Stability of domain walls in coupled systems

Dmitry Pelinovsky
Department of Mathematics, McMaster University

SIAM PDE Conference, Baltimore, December 2017

Joint work with:
S. Alama, L. Bronsard, A. Contreras *(ARMA 2015)*
A. Contreras, M. Plum *(SIMA 2018)*

Domain walls

- Bulk energy with stable states

$$W: \ \mathbb{R}^2 \to \mathbb{R}, \quad W(u) \geq 0, \quad W(p_+) = W(p_-) = 0$$

Domain walls

- Bulk energy with stable states

$$W: \mathbb{R}^2 \to \mathbb{R}, \quad W(u) \geq 0, \quad W(p_+) = W(p_-) = 0$$

- The total energy

$$E(u) = \int_{\mathbb{R}} \left[\frac{1}{2} |\nabla u|^2 + W(u) \right] dx$$

Domain walls

- Bulk energy with stable states

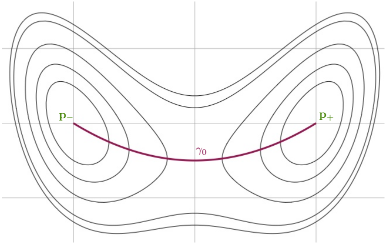
$$W : \mathbb{R}^2 \to \mathbb{R}, \quad W(u) \geq 0, \quad W(p_+) = W(p_-) = 0$$

- The total energy

$$E(u) = \int_{\mathbb{R}} \left[\frac{1}{2} |\nabla u|^2 + W(u) \right] dx$$

- Domain walls are stationary layers with profile U connecting $p_\pm$:

$$-U'' + DW(U) = 0, \qquad U \to p_\pm \text{ as } x \to \pm\infty$$



Example: Gross-Pitaevskii System

Motivated by two-component Bose-Einstein condensates,

$$\begin{aligned} i\partial_t \psi_1 &= -\partial_x^2 \psi_1 + (g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2)\psi_1, \\ i\partial_t \psi_2 &= -\partial_x^2 \psi_2 + (g_{12}|\psi_1|^2 + g_{22}|\psi_2|^2)\psi_2, \end{aligned}$$

with $\psi_{1,2}(x,t) \in \mathbb{C}$, $g_{11}, g_{22} > 0$, and $g_{12} > \sqrt{g_{11} g_{22}}$

Example: Gross-Pitaevskii System

Motivated by two-component Bose-Einstein condensates,

$$i\partial_t \psi_1 = -\partial_x^2 \psi_1 + (g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2)\psi_1,$$
$$i\partial_t \psi_2 = -\partial_x^2 \psi_2 + (g_{12}|\psi_1|^2 + g_{22}|\psi_2|^2)\psi_2,$$

with $\psi_{1,2}(x,t) \in \mathbb{C}$, $g_{11}, g_{22} > 0$, and $g_{12} > \sqrt{g_{11}g_{22}}$

For $g_{11} = g_{22} = 1$, $g_{12} = \gamma > 1$, and $\psi_j(x,t) = e^{-it}u_j(x)$,

$$-u_1'' + (u_1^2 + \gamma u_2^2 - 1)u_1 = 0,$$
$$-u_2'' + (\gamma u_1^2 + u_2^2 - 1)u_2 = 0.$$

The bulk energy is:

$$W(u_1, u_2) = \frac{1}{2}\left(|u_1|^2 + |u_2|^2 - 1\right)^2 + (\gamma - 1)|u_1|^2|u_2|^2.$$

Barankov (2002), Dror-Malomed-Zeng (2011), Filatrella–Malomed (2014)

Domain walls versus black solitons

Domain walls (u_1, u_2) satisfy the boundary-value problem:

$$-u_1'' + (u_1^2 + \gamma u_2^2 - 1)u_1 = 0,$$
$$-u_2'' + (\gamma u_1^2 + u_2^2 - 1)u_2 = 0,$$

with $(u_1, u_2) \to (0, 1)$ as $x \to -\infty$, and $(u_1, u_2) \to (1, 0)$ as $x \to +\infty$.

