Domain walls with and without external potentials

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Joint work with:
S. Alama, L. Bronsard, A. Contreras *(ARMA 2015)*
A. Contreras, M. Plum *(SIMA 2018)*
A. Contreras, V. Slastikov *(Calc Var PDE 2022)*

I: Existence of domain walls without potentials

Solitary waves in nonlinear PDEs

Bright soliton $\psi(t,x) = e^{it}\mathrm{sech}(x)$
of the focusing NLS equation

$$i\partial_t\psi + \partial_x^2\psi + 2|\psi|^2\psi = 0$$

satisfying $|\psi(t,x)| \to 0$ as $|x| \to \infty$

Dark soliton $\psi(t,x) = e^{-2it}\tanh(x)$
of the defocusing NLS equation

$$i\partial_t\psi + \partial_x^2\psi - 2|\psi|^2\psi = 0$$

satisfying $|\psi(t,x)| \to 1$ as $|x| \to \infty$

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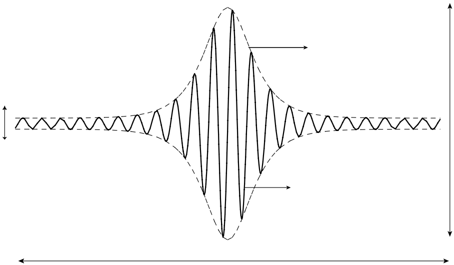

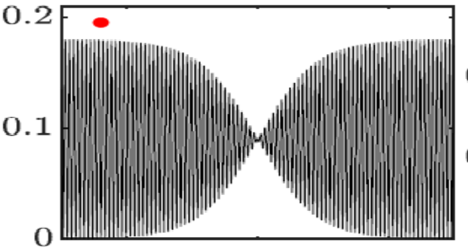

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satisfying $|\psi(t,x)| \to 1$ as $|x| \to \infty$

Coupled NLS models have bright-bright, bright-dark, dark-dark solitons:

$$\begin{aligned} i\partial_t\psi_1 + \partial_x^2\psi_1 + (\pm|\psi_1|^2 \pm |\psi_2|^2)\psi_1 = 0, \\ i\partial_t\psi_2 + \partial_x^2\psi_2 + (\pm|\psi_1|^2 \pm |\psi_2|^2)\psi_2 = 0. \end{aligned}$$

Domain walls satisfy

$$\begin{aligned} |\psi_1(t,x)| \to 0, \quad |\psi_2(t,x)| \to 1, \quad \text{as} \;\; x \to \mp\infty \\ |\psi_1(t,x)| \to 1, \quad |\psi_2(t,x)| \to 0, \quad \text{as} \;\; x \to \pm\infty \end{aligned}$$

B. Malomed, "Past and present trends in the development of the pattern-formation theory", Physics 2021, 3(4), 1015–1045

Domain walls from the energetic point of view

- Bulk energy with stable states

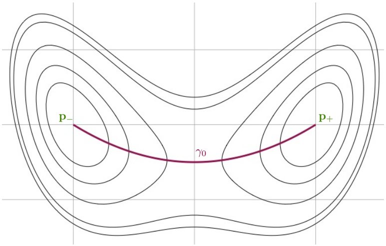
$$W : \mathbb{R}^2 \to \mathbb{R}, \quad W(u) \geq 0, \quad W(p_+) = W(p_-) = 0$$

- The total energy

$$E(u) = \int_{\mathbb{R}} \left[\frac{1}{2} |\nabla u|^2 + W(u) \right] dx$$

- Domain walls are heteroclinic orbits with profile U connecting $p_\pm$:

$$-U'' + DW(U) = 0, \qquad U \to p_\pm \text{ as } x \to \pm\infty$$



Example: Gross-Pitaevskii system without potentials

Motivated by two-component (repulsive) Bose-Einstein condensates,

$$i\partial_t \psi_1 = -\partial_x^2 \psi_1 + (g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2)\psi_1,$$
$$i\partial_t \psi_2 = -\partial_x^2 \psi_2 + (g_{12}|\psi_1|^2 + g_{22}|\psi_2|^2)\psi_2,$$

with $g_{11} > 0$, $g_{22} > 0$, and $g_{12} > \sqrt{g_{11} g_{22}}$.

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\end{aligned}
$$

with $g_{11} > 0$, $g_{22} > 0$, and $g_{12} > \sqrt{g_{11} g_{22}}$.

With normalization $g_{11} = g_{22} = 1$, $g_{12} = \gamma > 1$, the standing waves $\psi_j(t,x) = e^{-it} u_j(x)$ satisfy

$$
\begin{aligned}
-u_1'' + (u_1^2 + \gamma u_2^2 - 1)u_1 &= 0, \\
-u_2'' + (\gamma u_1^2 + u_2^2 - 1)u_2 &= 0,
\end{aligned}
$$

with the bulk energy

$$
W(u_1, u_2) = \frac{1}{2}\left(|u_1|^2 + |u_2|^2 - 1\right)^2 + (\gamma - 1)|u_1|^2 |u_2|^2.
$$

Barankov (2002), Dror-Malomed-Zeng (2011), Filatrella–Malomed (2014)

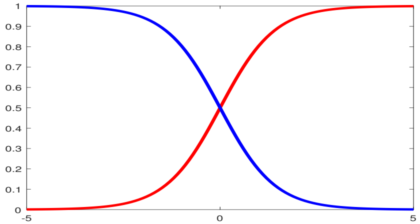
Domain walls satisfy the boundary-value problem:

$$-u_1'' + (u_1^2 + \gamma u_2^2 - 1)u_1 = 0,$$
$$-u_2'' + (\gamma u_1^2 + u_2^2 - 1)u_2 = 0,$$

with $(u_1, u_2) \to (0, 1)$ as $x \to \mp\infty$, and $(u_1, u_2) \to (1, 0)$ as $x \to \pm\infty$.

