D + \partial_\xi^3 D + \partial_\xi^{-1}\partial_\eta^2(D + A^2) = 0.$$

Approximation result between sequences on $\mathbb{Z}^2$ and functions on $\mathbb{R}^2$

Lemma (Hristov–P, 2021)

Let $u_{j,k} = U(\varepsilon j, \varepsilon^2 k)$*, with* $U \in H^s(\mathbb{R}^2)$*,* $s > 1$*. Then, there is a constant* $C_s > 0$*, such that for every* $\varepsilon \in (0,1)$ *we have*

$$\|u\|_{\ell^2(\mathbb{Z}^2)} \leq C_s \varepsilon^{-3/2} \|U\|_{H^s(\mathbb{R}^2)}, \qquad \forall U \in H^s\left(\mathbb{R}^2\right).$$

- One-dimensional result loses only $\varepsilon^{-1/2}$, extra power of epsilon due to the ε^2 scaling of η.
- The result is an exercise on Fourier transforms on $\mathbb{Z}^2$ and on $\mathbb{R}^2$.

Expansions to satisfy the compatibility conditions

Here is the starting system of equations of motion:

$$\dot{u}^{(1)}_{j,k} = w_{j+1,k} - w_{j,k}, \qquad \dot{u}^{(2)}_{j,k} = w_{j,k+1} - w_{j,k},$$
$$\dot{v}^{(1)}_{j,k} = z_{j+1,k} - z_{j,k}, \qquad \dot{v}^{(2)}_{j,k} = z_{j,k+1} - z_{j,k},$$

and

$$\begin{aligned}
\dot{w}_{j,k} =& c_1^2 \left(u^{(1)}_{j,k} - u^{(1)}_{j-1,k} \right) + c_2^2 \left(u^{(2)}_{j,k} - u^{(2)}_{j,k-1} \right) + \alpha_1 \left[\left(u^{(1)}_{j,k} \right)^2 - \left(u^{(1)}_{j-1,k} \right)^2 \right] \\
& + \alpha_2 \left[u^{(2)}_{j,k} v^{(2)}_{j,k} - u^{(2)}_{j,k-1} v^{(2)}_{j,k-1} + \frac{1}{2} \left(v^{(1)}_{j,k} \right)^2 - \frac{1}{2} \left(v^{(1)}_{j-1,k} \right)^2 \right] \\
\dot{z}_{j,k} =& c_1^2 \left(v^{(2)}_{j,k} - v^{(2)}_{j,k-1} \right) + c_2^2 \left(v^{(1)}_{j,k} - v^{(1)}_{j-1,k} \right) + \alpha_1 \left[\left(v^{(2)}_{j,k} \right)^2 - \left(v^{(2)}_{j,k-1} \right)^2 \right] \\
& + \alpha_2 \left[u^{(1)}_{j,k} v^{(1)}_{j,k} - u^{(1)}_{j-1,k} v^{(1)}_{j-1,k} + \frac{1}{2} \left(u^{(2)}_{j,k} \right)^2 - \frac{1}{2} \left(u^{(2)}_{j,k-1} \right)^2 \right]
\end{aligned}$$

Expansions to satisfy the compatibility conditions

- We introduce the following decomposition:

$$\begin{aligned}
u^{(1)}_{j,k} &= \varepsilon^2 A(\xi,\eta,\tau) + \varepsilon^2 U^{(1)}_{j,k}\\
u^{(2)}_{j,k} &= \varepsilon^2 B_\varepsilon(\xi,\eta,\tau) + \varepsilon^2 U^{(2)}_{j,k}\\
v^{(1)}_{j,k} &= \varepsilon^2 V^{(1)}_{j,k}\\
v^{(2)}_{j,k} &= \varepsilon^2 V^{(2)}_{j,k}\\
w_{j,k} &= \varepsilon^2 W_\varepsilon(\xi,\eta,\tau) + \varepsilon^2 W_{j,k}\\
z_{j,k} &= \varepsilon^2 Z_{j,k},
\end{aligned}$$

where $\xi = \varepsilon(j - c_1 t), \eta = \varepsilon^2 k, \tau = \varepsilon^3 t$