Exact solution for $\gamma = 3$:

$$u_1(x) = \frac{1}{2}\left[1 + \tanh\left(\frac{x}{\sqrt{2}}\right)\right], \quad u_2(x) = \frac{1}{2}\left[1 - \tanh\left(\frac{x}{\sqrt{2}}\right)\right].$$

Domain walls versus black solitons

Domain walls (u_1, u_2) satisfy the boundary-value problem:

$$-u_1'' + (u_1^2 + \gamma u_2^2 - 1)u_1 = 0,$$
$$-u_2'' + (\gamma u_1^2 + u_2^2 - 1)u_2 = 0,$$

with $(u_1, u_2) \to (0, 1)$ as $x \to -\infty$, and $(u_1, u_2) \to (1, 0)$ as $x \to +\infty$.

Exact solution for $\gamma = 3$:

$$u_1(x) = \frac{1}{2}\left[1 + \tanh\left(\frac{x}{\sqrt{2}}\right)\right], \quad u_2(x) = \frac{1}{2}\left[1 - \tanh\left(\frac{x}{\sqrt{2}}\right)\right].$$

Black solitons satisfy the same problem and exist for all values of γ:

$$(u_1, u_2) = (u_b, 0) \quad \text{and} \quad (u_1, u_2) = (0, u_b)$$

where $u_b = \tanh(x/\sqrt{2})$.

Existence Theorem

Recall the energy $E(U) = \int_{\mathbb{R}} [\frac{1}{2}|U'|^2 + W(U)]dx$ with $U = (u_1, u_2)$ and

$$W(U) = \frac{1}{2}\left(|u_1|^2 + |u_2|^2 - 1\right)^2 + (\gamma - 1)|u_1|^2|u_2|^2.$$

Theorem (Alama-Bronsard-Contreras-P., 2015)

- *The infimum of $E(U)$ is attained at a solution with $U(x) \to e_\pm$ as $x \to \pm\infty$, where $e_+ = (1, 0)$ and $e_- = (0, 1)$.*
- *Every minimizer $U = (u_1, u_2)$ satisfies*
  (a) $u_1(x) = u_2(-x)$ *for all* $x \in \mathbb{R}$.
  (b) $u_1^2(x) + u_2^2(x) \leq 1$ *for all* $x \in \mathbb{R}$.
  (c) $u_1'(x) > 0$ *and* $u_2'(x) < 0$ *for all* $x \in \mathbb{R}$.
  (d) $0 < u_{1,2}(x) < 1$ *with exponential convergence to constant states.*

Uniqueness was proven Aftalion-Sourdis (2016); Farina-Sciunzi-Soave (2017).

Spaces for Minimization

Recall the energy $E(U) = \int_{\mathbb{R}} [\frac{1}{2}|U'|^2 + W(U)]dx$ with $U = (u_1, u_2)$ and

$$W(U) = \frac{1}{2}\left(|u_1|^2 + |u_2|^2 - 1\right)^2 + (\gamma - 1)|u_1|^2|u_2|^2.$$

A minimizing sequence belongs to the energy space

$$\mathcal{D} = \left\{ U \in H^1_{loc}(\mathbb{R}) : \quad |U(x)| \to e_{\pm} \quad \text{as} \quad x \to \pm\infty \right\},$$

equipped with the family of distances parameterized by $A > 0$:

$$\rho_A(\Psi, \Phi) := \sum_{j=1,2} \left[\left\| \psi_j' - \varphi_j' \right\|_{L^2(\mathbb{R})} + \left\| |\psi_j| - |\varphi_j| \right\|_{L^2(\mathbb{R})} + \left\| \psi_j - \varphi_j \right\|_{L^\infty(-A,A)} \right]$$

Complex phases are not controlled far away from the domain wall.