Example: exact solution for $\gamma = 3$:

$$u_1(x) = \frac{1}{2}\left[1 + \tanh\left(\frac{x}{\sqrt{2}}\right)\right], \quad u_2(x) = \frac{1}{2}\left[1 - \tanh\left(\frac{x}{\sqrt{2}}\right)\right].$$



Domain walls among other stationary states

Other positive solutions exist in the coupled system:

$$
\begin{aligned}
-u_1'' + (u_1^2 + \gamma u_2^2 - 1)u_1 &= 0, \\
-u_2'' + (\gamma u_1^2 + u_2^2 - 1)u_2 &= 0.
\end{aligned}
$$

- Uncoupled states $(u_1, u_2) = (1, 0)$ and $(u_1, u_2) = (0, 1)$
- Coupled symmetric state $(u_1, u_2) = (1 + \gamma)^{-1/2}(1, 1)$.

Recall $\boxed{W(u_1, u_2) = \frac{1}{2}\left(u_1^2 + u_2^2 - 1\right)^2 + (\gamma - 1)u_1^2 u_2^2.}$

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Recall $\boxed{W(u_1, u_2) = \frac{1}{2}\left(u_1^2 + u_2^2 - 1\right)^2 + (\gamma - 1)u_1^2 u_2^2.}$

For $\gamma \in (0, 1)$,

$$W(u_1, u_2) \geq -\frac{\gamma(1-\gamma)}{2(1+\gamma)^2} = W((1+\gamma)^{-1/2}, (1+\gamma)^{-1/2})$$

hence the symmetric state is the minimizer of $W(u_1, u_2)$ for $\gamma \in (0, 1)$.

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$$-u_1'' + (u_1^2 + \gamma u_2^2 - 1)u_1 = 0,$$
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For $\gamma > 1$,

$$W(u_1, u_2) \geq 0 = W(1, 0) = W(0, 1)$$

hence the uncoupled states are the minimizers of $W(u_1, u_2)$ for $\gamma > 1$.

Domain walls among other stationary states

Other positive solutions exist in the coupled system:

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Recall $\boxed{W(u_1, u_2) = \frac{1}{2}\left(u_1^2 + u_2^2 - 1\right)^2 + (\gamma - 1)u_1^2 u_2^2.}$

Domain walls are the minimizers of the energy $E(U) = \frac{1}{2}\int_{\mathbb{R}}[|u_1'|^2 + |u_2'|^2 + W(u_1, u_2)]dx$ in the energy space

$$\mathcal{D} = \left\{ U = (u_1, u_2) \in H^1_{loc}(\mathbb{R}) : \quad |U(x)| \to e_{\pm} \quad \text{as} \quad x \to \pm\infty \right\}.$$

Such minimizers only exist for $\gamma > 1$.

Existence of domain walls

Recall the energy $E(U) = \frac{1}{2}\int_{\mathbb{R}}[|U'|^2 + W(U)]dx$ with $U = (u_1, u_2)$ and

$$W(U) = \frac{1}{2}\left(|u_1|^2 + |u_2|^2 - 1\right)^2 + (\gamma - 1)|u_1|^2|u_2|^2.$$

Theorem (Alama–Bronsard–Contreras–P., 2015)

For $\gamma > 1$,

- *The infimum of $E(U)$ is attained among functions $U \in H^1_{\rm loc}(\mathbb{R})$ with $U(x) \to e_\pm$ as $x \to \pm\infty$, where $e_+ = (1,0)$ and $e_- = (0,1)$.*
- *Every minimizer $U = (u_1, u_2)$ satisfies*
  - (a) $u_1(x) = u_2(-x)$ *for all* $x \in \mathbb{R}$.
  - (b) $u_1^2(x) + u_2^2(x) \leq 1$ *for all* $x \in \mathbb{R}$.
  - (c) $u_1'(x) > 0$ *and* $u_2'(x) < 0$ *for all* $x \in \mathbb{R}$.
  - (d) $0 < u_{1,2}(x) < 1$ *with exponential convergence to constant states.*

Uniqueness in Aftalion-Sourdis (2016); Farina-Sciunzi-Soave (2017).

Bifurcations of domain walls

For $\gamma = 1$, there exists rotational state $(u_1, u_2) = (\sin\theta, \cos\theta)$ for which

$$W(u_1, u_2) \geq 0 = W(\sin\theta, \cos\theta).$$

Bifurcation of domain walls can be anticipated from the rotational state.

A useful asymptotic approximation $(u_1, u_2) = (\sin\theta, \cos\theta)$ on $u_1^2 + u_2^2 = 1$ for $\gamma \approx 1$ (Contreras-P-Slastikov, 2022):

$$E(U) = \frac{1}{2}\int_{\mathbb{R}} [|\theta'|^2 + (\gamma - 1)\sin^2\theta\cos^2\theta]dx$$

Minimizers satisfy $-\theta''(x) + \frac{1}{4}(\gamma - 1)\sin^2(4\theta) = 0$ and exist for $\gamma > 1$ in the exact form: $\theta(x) = \frac{\pi}{2} - \arctan(e^{-\sqrt{\gamma-1}x})$.

II: Orbital stability of domain walls without potentials

Decomposition of energy for stability argument

Energy

$$E(U) = \frac{1}{2}\int_{\mathbb{R}} (|\psi_1'|^2 + |\psi_2|^2 + \frac{1}{2}(|\psi_1|^2 + |\psi_2|^2 - 1)^2 + (\gamma - 1)|\psi_1|^2|\psi_2|^2) dx$$

is decomposed in the form:

$$E(U+V+iW) - E(U) = (L_+ V, V)_{L^2} + (L_- W, W)_{L^2} + \mathcal{O}(\|V + iW\|^3_{H^1(\mathbb{R})}),$$

where $L_\pm$ are matrix Schrödinger operators for Hessians. However, cubic terms cannot be controlled in $H^1(\mathbb{R})$ because of phase modulations.