- Here B_ε, and W_ε are introduced to satisfy the linear equations of motion:

$$\dot{u}^{(1)}_{j,k} = w_{j+1,k} - w_{j,k}, \qquad \dot{u}^{(2)}_{j,k} = w_{j,k+1} - w_{j,k},$$

Expansions to satisfy the compatibility conditions

- These equations are satisfied up to $\mathcal{O}(\varepsilon^5)$ order:

$$W_\varepsilon(\xi+\varepsilon,\eta) - W_\varepsilon(\xi,\eta) = -\varepsilon c_1 \partial_\xi A(\xi,\eta) + \varepsilon^3 \partial_\tau A(\xi,\eta),$$
$$W_\varepsilon(\xi,\eta+\varepsilon^2) - W_\varepsilon(\xi,\eta) = -\varepsilon c_1 \partial_\xi B_\varepsilon(\xi,\eta) + \varepsilon^3 \partial_\tau B_\varepsilon(\xi,\eta).$$

- We seek an approximate solution by expanding $W_\varepsilon, B_\varepsilon$ in orders of ε
- $W_\varepsilon = -c_1 A + \varepsilon\left(\frac{c_1}{2}\partial_\xi A\right) + \varepsilon^2\left(\partial_\xi^{-1}\partial_\tau A - \frac{c_1}{12}\partial_\xi^2 A\right) - \varepsilon^3\left(\frac{1}{2}\partial_\tau A\right)$
- $B_\varepsilon = \varepsilon\partial_\xi^{-1}\partial_\eta A - \varepsilon^2\left(\frac{1}{2}\partial_\eta A\right) + \varepsilon^3\left(\frac{1}{2}\partial_\xi^{-1}\partial_\eta^2 A + \frac{1}{12}\partial_\xi\partial_\eta A\right)$
- By construction of terms W_ε and B_ε, the residual terms of the two equations vanish to $\mathcal{O}(\varepsilon^5)$.

Control of residual terms

The last two remaining equations are

$$\dot{W}_{j,k} = c_1^2 \left[U^{(1)}_{j,k} - U^{(1)}_{j-1,k} \right] + c_2^2 \left[U^{(2)}_{j,k} - U^{(2)}_{j,k-1} \right]$$

$$+ \alpha_1 \varepsilon^2 \left[2AU^{(1)}_{j,k} - 2A(\xi - \varepsilon, \eta) U^{(1)}_{j-1,k} + \left(U^{(1)}_{j,k} \right)^2 - \left(U^{(1)}_{j-1,k} \right)^2 \right]$$

$$+ \alpha_2 \varepsilon^2 \left[B_\varepsilon(\xi, \eta) V^{(2)}_{j,k} - B_\varepsilon(\xi, \eta - \varepsilon^2) V^{(2)}_{j,k-1} + U^{(2)}_{j,k} V^{(2)}_{j,k} - U^{(2)}_{j,k-1} V^{(2)}_{j,k-1} \right]$$

$$+ \alpha_2 \varepsilon^2 \left[\frac{1}{2} \left(V^{(1)}_{j,k} \right)^2 - \frac{1}{2} \left(V^{(1)}_{j-1,k} \right)^2 \right] + Res^W_{j,k}$$

$$\dot{Z}_{j,k} = c_2^2 \left[V^{(1)}_{j,k} - V^{(1)}_{j-1,k} \right] + c_1^2 \left[V^{(2)}_{j,k} - V^{(2)}_{j,k-1} \right]$$