F. Bethuel, P. Gravejat, J.C. Saut, D. Smets (2008)

Spectral stability of domain walls

Define the second variation of energy $D^2E(U)$ at a minimizer U. For the perturbation term $\Phi = V + iW$, the second variation is diagonalized:

$$D^2E(U)[\Phi] = \langle L_+V, V\rangle + \langle L_-W, W\rangle.$$

Theorem (ABCP, 2015)

- *Each operator L_+ and L_- is positive semi-definite in $H^1(\mathbb{R})$.*
- *Zero is a simple eigenvalue of L_+, with eigenfunction $U'(x)$.*
- $\sigma_{ess}(L_-) = [0, \infty)$, *and* $\exists\ \Sigma_0 > 0$ *with* $\sigma_{ess}(L_+) = [\Sigma_0, \infty)$.
- $L_-U_1 = L_-U_2 = 0$ *with* $U_1 = (u_1, 0)$ *and* $U_2 = (0, u_2)$.

As a consequence, the domain walls are spectrally stable: eigenvalues of the linearized flow satisfy $\mathrm{Re}\,(\lambda) = 0$.
DiMenza-Gallo (2007).

Domain walls versus black solitons

- Domain walls (u_1, u_2) are minimizers of energy with positive semi-definite second variation.

- Black solitons

$$(u_1, u_2) = (u_b, 0) \quad \text{and} \quad (u_1, u_2) = (0, u_b)$$

are saddle points of energy with a simple negative eigenvalue of the second variation. They are constrained minimizers under the conservation of the renormalized momentum.

Domain walls versus black solitons

- Domain walls (u_1, u_2) are minimizers of energy with positive semi-definite second variation.

- Black solitons

$$(u_1, u_2) = (u_b, 0) \quad \text{and} \quad (u_1, u_2) = (0, u_b)$$

  are saddle points of energy with a simple negative eigenvalue of the second variation. They are constrained minimizers under the conservation of the renormalized momentum.

For black solitons of the scalar NLS, it was shown that the energy functional is coercive in a weighted H^1 space. The family of distance ρ_A is abundant, since the weighted H^1 metric is introduced uniformly on $\mathbb{R}$. Both orbital stability and asymptotic stability is deduced from coercivity: Gravejat–Smets (2015).

Decomposition of energy

Recall the second variation of energy $E(U)$ for $\Phi = V + iW$:

$$D^2 E(U)[\Phi] = \langle L_+ V, V \rangle + \langle L_- W, W \rangle,$$

where

$$(L_+ V, V)_{L^2} \geq C_0 \|V\|^2_{H^1} \quad \text{for every} \quad V \in H^1(\mathbb{R}) : \quad (V, \partial_x U)_{L^2} = 0$$

but

$$(L_- W, W)_{L^2} \geq 0, \quad \text{with} \quad L_- U_1 = L_- U_2 = 0$$

with $U_1 = (u_1, 0)$ and $U_2 = (0, u_2)$.

Decomposition of energy

Recall the second variation of energy $E(U)$ for $\Phi = V + iW$:

$$D^2E(U)[\Phi] = \langle L_+ V, V\rangle + \langle L_- W, W\rangle,$$

where

$$(L_+ V, V)_{L^2} \geq C_0 \|V\|^2_{H^1} \quad \text{for every} \quad V \in H^1(\mathbb{R}) : \quad (V, \partial_x U)_{L^2} = 0$$

but

$$(L_- W, W)_{L^2} \geq 0, \quad \text{with} \quad L_- U_1 = L_- U_2 = 0$$

with $U_1 = (u_1, 0)$ and $U_2 = (0, u_2)$.

Energy can be decomposed in the form:

$$E(U + V + iW) - E(U) = (L_+ V, V)_{L^2} + (L_- W, W)_{L^2} + \mathcal{O}(\|V + iW\|^3_{H^1}),$$

The cubic terms cannot be controlled in H^1 norm because of modulations.

