Periodic waves in discrete MKDV equation: modulational instability and rogue waves

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The rogue wave of the cubic NLS equation

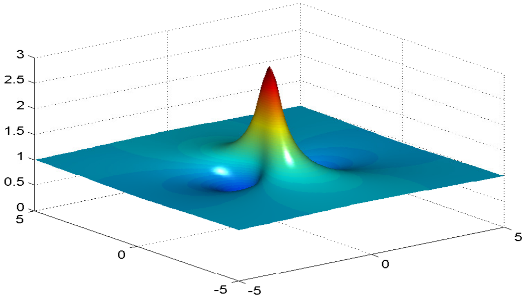
The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

admits the exact solution

$$\psi(x,t) = \left[1 - \frac{4(1+2it)}{1+4x^2+4t^2}\right] e^{it}.$$

It was discovered by H. Peregrine (1983) and was labeled as *the rogue wave*.



Traveling periodic waves

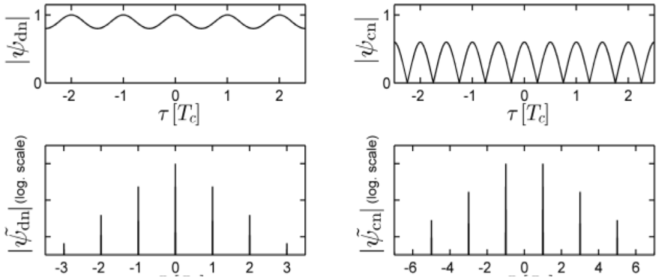
The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

admits the periodic traveling and standing wave solutions, e.g. the dnoidal and cnoidal waves:

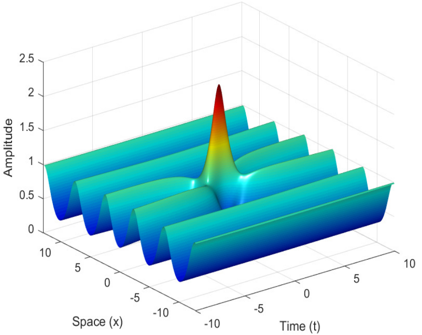
$$\psi_{\rm dn}(x,t) = {\rm dn}(x;k)e^{i(1-k^2/2)t}, \quad \psi_{\rm cn}(x,t) = k{\rm cn}(x;k)e^{i(k^2-1/2)t},$$

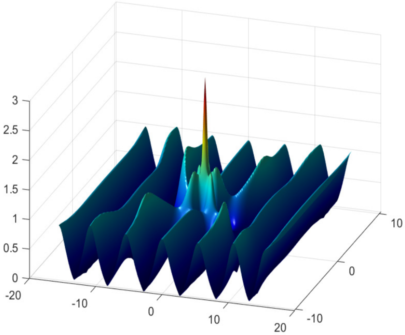
where $k \in (0,1)$ is elliptic modulus.



Rogue wave on background of periodic waves

J. Chen, D. Pelinovsky, Proceedings A **474** (2018) 20170814
J. Chen, D. Pelinovsky, R. White, Physica D **405** (2020) 132378





Other examples of integrable Hamiltonian systems

- Modified Korteweg–de Vries equation

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

Dnoidal periodic waves are modulationally stable.
Cnoidal periodic waves are modulationally unstable.
J. Chen & D. Pelinovsky, Nonlinearity **31** (2018) 1955–1980

- Sine–Gordon equation

$$u_{tt} - u_{xx} + \sin(u) = 0$$

Same conclusion.
D. Pelinovsky & R. White, Proceedings A **476** (2020) 20200490

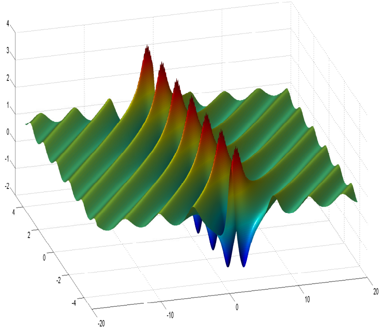
- Derivative NLS equation

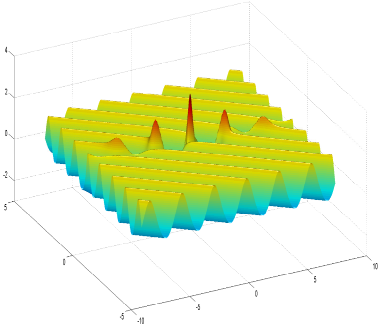
$$i\psi_t + \psi_{xx} + i(|\psi|^2\psi)_x = 0.$$