$$+ \alpha_2 \varepsilon^2 \left[B_\varepsilon(\xi, \eta) U^{(2)}_{j,k} - B_\varepsilon(\xi, \eta - \varepsilon^2) U^{(2)}_{j,k-1} + \frac{1}{2} \left(U^{(2)}_{j,k} \right)^2 - \frac{1}{2} \left(U^{(2)}_{j,k-1} \right)^2 \right]$$

$$+ \alpha_2 \varepsilon^2 \left[A(\xi, \eta) V^{(1)}_{j,k} - A(\xi - \varepsilon, \eta) V^{(1)}_{j-1,k} + V^{(1)}_{j,k} U^{(1)}_{j,k} - V^{(1)}_{j-1,k} U^{(1)}_{j-1,k} \right]$$

$$+ \alpha_1 \varepsilon^2 \left[\left(V^{(2)}_{j,k} \right)^2 - \left(V^{(2)}_{j,k-1} \right)^2 \right] + Res^Z_{j,k}$$

Control of residual terms

- Residuals are given by:

$$
\begin{aligned}
Res^{W}_{j,k} :=& c_1 \varepsilon \partial_\xi W_\varepsilon - \varepsilon^3 \partial_\tau W_\varepsilon + c_1^2 \left[A(\xi,\eta) - A(\xi-\varepsilon,\eta) \right] \\
& + c_2^2 \left[B_\varepsilon(\xi,\eta) - B_\varepsilon(\xi,\eta-\varepsilon^2) \right] + \alpha_1 \varepsilon^2 \left[A(\xi,\eta)^2 - A(\xi-\varepsilon,\eta)^2 \right], \\
Res^{Z}_{j,k} :=& \frac{\alpha_2 \varepsilon^2}{2} \left[B_\varepsilon(\xi,\eta)^2 - B_\varepsilon(\xi,\eta-\varepsilon^2)^2 \right].
\end{aligned}
$$

- Expanding Res^W gives the following formal expansion:

$$
\begin{aligned}
Res^{W}_{j,k} = & \quad \varepsilon^3 \left[2c_1 \partial_\tau A + \frac{c_1^2}{12} \partial_\xi^3 A + c_2^2 \partial_\xi^{-1} \partial_\eta^2 A + \alpha_1 \partial_\xi \left(A^2\right) \right] \\
& - \varepsilon^4 \left[c_1 \partial_\xi \partial_\tau A + \frac{c_1^2}{24} \partial_\xi^4 A + \frac{c_2^2}{2} \partial_\eta^2 A + \frac{\alpha_1}{2} \partial_\xi^2 \left(A^2\right) \right] + \mathcal{O}(\varepsilon^5).
\end{aligned}
$$

- Res^Z has a formal order of $\mathcal{O}(\varepsilon^6)$

Control of residual terms

Lemma

Let $A \in C^0(\mathbb{R}, H^s)$ be a solution to the KP-II equation (KP) with $s \geq 9$. There is a positive constant C that depend on A such that for all $\varepsilon \in (0,1]$, we have

$$\left\| Res_{j,k}^{U^{(1)}} \right\|_{\ell^2} + \left\| Res_{j,k}^{U^{(2)}} \right\|_{\ell^2} + \left\| Res_{j,k}^{W} \right\|_{\ell^2} + \left\| Res_{j,k}^{Z} \right\|_{\ell^2} \leq C\varepsilon^{\frac{7}{2}}.$$

- The formal expansions of the residual terms are handled using Taylor's theorem, e.g.

$$A(\xi+\varepsilon,\eta)-A(\xi,\eta) = \varepsilon\partial_\xi A+\frac{1}{2}\varepsilon^2\partial_\xi^2 A+\frac{1}{3!}\varepsilon^3\partial_\xi^3 A+\frac{1}{4!}\varepsilon^4\partial_\xi^4 A+\frac{1}{4!}\varepsilon^5\int_0^1 (1-r)^4\partial_\xi^5 A(\varepsilon(j+r),\varepsilon^2 k,\varepsilon^3 t)dr$$

- The integral residual terms is estimated in ℓ^2-norm for every r on $[0,1]$
- Since the rigorous bound loses $\mathcal{O}(\varepsilon^{-3/2})$, the formal bound of $\mathcal{O}(\varepsilon^5)$ yields $\mathcal{O}(\varepsilon^{7/2})$ in the ℓ^2-norm.