Alternative decomposition of energy

Energy can be decomposed in the equivalent way [Gravejat–Smets (2015)]:

$$E(U + V + iW) - E(U) = (L_- V, V)_{L^2} + (L_- W, W)_{L^2} + \frac{1}{2}(M\Upsilon, \Upsilon)_{L^2},$$

where $\Upsilon = (\eta_1, \eta_2)$ with $\eta_j := |u_j + v_j + iw_j|^2 - u_j^2 = 2u_j v_j + v_j^2 + w_j^2$ and

$$M = \begin{bmatrix} 1 & \gamma \\ \gamma & 1 \end{bmatrix} : \quad \det(M) = 1 - \gamma^2 < 0.$$

Alternative decomposition of energy

Energy can be decomposed in the equivalent way [Gravejat–Smets (2015)]:

$$E(U+V+iW) - E(U) = (L_- V, V)_{L^2} + (L_- W, W)_{L^2} + \frac{1}{2}(M\Upsilon, \Upsilon)_{L^2},$$

where $\Upsilon = (\eta_1, \eta_2)$ with $\eta_j := |u_j + v_j + iw_j|^2 - u_j^2 = 2u_j v_j + v_j^2 + w_j^2$ and

$$M = \begin{bmatrix} 1 & \gamma \\ \gamma & 1 \end{bmatrix} : \quad \det(M) = 1 - \gamma^2 < 0.$$

- The first two quadratic forms are coercive in $\mathcal{H}$ under two constraints:

$$\langle \Psi, \Phi \rangle_{\mathcal{H}} := \sum_{j=1}^{2} \int_{\mathbb{R}} \left[\frac{d\psi_j}{dx} \frac{d\bar{\varphi}_j}{dx} + (\gamma - 1)(1 - u_j^2)\psi_j \bar{\varphi}_j \right] dx,$$

  where $\gamma > 1$ and $1 - u_j^2 > 0$. Then, $\|\Psi\|_{\mathcal{H}} \leq C\|\Psi\|_{H^1}$.
- Only one constraint can be set on V.
- The third quadratic form is sign-indefinite.

Orbital Stability

Weighted H^1 space:

$$\langle \Psi, \Phi \rangle_{\mathcal{H}} := \sum_{j=1}^{2} \int_{\mathbb{R}} \left[\frac{d\psi_j}{dx} \frac{d\bar{\varphi}_j}{dx} + (\gamma - 1)(1 - u_j^2)\psi_j \bar{\varphi}_j \right] dx,$$

equipped $\mathcal{H}$ with the family of distances parameterized by $R > 0$:

$$\rho_R(\Psi, \Phi) := \left\| \Psi - \Phi \right\|_{\mathcal{H}} + \sum_{j=1,2} \left\| |\psi_j|^2 - |\varphi_j|^2 \right\|_{L^2(|x| \geq R)}.$$

Theorem (Contreras-P-Plum, 2018)

Let $\Psi_0 \in \mathcal{D} \cap L^\infty(\mathbb{R})$. *There exists* $R_0 > 0$ *such that for any* $R > R_0$ *and for every* $\varepsilon > 0$, *there is* $\delta > 0$ *and real functions* $\alpha(t), \theta_1(t), \theta_2(t)$ *such that if* $\rho_R(\Psi_0, U) \leq \delta$, *then* $\sup_{t \in \mathbb{R}} \rho_R(\Psi(t), U_{\alpha(t),\theta_1(t),\theta_2(t)}) \leq \varepsilon$, *where*

$$U_{\alpha(t),\theta_1(t),\theta_2(t)} = (e^{-i\theta_1(t)} u_1(\cdot - \alpha(t)), e^{-i\theta_2(t)} u_2(\cdot - \alpha(t))).$$

Remarks

- Modulation parameters α, θ_1, and θ_2 in the orbit of domain walls

$$U_{\alpha(t),\theta_1(t),\theta_2(t)} = (e^{-i\theta_1(t)} u_1(\cdot - \alpha(t)), e^{-i\theta_2(t)} u_2(\cdot - \alpha(t)))$$

  are uniquely determined by the projection algorithm.