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The second variation satisfies the following properties:

- Self-adjoint operator L_+ and L_- are positive semi-definite in $H^1(\mathbb{R})$.
- $\exists\ \Sigma_0 > 0 : \sigma_{ess}(L_+) = [\Sigma_0, \infty)$. $\sigma_{ess}(L_-) = [0, \infty)$
- Zero is a simple eigenvalue of L_+, with eigenfunction $\partial_x U > 0$.
- $L_- U_1 = L_- U_2 = 0$ with $U_1 = (u_1, 0)$ and $U_2 = (0, u_2)$.

Decomposition of energy for stability argument

Energy

$$E(U) = \frac{1}{2}\int_{\mathbb{R}} (|\psi_1'|^2 + |\psi_2|^2 + \frac{1}{2}(|\psi_1|^2 + |\psi_2|^2 - 1)^2 + (\gamma - 1)|\psi_1|^2|\psi_2|^2) dx$$

is decomposed in the form:

$$E(U + V + iW) - E(U) = (L_+ V, V)_{L^2} + (L_- W, W)_{L^2} + \mathcal{O}(\|V + iW\|^3_{H^1(\mathbb{R})}),$$

where $L_\pm$ are matrix Schrödinger operators for Hessians. However, cubic terms cannot be controlled in $H^1(\mathbb{R})$ because of phase modulations.
As a result, we have coercivity

$$(L_+ V, V)_{L^2} \geq C_0 \|V\|^2_{H^1} \quad \text{for every} \quad V \in H^1(\mathbb{R}) : \quad (V, \partial_x U)_{L^2} = 0$$

and the lack of coercivity:

$$(L_- W, W)_{L^2} \geq 0, \quad \text{with} \quad L_- U_1 = L_- U_2 = 0.$$

The problem of phase modulations

Similar problems are known for dark solitons in the scalar NLS equation

$$i\psi_t + \psi_{xx} - |\psi|^2\psi = 0.$$

Perturbations in the energy space can be considered with the distance:

$$\rho_A(\psi, \varphi) := \left[\|\psi' - \varphi'\|_{L^2(\mathbb{R})} + \||\psi|^2 - |\varphi|^2\|_{L^2(\mathbb{R})} + \|\psi - \varphi\|_{L^\infty(-A,A)}\right]$$

for some $A > 0$
Bethuel–Gravejat–Saut–Smets (2008)
or with the exponentially weighted distance:

$$\rho(\psi, \varphi) = \left[\|\psi' - \varphi'\|_{L^2(\mathbb{R})} + \||\psi|^2 - |\varphi|^2\|_{L^2(\mathbb{R})} + \|\mathrm{sech}(\cdot)(\psi - \varphi)\|_{L^2(\mathbb{R})}\right]$$

Gravejat–Smets (2015)

Alternative decomposition of energy

Energy can be decomposed in the equivalent way:

$$E(U+V+iW)-E(U)=(L_- V,V)_{L^2}+(L_- W,W)_{L^2}+\frac{1}{2}(M\Upsilon,\Upsilon)_{L^2},$$

where $\Upsilon=(\eta_1,\eta_2)$ with $\eta_j:=|u_j+v_j+iw_j|^2-u_j^2=2u_jv_j+v_j^2+w_j^2$ and

$$M=\begin{bmatrix}1 & \gamma\\ \gamma & 1\end{bmatrix}: \quad \det(M)=1-\gamma^2<0.$$

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$$E(U+V+iW)-E(U)=(L_-V,V)_{L^2}+(L_-W,W)_{L^2}+\frac{1}{2}(M\Upsilon,\Upsilon)_{L^2},$$

One can introduce weighted H^1 space:

$$\langle\Psi,\Phi\rangle_{\mathcal{H}}:=\sum_{j=1}^{2}\int_{\mathbb{R}}\left[\frac{d\psi_j}{dx}\frac{d\bar{\varphi}_j}{dx}+(\gamma-1)(1-u_j^2)\psi_j\bar{\varphi}_j\right]dx$$

and write

$$(L_-W,W)_{L^2}=\|W\|_{\mathcal{H}}^2-\gamma\langle TW,W\rangle_{\mathcal{H}},$$

where $T:\mathcal{H}\to\mathcal{H}$ is the compact positive operator defined by

$$\langle T\Psi,\Phi\rangle_{\mathcal{H}}:=\int_{\mathbb{R}}\left(1-u_1^2-u_2^2\right)\left(\psi_1\bar{\varphi}_1+\psi_2\bar{\varphi}_2\right)dx.$$

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$$E(U + V + iW) - E(U) = (L_- V, V)_{L^2} + (L_- W, W)_{L^2} + \frac{1}{2}(M\Upsilon, \Upsilon)_{L^2},$$

Then,

- The spectrum of L_- in $\mathcal{H}$ consists of isolated eigenvalues accumulating to 1.
- The smallest eigenvalue of L_- is a double zero with $U_1 = (u_1, 0) \in \mathcal{H}$ and $U_2 = (0, u_2) \in \mathcal{H}$.