Energy Estimates

- Recall the total energy of the FPU system in strain variables

$$H = \frac{1}{2} \sum_{(j,k)\in\mathbb{Z}^2} w_{j,k}^2 + z_{j,k}^2 + c_1^2 (u_{j,k}^{(1)})^2 + c_2^2 (u_{j,k}^{(2)})^2 + c_1^2 (v_{j,k}^{(1)})^2 + c_2^2 (v_{j,k}^{(2)})^2$$
$$+ \frac{1}{3}\alpha_1 \sum_{(j,k)\in\mathbb{Z}^2} (u_{j,k}^{(1)})^3 + (v_{j,k}^{(2)})^3 + \frac{1}{2}\alpha_2 \sum_{(j,k)\in\mathbb{Z}^2} (u_{j,k}^{(1)})(v_{j,k}^{(1)})^2 + (u_{j,k}^{(2)})^2 (v_{j,k}^{(2)}).$$

- This suggests the following energy quantity to control the growth of the approximation error:

$$E(t) = \frac{1}{2} \sum_{j,k\in\mathbb{Z}^2} W_{j,k}^2 + Z_{j,k}^2 + c_1^2 \left(U_{j,k}^{(1)}\right)^2 + c_2^2 \left(U_{j,k}^{(2)}\right)^2 + c_1^2 \left(V_{j,k}^{(1)}\right)^2 + c_2^2 \left(V_{j,k}^{(2)}\right)^2$$
$$+ \frac{1}{3}\alpha_1 \varepsilon^2 \sum_{j,k\in\mathbb{Z}^2} \left[3A \left(U_{j,k}^{(1)}\right)^2 + \left(U_{j,k}^{(1)}\right)^3 + \left(V_{j,k}^{(2)}\right)^3 \right]$$
$$+ \frac{1}{2}\alpha_2 \varepsilon^2 \sum_{j,k\in\mathbb{Z}^2} \left[A \left(V_{j,k}^{(1)}\right)^2 + U_{j,k}^{(1)} \left(V_{j,k}^{(1)}\right)^2 + \left(U_{j,k}^{(2)}\right)^2 V_{j,k}^{(2)} + 2B_\varepsilon U_{j,k}^{(2)} V_{j,k}^{(2)} \right].$$

Energy Estimates

- Here we can recall the decomposition

$$\begin{aligned} u^{(1)}_{j,k} &= \varepsilon^2 A(\xi,\eta,\tau) + \varepsilon^2 U^{(1)}_{j,k}, & v^{(1)}_{j,k} &= \varepsilon^2 V^{(1)}_{j,k} \\ u^{(2)}_{j,k} &= \varepsilon^2 B_\varepsilon(\xi,\eta,\tau) + \varepsilon^2 U^{(2)}_{j,k}, & v^{(2)}_{j,k} &= \varepsilon^2 V^{(2)}_{j,k} \\ w_{j,k} &= \varepsilon^2 W_\varepsilon(\xi,\eta,\tau) + \varepsilon^2 W_{j,k}, & z_{j,k} &= \varepsilon^2 Z_{j,k}, \end{aligned}$$

- The ε-dependent terms in the energy $E(t)$ are chosen such that the growth rate $E'(t)$ does not contain terms up to the formal order $\mathcal{O}(\varepsilon^2)$
- The energy is used to control the approximation errors in the following sense. Assume that $E(t) \leq E_0$ for some ε-independent constant $E_0 > 0$ for every $t \in [-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}]$. There exist some constants $\varepsilon_0 > 0$ and $K_0 > 0$ that depend on A such that

$$K_0 E(t) \leq \|W\|^2_{\ell^2} + \|Z\|^2_{\ell^2} + \left\|U^{(1)}\right\|^2_{\ell^2} + \left\|U^{(2)}\right\|^2_{\ell^2} + \left\|V^{(1)}\right\|^2_{\ell^2} + \left\|V^{(2)}\right\|^2_{\ell^2} \leq 2K_0 E(t),$$

for each $\varepsilon \in (0,\varepsilon_0)$ and $t \in [-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}]$.