There exist modulationally stable periodic waves.
J. Chen, D. Pelinovsky, & J. Upsal, J. Nonlinear Science **31** (2021) 58

Rogue wave for the modified KdV equation

J. Chen & D. Pelinovsky, Journal of Nonlinear Science **29** (2019) 2797–2843





Discrete modified KdV equation

It is considered to be the third-order flow in the Ablowitz–Ladik hierarchy:

$$\dot{u}_n = (1 + u_n^2)(u_{n+1} - u_{n-1}), \quad n \in \mathbb{Z},$$

where $u_n = u_n(t)$ is real.

In the continuum limit, long waves of small amplitudes can be modeled by

$$u_n(t) = \varepsilon \mathfrak{u}(\varepsilon(n+2t), \frac{1}{3}\varepsilon^3 t),$$

satisfy the continuous the mKdV equation

$$\mathfrak{u}_\tau = 6\mathfrak{u}^2 \mathfrak{u}_\xi + \mathfrak{u}_{\xi\xi\xi},$$

where $\mathfrak{u} = \mathfrak{u}(\xi, \tau)$ with $\xi := \varepsilon(n + 2t)$ and $\tau := \frac{1}{3}\varepsilon^3 t$, and ε is small parameter.

Lax equations

DMKDV is a compatibility condition of the linear Lax system

$$\varphi_{n+1} = \frac{1}{\sqrt{1+u_n^2}} \begin{pmatrix} \lambda & u_n \\ -u_n & \lambda^{-1} \end{pmatrix} \varphi_n$$

and

$$\dot{\varphi}_n = \begin{pmatrix} \frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) & \lambda u_n + \lambda^{-1} u_{n-1} \\ -\lambda u_{n-1} - \lambda^{-1} u_n & -\frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) \end{pmatrix} \varphi_n.$$

There exists another Lax system representation:

$$\varphi_{n+1} = \begin{pmatrix} \lambda & u_n \\ -u_n & \lambda^{-1} \end{pmatrix} \varphi_n$$

and

$$\dot{\varphi}_n = \begin{pmatrix} \frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) + u_n u_{n-1} & \lambda u_n + \lambda^{-1} u_{n-1} \\ -\lambda u_{n-1} - \lambda^{-1} u_n & -\frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) + u_n u_{n-1} \end{pmatrix} \varphi_n.$$

Nonlinearization method

If $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ is a solution of Lax system for $\lambda = \lambda_1$,
then $\varphi_n = (-q_n, p_n)^{\mathrm{T}}$ is a solution for $\lambda = \lambda_1^{-1}$.

Assume the relation between solutions of the DMKV and Lax systems:

$$u_n = \lambda_1 p_n^2 + \lambda_1^{-1} q_n^2, \qquad n \in \mathbb{Z}.$$

Then, $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ satisfies the nonlinear symplectic map

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \frac{1}{\sqrt{1 + (\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2)^2}} \begin{pmatrix} \lambda_1 p_n + (\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2) q_n \\ \lambda_1^{-1} q_n - (\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2) p_n \end{pmatrix}$$

and the nonlinear Hamiltonian system

$$\frac{dp_n}{dt} = \frac{\partial H}{\partial q_n}, \quad \frac{dq_n}{dt} = -\frac{\partial H}{\partial p_n},$$

with

$$H(p_n, q_n) = \frac{1}{2}(\lambda_1^2 - \lambda_1^{-2}) p_n q_n + \frac{1}{2}(\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2)(\lambda_1^{-1} p_n^2 + \lambda_1 q_n^2).$$

Nonlinearization method

If $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ is a solution of Lax system for $\lambda = \lambda_1$,
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Then, $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ satisfies the nonlinear symplectic map