As a result, the quadratic form is coercive under the two constraints

$$(L_- W, W)_{L^2} \geq C\|W\|^2_{\mathcal{H}} \quad \forall W \in \mathcal{H} : \quad \langle W, U_1\rangle_{\mathcal{H}} = \langle W, U_2\rangle_{\mathcal{H}} = 0.$$

Alternative decomposition of energy

Energy can be decomposed in the equivalent way:

$$E(U + V + iW) - E(U) = (L_- V, V)_{L^2} + (L_- W, W)_{L^2} + \frac{1}{2} (M\Upsilon, \Upsilon)_{L^2},$$

However,

- Only one constraint can be set on V in $(L_- V, V)_{L^2}$.
- The nonlinear part $(M\Upsilon, \Upsilon)_{L^2}$ is sign-indefinite since

$$M = \begin{bmatrix} 1 & \gamma \\ \gamma & 1 \end{bmatrix} : \quad \det(M) = 1 - \gamma^2 < 0.$$

(Revised) alternative decomposition of energy

We introduce the family of distances parameterized by (large) $R > 0$:

$$\rho_R(\Psi, \Phi) := \left\| \Psi - \Phi \right\|_{\mathcal{H}} + \sum_{j=1,2} \left\| |\psi_j|^2 - |\varphi_j|^2 \right\|_{L^2(|x| \geq R)}.$$

in addition to

$$\langle \Psi, \Phi \rangle_{\mathcal{H}} := \sum_{j=1}^{2} \int_{\mathbb{R}} \left[\frac{d\psi_j}{dx} \frac{d\bar{\varphi}_j}{dx} + (\gamma - 1)(1 - u_j^2)\psi_j \bar{\varphi}_j \right] dx.$$

(Revised) alternative decomposition of energy

The revised alternative decomposition can be controlled in ρ_R:

$$\begin{aligned} & E(U+V+iW) - E(U) = (L_R V, V)_{L^2} + (L_- W, W)_{L^2} \\ & + \int_{-R}^{R} [N_3(V,W) + N_4(V,W)]\, dx + \frac{1}{2}\left(\int_{-\infty}^{-R} + \int_{R}^{\infty}\right)(\eta_1^2 + \eta_2^2) dx \\ & + \gamma \int_{-\infty}^{-R} \eta_2 (2u_1 v_1 + v_1^2 + w_1^2) dx + \gamma \int_{R}^{\infty} \eta_1 (2u_2 v_2 + v_2^2 + w_2^2) dx, \end{aligned}$$

where

$$\begin{aligned} L_R &= L_- + 2 \begin{bmatrix} u_1^2 & \gamma u_1 u_2 \\ \gamma u_1 u_2 & u_2^2 \end{bmatrix} \chi_{[-R,R]} \\ &= L_+ - 2 \begin{bmatrix} u_1^2 & \gamma u_1 u_2 \\ \gamma u_1 u_2 & u_2^2 \end{bmatrix} \chi_{(-\infty,-R)\cup(R,\infty)}. \end{aligned}$$

As $R \to \infty$, $L_R \to L_+$ and $L_+ \partial_x U = 0$ with $\partial_x U \in \mathcal{H}$.

(Revised) alternative decomposition of energy

- The spectrum of L_R in $\mathcal{H}$ consists of isolated eigenvalues accumulating to 1.
- The zero eigenvalue is shifted for $R < \infty$ but is near 0 if R is large.

As a result, the quadratic form is coercive under one constraint

$$(L_R V, V)_{L^2} \geq C \|V\|^2_{\mathcal{H}} \quad \forall V \in \mathcal{H} : \quad \langle V, \partial_x U \rangle_{\mathcal{H}} = 0.$$

The nonlinear terms can be controlled inside and outside of $[-R, R]$, e.g.

$$\|V + iW\|_{H^1(-R,R)} \leq C e^{\kappa R} \|V + iW\|_{\mathcal{H}}$$

and

$$\left| \int_R^\infty \eta_1 (2u_2 v_2 + v_2^2 + w_2^2) dx \right| \leq C e^{-\kappa R} \|V + iW\|_{\mathcal{H}} \|\eta_1\|_{L^2(|x| \geq R)}.$$

Orbital Stability

Theorem (Contreras–P–Plum, 2018)

Let $\Psi_0 \in \mathcal{D} \cap L^\infty(\mathbb{R})$*. There exists* $R_0 > 0$ *such that for any* $R > R_0$ *and for every* $\varepsilon > 0$*, there is* $\delta > 0$ *and real functions* $\alpha(t), \theta_1(t), \theta_2(t)$ *such that if* $\rho_R(\Psi_0, U) \leq \delta$*, then* $\sup_{t\in\mathbb{R}} \rho_R(\Psi(t), U_{\alpha(t),\theta_1(t),\theta_2(t)}) \leq \varepsilon$*, where*

$$U_{\alpha(t),\theta_1(t),\theta_2(t)} = (e^{-i\theta_1(t)} u_1(\cdot - \alpha(t)), e^{-i\theta_2(t)} u_2(\cdot - \alpha(t))).$$

Here

$$\rho_R(\Psi, \Phi) := \left\| \Psi - \Phi \right\|_{\mathcal{H}} + \sum_{j=1,2} \left\| |\psi_j|^2 - |\varphi_j|^2 \right\|_{L^2(|x|\geq R)}$$

and

$$\langle \Psi, \Phi \rangle_{\mathcal{H}} := \sum_{j=1}^{2} \int_{\mathbb{R}} \left[\frac{d\psi_j}{dx} \frac{d\bar{\varphi}_j}{dx} + (\gamma - 1)(1 - u_j^2)\psi_j \bar{\varphi}_j \right] dx.$$

Remarks

- Modulation parameters α, θ_1, and θ_2 in the orbit of domain walls

$$U_{\alpha(t),\theta_1(t),\theta_2(t)} = (e^{-i\theta_1(t)} u_1(\cdot - \alpha(t)), e^{-i\theta_2(t)} u_2(\cdot - \alpha(t)))$$

  are uniquely determined by the projection algorithm.

- The time evolution of the modulation parameters is controlled:

$$|\alpha(t)| + |\theta_1(t)| + |\theta_2(t)| \leq C\varepsilon(1 + |t|), \quad t \in \mathbb{R}$$

  for some $C > 0$.

- If R is large, then δ and ε are exponentially small in R.