Energy Estimates

By differentiating $E(t)$, we obtain

$$E'(t) = \sum_{j,k\in\mathbb{Z}^2} W_{j,k} Res^W_{j,k} + Z_{j,k} Res^Z_{j,k} + \cdots$$
$$+ \alpha_1 \varepsilon^2 \left(-c_1 \varepsilon \partial_\xi A + \varepsilon^3 \partial_\tau A\right) \left(U^{(1)}_{j,k}\right)^2$$
$$+ \alpha_2 \varepsilon^2 \left(-c_1 \varepsilon \partial_\xi B_\varepsilon + \varepsilon^3 \partial_\tau B_\varepsilon\right) U^{(2)}_{j,k} V^{(2)}_{j,k} + \alpha_2 \varepsilon^2 \left(-c_1 \varepsilon \partial_\xi A + \varepsilon^3 \partial_\tau A\right) \left(V^{(1)}_{j,k}\right)^2.$$

From here, Cauchy-Schwartz and the estimates for the residual terms and for the approximation errors gives the differential inequality:

$$|E'(t)| \leq C \left(\varepsilon^{7/2} E(t)^{1/2} + \varepsilon^3 E(t)\right),$$

for some $C_0 > 0$ as long as $E(t) \leq E_0$ for some $E_0 > 0$.

Bounds on the approximation error in the time evolution

- By making the substitution $E(t) := \frac{1}{2}Q(t)^2$, we obtain:

$$|Q'(t)| \le C\left(\varepsilon^{7/2} + \varepsilon^3 Q\right) \tag{QB}$$

- We obtain a bound on Q(t) from the Gronwall lemma

Lemma

Assume that $Q(t)$ satisfies (QB) for $t \in [-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}]$ and $Q(0) \le C_0\varepsilon^{1/2}$ for some ε-independent constant C_0. There exists $\varepsilon_0 > 0$ such that

$$Q(t) \le \varepsilon^{1/2}(1 + C_0)\exp(C\tau_0)$$

for each $\varepsilon \in (0, \varepsilon_0)$ and $t \in [-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}]$.

Bounds on the approximation error in the time evolution

- Recall the energy bounds:

$$K_0 E(t) \leq \|W\|_{\ell^2}^2 + \|Z\|_{\ell^2}^2 + \left\|U^{(1)}\right\|_{\ell^2}^2 + \left\|U^{(2)}\right\|_{\ell^2}^2 + \left\|V^{(1)}\right\|_{\ell^2}^2 + \left\|V^{(2)}\right\|_{\ell^2}^2 \leq 2K_0 E(t),$$