Remarks

- Modulation parameters α, θ_1, and θ_2 in the orbit of domain walls
$$U_{\alpha(t),\theta_1(t),\theta_2(t)} = (e^{-i\theta_1(t)}u_1(\cdot - \alpha(t)), e^{-i\theta_2(t)}u_2(\cdot - \alpha(t)))$$
are uniquely determined by the projection algorithm.
- The time evolution of the modulation parameters is controlled:
$$|\alpha(t)| + |\theta_1(t)| + |\theta_2(t)| \leq C\varepsilon(1 + |t|), \quad t \in \mathbb{R}$$
for some $C > 0$.

Remarks

- Modulation parameters α, θ_1, and θ_2 in the orbit of domain walls

$$U_{\alpha(t),\theta_1(t),\theta_2(t)} = (e^{-i\theta_1(t)} u_1(\cdot - \alpha(t)), e^{-i\theta_2(t)} u_2(\cdot - \alpha(t)))$$

  are uniquely determined by the projection algorithm.
- The time evolution of the modulation parameters is controlled:

$$|\alpha(t)| + |\theta_1(t)| + |\theta_2(t)| \leq C\varepsilon(1 + |t|), \quad t \in \mathbb{R}$$

  for some $C > 0$.
- If R is large, then δ and ε are exponentially small in R.

Remarks

- Modulation parameters α, θ_1, and θ_2 in the orbit of domain walls
$$U_{\alpha(t),\theta_1(t),\theta_2(t)} = (e^{-i\theta_1(t)} u_1(\cdot - \alpha(t)), e^{-i\theta_2(t)} u_2(\cdot - \alpha(t)))$$
are uniquely determined by the projection algorithm.
- The time evolution of the modulation parameters is controlled:
$$|\alpha(t)| + |\theta_1(t)| + |\theta_2(t)| \leq C\varepsilon(1 + |t|), \quad t \in \mathbb{R}$$
for some $C > 0$.
- If R is large, then δ and ε are exponentially small in R.
- The distances ρ_A and ρ_R are not comparable:
$$\rho_A(\Psi, \Phi) := \sum_{j=1,2} \left[\|\psi_j' - \varphi_j'\|_{L^2(\mathbb{R})} + \||\psi_j| - |\varphi_j|\|_{L^2(\mathbb{R})} + \|\psi_j - \varphi_j\|_{L^\infty(-A,A)} \right]$$
and
$$\rho_R(\Psi, \Phi) := \|\Psi - \Phi\|_{\mathcal{H}} + \sum_{j=1,2} \||\psi_j|^2 - |\varphi_j|^2\|_{L^2(|x| \geq R)}.$$

Coercivity of $(L_- W, W)_{L^2}$ in $\mathcal{H}$

Consider

$$(L_- W, W)_{L^2} = \|W\|^2_{\mathcal{H}} - \gamma \langle TW, W \rangle_{\mathcal{H}},$$

where $T : \mathcal{H} \to \mathcal{H}$ is the compact positive operator defined by

$$\langle T\Psi, \Phi \rangle_{\mathcal{H}} := \int_{\mathbb{R}} \left(1 - u_1^2 - u_2^2\right) \left(\psi_1 \bar{\varphi}_1 + \psi_2 \bar{\varphi}_2\right) dx.$$

Coercivity of $(L_- W, W)_{L^2}$ in $\mathcal{H}$

Consider

$$(L_- W, W)_{L^2} = \|W\|^2_{\mathcal{H}} - \gamma \langle TW, W\rangle_{\mathcal{H}},$$

where $T : \mathcal{H} \to \mathcal{H}$ is the compact positive operator defined by

$$\langle T\Psi, \Phi\rangle_{\mathcal{H}} := \int_{\mathbb{R}} \left(1 - u_1^2 - u_2^2\right)\left(\psi_1 \bar{\varphi}_1 + \psi_2 \bar{\varphi}_2\right) dx.$$