III: Domain walls in external potentials

Domain walls in small bounded potentials

Consider the domain walls in small bounded potentials:

$$i\partial_t \psi_1 = -\partial_x^2 \psi_1 + \varepsilon V(x)\psi_1 + (|\psi_1|^2 + \gamma|\psi_2|^2)\psi_1,$$
$$i\partial_t \psi_2 = -\partial_x^2 \psi_2 + \varepsilon V(x)\psi_2 + (\gamma|\psi_1|^2 + |\psi_2|^2)\psi_2,$$

where $V \in W^{2,\infty}(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\varepsilon \ll 1$ is small.

Domain walls (u_1, u_2) are pinned to the extremal points of the potential V and the pinning is stable at the maximum of the potential:

$$\exists x_0 \in \mathbb{R} : \qquad V'(x_0) = 0, \quad V''(x_0) < 0.$$

Dror-Malomed-Zeng 2011, Alama–Bronsard–Contreras–P 2015

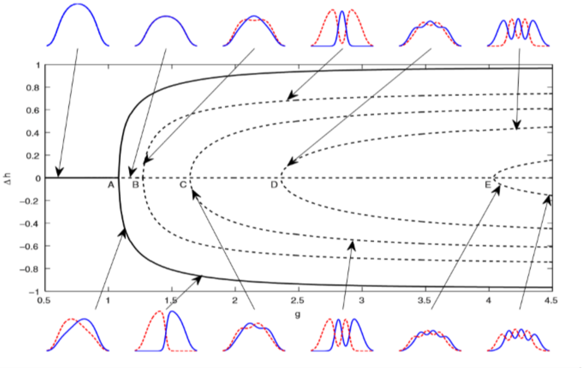
For applications to Bose–Einstein condensates in magnetic traps, we need to consider $V(x) = x^2$ which violates assumptions on $V(x)$.

Numerical results (motivations)

Consider the domain walls in the semi-classical limit:

$$i\partial_t\psi_1 = -\varepsilon^2\partial_x^2\psi_1 + x^2\psi_1 + (|\psi_1|^2 + \gamma|\psi_2|^2)\psi_1,$$
$$i\partial_t\psi_2 = -\varepsilon^2\partial_x^2\psi_2 + x^2\psi_2 + (\gamma|\psi_1|^2 + |\psi_2|^2)\psi_2.$$

where $\varepsilon \ll 1$ is small.



Navarro–Carretero-Gonzále–Kevrekidis 2008

Stationary states in the harmonic potentials

Existence is found from the stationary Gross–Pitaevskii equations:

$$\begin{aligned} -\varepsilon^2 \partial_x^2 \psi_1 + x^2 \psi_1 + (\psi_1^2 + \gamma \psi_2^2 - 1)\psi_1 = 0, \\ -\varepsilon^2 \partial_x^2 \psi_2 + x^2 \psi_2 + (\gamma \psi_1^2 + \psi_2^2 - 1)\psi_2 = 0. \end{aligned}$$

All solutions (ψ_1, ψ_2) decay like Hermite–Gauss functions at infinity.
How to define domain walls among the localized solutions?

If we are to study domain walls variationally, we need the energy of the Gross–Pitaevskii system which is defined in $H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$:

$$\begin{aligned} G_\varepsilon(\Psi) = \frac{1}{2} \int_{\mathbb{R}} \big[\varepsilon^2 (\psi_1')^2 + \varepsilon^2 (\psi_2')^2 + (x^2 - 1)(\psi_1^2 + \psi_2^2) \\ + \frac{1}{2}(\psi_1^2 + \psi_2^2)^2 + (\gamma - 1)\psi_1^2 \psi_2^2 \big] \, dx. \end{aligned}$$

Ground state of the scalar Gross–Pitaevskii theory

The scalar stationary Gross–Pitaevskii equation

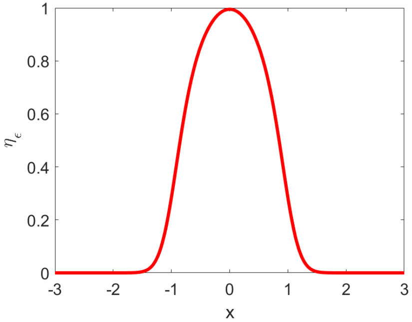
$$-\varepsilon^2\eta_\varepsilon''(x) + (x^2 + \eta_\varepsilon^2(x) - 1)\eta_\varepsilon(x) = 0,$$

admits the ground state of energy with $\eta_\varepsilon(x) > 0$.

The limiting Thomas–Fermi approximation:

$$\lim_{\varepsilon\to 0} \eta_\varepsilon(x) = \sqrt{1-x^2}\, 1_{\{|x|<1\}}$$

with the convergence:
$\|\eta_\varepsilon - \eta_0\|_{L^\infty} \leq C\varepsilon^{1/3}$,
$\|\eta_\varepsilon'\|_{L^\infty} \leq C\varepsilon^{-1/3}$.
Ignat–Millot (2006); Gallo–P (2011)



Domain walls in harmonic potentials

By using the transformation $\psi_{1,2}(x) = \eta_\varepsilon(x)\phi_{1,2}(x/\varepsilon)$ and changing the variables $x \to z := x/\varepsilon$, we obtain $G_\varepsilon(\Psi) = G_\varepsilon(\Upsilon_\varepsilon) + \varepsilon J_\varepsilon(\Phi)$, where

$$J_\varepsilon(\Phi) = \frac{1}{2}\int_{\mathbb{R}} \eta_\varepsilon(\varepsilon z)^2 \left[(\phi_1')^2 + (\phi_2')^2\right] dz + \frac{1}{2}\int_{\mathbb{R}} \eta_\varepsilon(\varepsilon z)^4 \left[\frac{1}{2}(\phi_1^2 + \phi_2^2 - 1)^2 + (\gamma - 1)\phi_1^2\phi_2^2\right] dz.$$

$\Psi \in H^1 \cap L^{2,1}$ is a minimizer of G_ε if and only if Φ is a minimizer of J_ε.