- Hence the initial bound gives

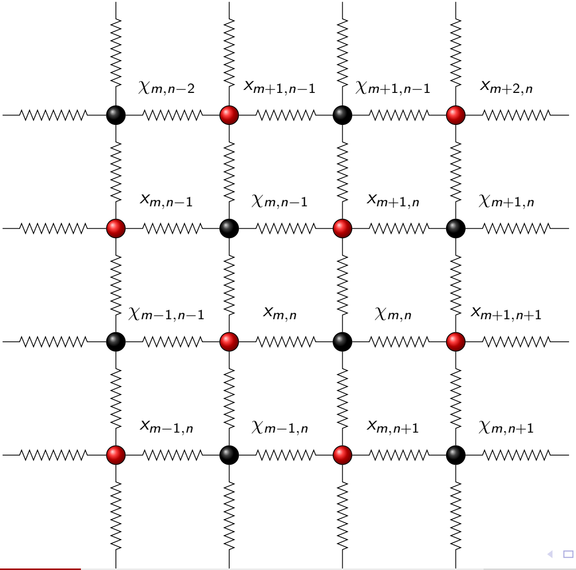
$$Q(0) \leq C_A \left(\left\|U_{in}^{(1)}\right\|_{\ell^2} + \left\|U_{in}^{(2)}\right\|_{\ell^2} + \|W_{in}\|_{\ell^2} + \left\|V_{in}^{(1)}\right\|_{\ell^2} + \left\|V_{in}^{(2)}\right\|_{\ell^2} + \|Z_{in}\|_{\ell^2} \right) \leq C_0 \varepsilon^{1/2}$$

- Gronwall's lemma and the decomposition then yield

$$\begin{aligned} &\left\| u^{(1)}(t) - \varepsilon^2 A\left(\varepsilon(j - c_1 t), \eta^2 k, \varepsilon^3 t\right) \right\|_{\ell^2} + \left\| u^{(2)}(t) - \varepsilon^2 B_\varepsilon\left(\varepsilon(j - c_1 t), \eta^2 k, \varepsilon^3 t\right) \right\|_{\ell^2} \\ &+ \left\| w(t) - \varepsilon^2 W_\varepsilon\left(\varepsilon(j - c_1 t), \eta^2 k, \varepsilon^3 t\right) \right\|_{\ell^2} + \left\| v^{(1)}(t) \right\|_{\ell^2} + \left\| v^{(2)}(t) \right\|_{\ell^2} + \|z(t)\|_{\ell^2} \leq K_0 \varepsilon^2 Q(t), \end{aligned}$$

which is bounded by $C\varepsilon^{5/2}$. This completes the proof of the justification theorem.

Towards other directions of the propagation



KP-II equation for the diagonal propagation

- Diagonal propagation is similar to diatomic lattice

$$m = \frac{1}{2}(j+k), \quad n = \frac{1}{2}(j-k), \quad x_{m,n} := x_{j,k}, \quad \chi_{m,n} := x_{j+1,k}.$$

- Formal aproximating function

$$x_{m+1,n} - x_{m,n} = \varepsilon^2 A(\varepsilon\,(m - c_1^* t)\,, \varepsilon^2\,(n - c_2^* t)\,, \varepsilon^3 t) + error$$

  where $c_1^* = \frac{1}{2}\sqrt{c_1^2 + c_2^2}$ and $c_2^* = \frac{1}{2}\sqrt{c_1^2 - c_2^2}$

- However, it is hard to control error in general case because non-local terms related to KP-II solutions appear at lower orders in ε

- With N. Hristov, we only succeeded to justify the KP-II equation for the choice $c_1 = c_2$ and $\alpha_2 = 2\alpha_1$, for which the FPU system is satisfied by the invariant reduction $x_{j,k} = y_{j,k}$ and $c_2^* = 0$.

Other propagation directions

With G. Schneider, we achieved a better progress for arbitrary directions of propagation.

The first simplification is to work with the second-order equations, rather than the first-order equations. From

$$\begin{cases} \dot{u}_{m,n} = w_{m+1,n} - w_{m,n}, \\ \dot{v}_{m,n} = w_{m,n+1} - w_{m,n}, \\ \dot{w}_{m,n} = V'(u_{m,n}) - V'(u_{m-1,n}) + V'(v_{m,n}) - V'(v_{m,n-1}). \end{cases}$$

we eliminate $w_{m,n}$ and get

$$\begin{cases} \ddot{u}_{m,n} = V'(u_{m+1,n}) - 2V'(u_{m,n}) + V'(u_{m-1,n}) \\ \qquad\qquad + V'(v_{m+1,n}) - V'(v_{m+1,n-1}) - V'(v_{m,n}) + V'(v_{m,n-1}), \\ \ddot{v}_{m,n} = V'(v_{m,n+1}) - 2V'(v_{m,n}) + V'(v_{m,n-1}) \\ \qquad\qquad + V'(u_{m,n+1}) - V'(u_{m-1,n+1}) - V'(u_{m,n}) + V'(u_{m-1,n}), \end{cases}$$

There exists still a compatibility condition between $u_{m,n}$ and $v_{m,n}$.