Lemma

There exists $\Lambda_- > 0$ *such that*

$$(L_- W, W)_{L^2} \geq \Lambda_- \|W\|^2_{\mathcal{H}} \quad \forall W \in \mathcal{H} : \quad \langle W, U_1\rangle_{\mathcal{H}} = \langle W, U_2\rangle_{\mathcal{H}} = 0.$$

- The spectrum of L_- in $\mathcal{H}$ consists of isolated eigenvalues accumulating to 1.
- The smallest eigenvalue of L_- is a double zero with $U_1 = (u_1, 0) \in \mathcal{H}$ and $U_2 = (0, u_2) \in \mathcal{H}$.

Coercivity of $(L_+ V, V)_{L^2}$ in $\mathcal{H}$

Break $\mathbb{R}$ into $(-\infty,-R) \cup (-R,R) \cup (R,\infty)$. Then, define

$$\begin{aligned} L_R &= L_- + 2 \begin{bmatrix} u_1^2 & \gamma u_1 u_2 \\ \gamma u_1 u_2 & u_2^2 \end{bmatrix} \chi_{[-R,R]} \\ &= L_+ - 2 \begin{bmatrix} u_1^2 & \gamma u_1 u_2 \\ \gamma u_1 u_2 & u_2^2 \end{bmatrix} \chi_{(-\infty,-R)\cup(R,\infty)}. \end{aligned}$$

As $R \to \infty$, $L_R \to L_+$ and $L_+ \partial_x U = 0$ with $\partial_x U \in \mathcal{H}$.

Coercivity of $(L_+ V, V)_{L^2}$ in $\mathcal{H}$

Break $\mathbb{R}$ into $(-\infty, -R) \cup (-R, R) \cup (R, \infty)$. Then, define

$$\begin{aligned} L_R &= L_- + 2 \begin{bmatrix} u_1^2 & \gamma u_1 u_2 \\ \gamma u_1 u_2 & u_2^2 \end{bmatrix} \chi_{[-R,R]} \\ &= L_+ - 2 \begin{bmatrix} u_1^2 & \gamma u_1 u_2 \\ \gamma u_1 u_2 & u_2^2 \end{bmatrix} \chi_{(-\infty,-R)\cup(R,\infty)}. \end{aligned}$$

As $R \to \infty$, $L_R \to L_+$ and $L_+ \partial_x U = 0$ with $\partial_x U \in \mathcal{H}$.

Lemma

There exist $R_0 > 0$ and $\Lambda_+ > 0$ such that for every $R > R_0$,

$$(L_R V, V)_{L^2} \geq \Lambda_+ \|V\|^2_{\mathcal{H}} \quad \forall V \in \mathcal{H} : \quad \langle W, \partial_x U \rangle_{\mathcal{H}} = 0.$$

- The spectrum of L_R in $\mathcal{H}$ consists of isolated eigenvalues accumulating to 1.
- The spectrum of L_+ is only defined in $H^1(\mathbb{R})$ and includes a continuous part bounded from below by 1.

Energy estimates

Decomposition of energy:

$$
\begin{aligned}
E(U+V+iW)-E(U) &= (L_R V, V)_{L^2} + (L_- W, W)_{L^2} \\
&+ \int_{-R}^{R} [N_3(V,W) + N_4(V,W)]\, dx + \frac{1}{2}\left(\int_{-\infty}^{-R} + \int_{R}^{\infty}\right)(\eta_1^2 + \eta_2^2)dx \\
&+ \gamma \int_{-\infty}^{-R} \eta_2 (2u_1 v_1 + v_1^2 + w_1^2)dx + \gamma \int_{R}^{\infty} \eta_1 (2u_2 v_2 + v_2^2 + w_2^2)dx.
\end{aligned}
$$

The rest of the proof:

- Estimates of nonlinear terms with

$$
\|V+iW\|_{H^1(-R,R)} \leq Ce^{\kappa R}\|V+iW\|_{\mathcal{H}}
$$

- Estimates of the last terms with

$$
\left|\int_R^\infty \eta_1(2u_2v_2 + v_2^2 + w_2^2)dx\right| \leq Ce^{-\kappa R}\|V+iW\|_{\mathcal{H}}\|\eta_1\|_{L^2(|x|\geq R)}.
$$

Domain walls versus black solitons

Persistence of stationary solutions in the ε-perturbed system:

$$i\partial_t \psi_1 = -\partial_x^2 \psi_1 + \varepsilon V(x)\psi_1 + (|\psi_1|^2 + \gamma|\psi_2|^2)\psi_1,$$
$$i\partial_t \psi_2 = -\partial_x^2 \psi_2 + \varepsilon V(x)\psi_2 + (\gamma|\psi_1|^2 + |\psi_2|^2)\psi_2,$$

under a smooth and integrable potential $V \in C^2(\mathbb{R}) \cap L^1(\mathbb{R})$.

- Domain walls (u_1, u_2) are pinned to the extremal points of the potential V and the pinning is stable at the maximum of the potential. (Dror-Malomed-Zeng 2011, ABCP 2015).

- Black solitons

  $$(u_1, u_2) = (u_b, 0) \quad \text{and} \quad (u_1, u_2) = (0, u_b)$$

  are also pinned to the extremal points of the potential V but the pinning is unstable both at the maximum and minimum of the potential. (P–Kevrekidis 2008).

Persistence of domain walls

Persistence of domain walls in the ε-perturbed system:

$$i\partial_t \psi_1 = -\partial_x^2 \psi_1 + \varepsilon V(x)\psi_1 + (|\psi_1|^2 + \gamma |\psi_2|^2)\psi_1,$$
$$i\partial_t \psi_2 = -\partial_x^2 \psi_2 + \varepsilon V(x)\psi_2 + (\gamma |\psi_1|^2 + |\psi_2|^2)\psi_2.$$

Theorem (Alama-Bronsard-Contreras-P., 2015)

Let $U_0 = (u_1, u_2)$ be a domain wall with $\gamma > 1$ and $\varepsilon = 0$. For a given $V \in C^2(\mathbb{R}) \cap L^1(\mathbb{R})$, assume that there exists $x_0 \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} V'(x + x_0)(u_1^2 + u_2^2 - 1)dx = 0.$$

There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the system admits a unique branch of the domain walls $U = (u_1, u_2)$ such that

$$\sup_{x \in \mathbb{R}} |U(x) - U_0(x - x_0)| \leq C|\varepsilon|, \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

If V is even in x, then $x_0 = 0$.

Stability of domain walls

Theorem (ABCP, 2015)

The domain walls in the ε-perturbed system are spectrally stable if $\sigma > 0$ and unstable if $\sigma < 0$, where

$$\sigma := \frac{1}{2} \int_{\mathbb{R}} V''(x + x_0)(u_1^2 + u_2^2 - 1)dx \neq 0.$$

- $L_-(\varepsilon)$ remains semi-positive operator with no spectral gap.
- The isolated zero eigenvalue of $L_+(\varepsilon)$ becomes positive if $\sigma > 0$ and negative if $\sigma < 0$.

If V is even and $V''(x) > 0$, then the domain wall at $x_0 = 0$ is unstable.

Summary

Domain walls (u_1, u_2) (with $\gamma = 3$)

$$u_1(x) = \frac{1}{2}\left[1 + \tanh\left(\frac{x}{\sqrt{2}}\right)\right], \quad u_2(x) = \frac{1}{2}\left[1 - \tanh\left(\frac{x}{\sqrt{2}}\right)\right].$$

- minimizers of energy without any constraints
- orbitally stable
- persist under small perturbations
- are pinned to the maximum of potential
- asymptotically stable (conjecture)
- do not travel in space (conjecture)

Happy Holidays!

Joyeuse Fêtes!