Domain walls in harmonic potentials

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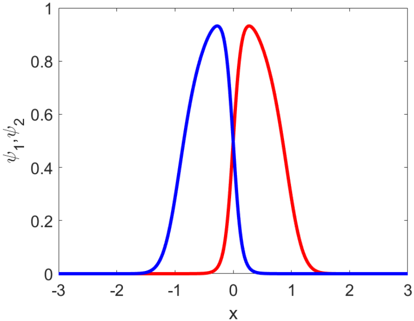
$\Psi \in H^1 \cap L^{2,1}$ is a minimizer of G_ε if and only if Φ is a minimizer of J_ε.

The limit $\varepsilon \to 0$ with $\eta_0(0) = 1$ recovers domain walls without harmonic potentials for $\gamma > 1$. By the Γ convergence theorem,

$$J_\varepsilon \to J_0 \text{ as } \varepsilon \to 0$$

Contreras–P–Slastikov (2022)



Domain walls among other stationary states

Other positive solutions exist in the coupled system:

$$
\begin{aligned}
-\varepsilon^2 \partial_x^2 \psi_1 + x^2 \psi_1 + (\psi_1^2 + \gamma \psi_2^2 - 1)\psi_1 = 0, \\
-\varepsilon^2 \partial_x^2 \psi_2 + x^2 \psi_2 + (\gamma \psi_1^2 + \psi_2^2 - 1)\psi_2 = 0,
\end{aligned}
$$

- Uncoupled states $(\psi_1, \psi_2) = (\eta_\varepsilon, 0)$ and $(\psi_1, \psi_2) = (0, \eta_\varepsilon)$
- Symmetric state $(\psi_1, \psi_2) = (1+\gamma)^{-1/2}(\eta_\varepsilon, \eta_\varepsilon)$.

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By the same reason for $\gamma \in (0,1)$,

$$
\begin{aligned}
W(u_1, u_2) &= \frac{1}{2}\left(u_1^2 + u_2^2 - 1\right)^2 + (\gamma - 1)u_1^2u_2^2\\
&\geq -\frac{\gamma(1-\gamma)}{2(1+\gamma)^2} = W((1+\gamma)^{-1/2}, (1+\gamma)^{-1/2})
\end{aligned}
$$

the symmetric state is the minimizer of G_ε for $\gamma \in (0,1)$.

Domain walls among other stationary states

Other positive solutions exist in the coupled system:

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- Uncoupled states $(\psi_1, \psi_2) = (\eta_\varepsilon, 0)$ and $(\psi_1, \psi_2) = (0, \eta_\varepsilon)$
- Symmetric state $(\psi_1, \psi_2) = (1+\gamma)^{-1/2}(\eta_\varepsilon, \eta_\varepsilon)$.

By the same reason for $\gamma > 1$,

$$W(u_1, u_2) = \frac{1}{2}\left(u_1^2 + u_2^2 - 1\right)^2 + (\gamma - 1)u_1^2 u_2^2$$
$$\geq 0 = W(1,0) = W(0,1)$$

the uncoupled states are the minimizers of G_ε for $\gamma > 1$.

Spaces for Minimization

Domain walls arise as minimizers of G_ε in the energy space with the symmetry:

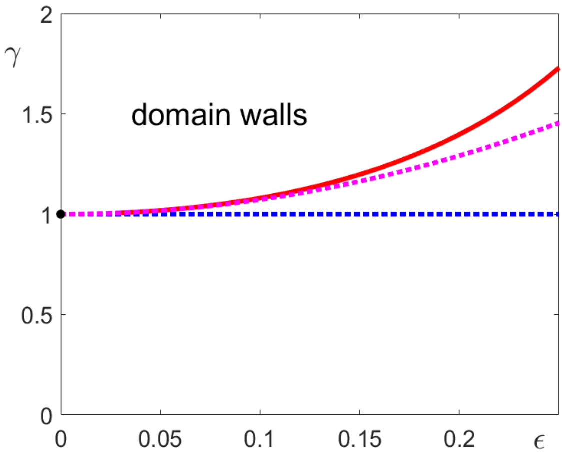
$$\mathcal{E}_s := \{\Psi \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R}) : \quad \psi_1(x) = \psi_2(-x), \;\; x \in \mathbb{R}\}.$$

Theorem (Contreras–P–Slastikov, 2022)

There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is $\gamma_0(\varepsilon) \in (1, \infty)$ such that the symmetric state is a global minimizer of the energy G_ε in $\mathcal{E}_s$ if $\gamma \in (0, \gamma_0(\varepsilon)]$ and a saddle point if $\gamma \in (\gamma_0(\varepsilon), \infty)$.

Domain wall states are global minimizers of the energy G_ε in $\mathcal{E}_s$ if $\gamma \in (\gamma_0(\varepsilon), \infty)$: one satisfies $\psi_1(x) > \psi_2(x) > 0$ for $x > 0$ and the other one obtained by the transformation $\psi_1 \leftrightarrow \psi_2$.

Bifurcation diagram



It follows that $\gamma_0(\varepsilon) \to 1$ as $\varepsilon \to 0$. Contreras–P–Slastikov (2022)

Second variation

Variational analysis is complemented by the study of the second variation at the symmetric state $\Psi = (1+\gamma)^{-1/2}(\eta_\varepsilon, \eta_\varepsilon)$:

$$G''_\varepsilon(\Psi) = \begin{pmatrix} -\varepsilon^2\partial_x^2 + x^2 + 3\psi_1^2 + \gamma\psi_2^2 - 1 & 2\gamma\psi_1\psi_2 \\ 2\gamma\psi_1\psi_2 & -\varepsilon^2\partial_x^2 + x^2 + \gamma\psi_1^2 + 3\psi_2^2 - 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\varepsilon^2\partial_x^2 + x^2 + \frac{3+\gamma}{1+\gamma}\eta_\varepsilon^2 - 1 & \frac{2\gamma}{1+\gamma}\eta_\varepsilon^2 \\ \frac{2\gamma}{1+\gamma}\eta_\varepsilon^2 & -\varepsilon^2\partial_x^2 + x^2 + \frac{3+\gamma}{1+\gamma}\eta_\varepsilon^2 - 1 \end{pmatrix}.$$