Other propagation directions

The second simplification is to use the Fourier transform and convert the system into the form:

$$\begin{cases} \partial_t^2 \widehat{u} = -(\omega_k^2 + \omega_l^2)\widehat{u} + \omega_k^2(\widehat{u} * \widehat{u}) - (e^{-ik} - 1)(1 - e^{il})(\widehat{v} * \widehat{v}), \\ \partial_t^2 \widehat{v} = -(\omega_k^2 + \omega_l^2)\widehat{v} + \omega_l^2(\widehat{v} * \widehat{v}) - (e^{-il} - 1)(1 - e^{ik})(\widehat{u} * \widehat{u}). \end{cases}$$

where $\omega_k^2 := 2 - 2\cos(k)$ and we use $V'(u) = u - u^2$ just for simplifications.

The compatibility condition between $u_{m,n}$ and $v_{m,n}$ can be expressed easier in the Fourier form as

$$(e^{-ik} - 1)\widehat{v}(k, l, t) = (e^{-il} - 1)\widehat{u}(k, l, t).$$

Other propagation directions

The leading order approximation for an arbitrary angle ϕ can be expressed by

$$u_{m,n}(t) = \varepsilon^2 A(X,Y,T), \qquad v_{m,n}(t) = \varepsilon^2 B(X,Y,T),$$

where

$$X = \varepsilon((\cos\phi)m + (\sin\phi)n - t), \quad Y = \varepsilon^2(-(\sin\phi)m + (\cos\phi)n), \quad T = \varepsilon^3 t.$$

This yields the extended KP-II equation

$$\begin{aligned} -2\partial_X\partial_T A = & \frac{1}{12}[(\cos\phi)^4 + (\sin\phi)^4]\partial_X^4 A + \partial_Y^2 A \\ & - (\cos\phi)^2\partial_X^2(A^2) - (\sin\phi)(\cos\phi))\partial_X^2(B^2) \\ & - \frac{1}{3}\varepsilon[(\cos\phi)^2 - (\sin\phi)^2](\cos\phi)(\sin\phi)\partial_X^3\partial_Y A + 2\varepsilon(\cos\phi)(\sin\phi)\partial_X\partial_Y(A^2) \\ & - \varepsilon[(\cos\phi)^2 - (\sin\phi)^2]\partial_X\partial_Y(B^2) \\ & - \frac{1}{2}\varepsilon[\cos\phi - \sin\phi](\cos\phi)(\sin\phi)\partial_X^3(B^2). \end{aligned} \tag{1}$$

and $(\cos\phi)\partial_X B = (\sin\phi)\partial_X A$ up to the leading order.

Other propagation directions

The extended KP-II equation splits into the KP-II equation and the linearized KP-II equation, where we need to control $\partial_X^{-1}\partial_Y(A^2)$ in Sobolev spaces. However, this is impossible on $\mathbb{R}^2$.

On other hand, working on torus $\mathbb{T}^2$ (Bourgain, 1993), if the mean value of A in X is independent of Y, then $\partial_X^{-3}\partial_Y^3 A$ is controllable in $H^s(\mathbb{T}^2)$ and so is $\partial_X^{-1}\partial_Y(A^2)$.

Thus, we will be able to justify the KP-II equation for arbitrary directions of propagations on $\mathbb{T}^2$, but not on $\mathbb{R}^2$ (P-Schneider, 2022).

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Thank you for your attention. Questions ???