Second variation

After rotation, $G''_\varepsilon(\Psi)$ becomes

$$\begin{pmatrix} L_+ & 0 \\ 0 & L_- + 2\frac{1-\gamma}{1+\gamma}\eta_\varepsilon^2 \end{pmatrix}, \qquad \begin{array}{l} L_+ := -\varepsilon^2\partial_x^2 + x^2 - 1 + 3\eta_\varepsilon^2, \\ L_- := -\varepsilon^2\partial_x^2 + x^2 - 1 + \eta_\varepsilon^2, \end{array}$$

where $L_+ > 0$ and $L_- \geq 0$ with $L_-\eta_\varepsilon = 0$ Gallo–P, 2011

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where $L_+ > 0$ and $L_- \geq 0$ with $L_-\eta_\varepsilon = 0$ Gallo–P, 2011

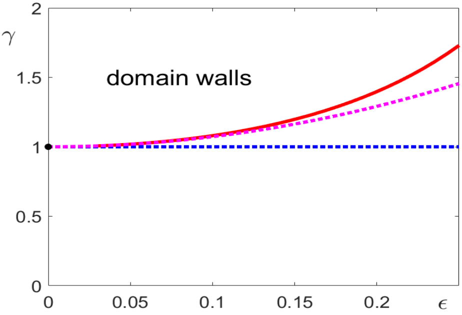
Under the symmetry constraint in

$$\mathcal{E}_s := \{\Psi \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R}) : \quad \psi_1(x) = \psi_2(-x), \;\; x \in \mathbb{R}\}.$$

and the rotation, bifurcation from the symmetric state to the domain wall corresponds to the second eigenvalue of L_γ crossing 0, where

$$L_\gamma := L_- + 2\frac{1-\gamma}{1+\gamma}\eta_\varepsilon^2.$$

More about bifurcation diagram



For $\gamma = 1$, there is rotational symmetry with 1-parameter family

$$\psi_1(x) = \cos\theta\ \eta_\varepsilon(x), \quad \psi_2(x) = \sin\theta\ \eta_\varepsilon(x).$$

If $\gamma \neq 1$, however, only solutions with $\theta = \{0, \frac{\pi}{4}, \frac{\pi}{2}\}$ bifurcate from the family and they correspond to the uncoupled and symmetric states.

Proper analysis of bifurcation at $\gamma_0(\varepsilon)$ as $\varepsilon \to 0$

Let us rescale near $(\varepsilon, \gamma) = (0, 1)$:

$$z \mapsto y := z\sqrt{\gamma - 1}, \quad \Phi(z) = U(y), \quad \varepsilon = \mu\sqrt{\gamma - 1},$$

so that $J_\varepsilon(\Phi) = \sqrt{\gamma - 1} I_{\mu,\gamma}(U)$ is given by

$$I_{\mu,\gamma}(U) = \frac{1}{2}\int_{\mathbb{R}} \eta_\varepsilon(\mu y)^2 \left[(u_1')^2 + (u_2')^2\right] dy$$
$$+ \frac{1}{4(\gamma - 1)}\int_{\mathbb{R}} \eta_\varepsilon(\mu y)^4 (u_1^2 + u_2^2 - 1)^2 dy + \frac{1}{2}\int_{\mathbb{R}} \eta_\varepsilon(\mu y)^4 u_1^2 u_2^2 dy.$$

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$$I_{\mu,\gamma}(U) = \frac{1}{2}\int_{\mathbb{R}} \eta_\varepsilon(\mu y)^2 \left[(u_1')^2 + (u_2')^2\right] dy + \frac{1}{4(\gamma - 1)}\int_{\mathbb{R}} \eta_\varepsilon(\mu y)^4 (u_1^2 + u_2^2 - 1)^2 dy + \frac{1}{2}\int_{\mathbb{R}} \eta_\varepsilon(\mu y)^4 u_1^2 u_2^2 dy.$$

The limit $\gamma \to 1$ gives in the sense of Γ convergence:

$$I_{\mu,\gamma}(U) \to I_{\mu,1}(\theta) = \frac{1}{2}\int_{-\mu^{-1}}^{\mu^{-1}} \left[\eta_0(\mu y)^2 (\theta')^2 + \frac{1}{4}\eta_0(\mu y)^4 \sin^2(2\theta)\right] dy,$$

where $(u_1, u_2) = (\sin(\theta), \cos(\theta))$.

Proper analysis of bifurcation at $\gamma_0(\varepsilon)$ as $\varepsilon \to 0$

$$I_{\mu,1}(\theta) = \frac{1}{2} \int_{-\mu^{-1}}^{\mu^{-1}} \left[\eta_0(\mu y)^2 (\theta')^2 + \frac{1}{4} \eta_0(\mu y)^4 \sin^2(2\theta) \right] dy.$$

Theorem (Contreras–P–Slastikov, 2022)

There exists $\mu_0 \in (0, \infty)$ such that $\theta = \frac{\pi}{4}$ (symmetric state) is a global minimizer of the energy $I_{\mu,1}$ if $\mu \in [\mu_0, \infty)$ and a saddle point of $I_{\mu,1}$ if $\mu \in (0, \mu_0)$. The domain wall states exist only if $\mu \in (0, \mu_0)$ and are global minimizers of the energy $I_{\mu,1}$.

Proper analysis of bifurcation at $\gamma_0(\varepsilon)$ as $\varepsilon \to 0$

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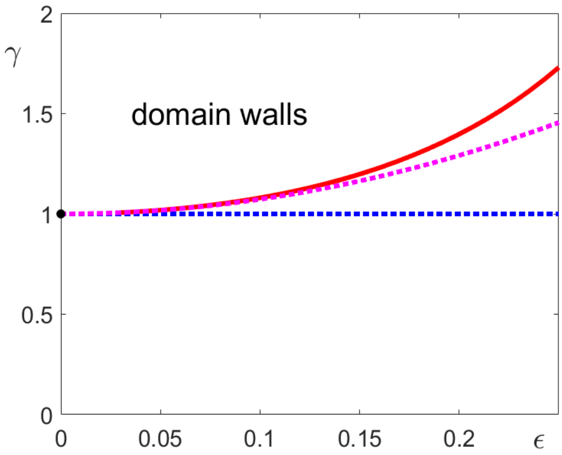
Bifurcation corresponds to the second eigenvalue $\nu = \mu^{-2}$

$$-\frac{d}{dx} \left[(1 - x^2) \frac{dv}{dx} \right] = \nu (1 - x^2)^2 v(x), \quad -1 < x < 1.$$

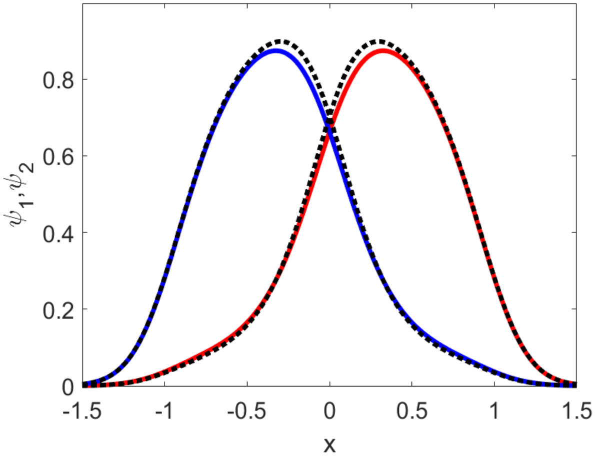
It is found at $\nu_0 \approx 7.29$ which determines $\gamma_0(\varepsilon) = 1 + \nu_0 \varepsilon^2 + \mathcal{O}(\varepsilon^4)$.

Bifurcation diagram



The magenta line corresponds to $\gamma_0(\varepsilon) = 1 + \nu_0 \varepsilon^2$.

Comparison between numerical and asymptotic approximations for $\varepsilon = 0.1$ and $\gamma = 1.2$



IV: Numerical approximations of domain walls in harmonic potentials

Numerical approximations of domain walls

We consider the half-line energy space:

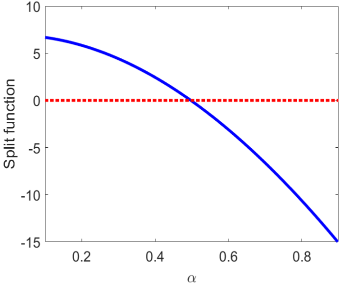
$$\mathcal{D}_\alpha := \{\Psi \in H^1(\mathbb{R}_+) \cap L^{2,1}(\mathbb{R}_+) : \quad \psi_1(0) = \psi_2(0) = \alpha\}.$$

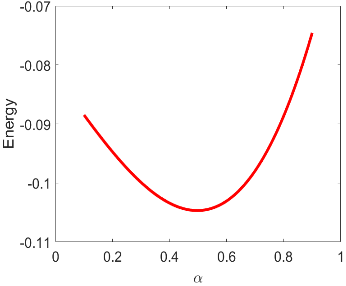
Such minimizers $\Psi_\alpha = (\psi_1, \psi_2)$ of energy G_ε always exist in $\mathcal{D}_\alpha$.
Ψ_α becomes the minimizer of G_ε on full line $\mathbb{R}$ with the symmetry $\psi_1(x) = \psi_2(-x)$ under the following two numerically detected conditions:

- Split function $S_\varepsilon(\alpha) := \psi_1'(0) + \psi_2'(0)$ vanishes
- Energy $G_\varepsilon(\Psi_\alpha)$ is minimal.

Numerical approximations of domain walls

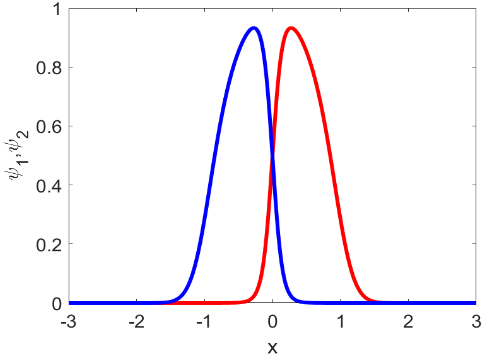
For fixed $\varepsilon = 0.1$ and $\gamma = 3$:





Numerical approximations of domain walls

For optimal α, we get the domain wall solution with $\psi_1(x) = \psi_2(-x)$:



Summary

Domain walls $\Psi = (\psi_1, \psi_2)$ of the coupled Gross–Pitaevskii equations:

$$-\varepsilon^2 \partial_x^2 \psi_1 + V(x)\psi_1 + (\psi_1^2 + \gamma \psi_2^2 - 1)\psi_1 = 0,$$
$$-\varepsilon^2 \partial_x^2 \psi_2 + V(x)\psi_2 + (\gamma \psi_1^2 + \psi_2^2 - 1)\psi_2 = 0,$$

- are minimizers of energy in the energy space with symmetry
- are orbitally stable in a weighted $H^1(\mathbb{R})$ energy space
- persist under small bounded potentials
- persist under harmonic potentials

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- persist under small bounded potentials
- persist under harmonic potentials

Conjectures:

- Domain walls are asymptotically stable,
- Domain walls do not travel in space with constant speed.

Summary

Domain walls $\Psi = (\psi_1, \psi_2)$ of the coupled Gross–Pitaevskii equations:

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Many thanks for your attention!