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \frac{1}{\sqrt{1 + (\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2)^2}} \begin{pmatrix} \lambda_1 p_n + (\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2) q_n \\ \lambda_1^{-1} q_n - (\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2) p_n \end{pmatrix}$$

and the nonlinear Hamiltonian system

$$\frac{dp_n}{dt} = \frac{\partial H}{\partial q_n}, \quad \frac{dq_n}{dt} = -\frac{\partial H}{\partial p_n},$$

with

$$H(p_n, q_n) = \frac{1}{2}(\lambda_1^2 - \lambda_1^{-2}) p_n q_n + \frac{1}{2}(\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2)(\lambda_1^{-1} p_n^2 + \lambda_1 q_n^2).$$

Restrictions on the class of admissible solutions

In addition to

$$u_n = \lambda_1 p_n^2 + \lambda_1^{-1} q_n^2, \qquad n \in \mathbb{Z},$$

one can easily prove the relation

$$u_{n-1} = \lambda_1^{-1} p_n^2 + \lambda_1 q_n^2, \qquad n \in \mathbb{Z}.$$

Since $F_1 = 2H(p_n, q_n) = (\lambda_1^2 - \lambda_1^{-2}) p_n q_n + (\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2)(\lambda_1^{-1} p_n^2 + \lambda_1 q_n^2)$ is independent of $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, we obtain

$$p_n^2 = \frac{\lambda_1 u_n - \lambda_1^{-1} u_{n-1}}{\lambda_1^2 - \lambda_1^{-2}}, \quad q_n^2 = \frac{\lambda_1 u_{n-1} - \lambda_1^{-1} u_n}{\lambda_1^2 - \lambda_1^{-2}}, \quad p_n q_n = \frac{F_1 - u_n u_{n-1}}{\lambda_1^2 - \lambda_1^{-2}}.$$

One can show that u_n satisfies the stationary discrete equation

$$(1 + u_n^2)(u_{n+1} + u_{n-1}) = \omega u_n, \quad n \in \mathbb{Z},$$

where $\omega := \lambda_1^2 + \lambda_1^{-2} + 2F_1$. Connection to traveling solutions of DMKDV?!

Integrability of the nonlinear Hamiltonian system

The Hamiltonian system for (p_n, q_n) is obtained from the Lax equations

$$W(p_{n+1}, q_{n+1}, \lambda) U(u_n, \lambda_1) - U(u_n, \lambda_1) W(p_n, q_n, \lambda) = 0$$

and

$$\frac{d}{dt} W(p_n, q_n, \lambda) = V(u_n, \lambda_1) W(p_n, q_n, \lambda) - W(p_n, q_n, \lambda) V(u_n, \lambda_1),$$

where

$$W(p_n, q_n, \lambda) = \begin{pmatrix} \frac{1}{2} - \frac{\lambda_1^2 p_n q_n}{\lambda^2 - \lambda_1^2} + \frac{\lambda_1^{-2} p_n q_n}{\lambda^2 - \lambda_1^{-2}} & \lambda \left(\frac{\lambda_1 p_n^2}{\lambda^2 - \lambda_1^2} + \frac{\lambda_1^{-1} q_n^2}{\lambda^2 - \lambda_1^{-2}} \right) \\ -\lambda \left(\frac{\lambda_1 q_n^2}{\lambda^2 - \lambda_1^2} + \frac{\lambda_1^{-1} p_n^2}{\lambda^2 - \lambda_1^{-2}} \right) & -\frac{1}{2} + \frac{\lambda_1^2 p_n q_n}{\lambda^2 - \lambda_1^2} - \frac{\lambda_1^{-2} p_n q_n}{\lambda^2 - \lambda_1^{-2}} \end{pmatrix}$$

satisfies

$$\det W(p_n, q_n, \lambda) = -\frac{1}{4} + \frac{\lambda^2 F_1}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_1^{-2})}.$$

Algebraic polynomial for the traveling periodic waves

Due to the squared eigenfunction constraints, we also have

$$W(p_n, q_n, \lambda) = \begin{pmatrix} \frac{1}{2} - \frac{\lambda^2 (F_1 - u_n u_{n-1})}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_1^{-2})} & \frac{\lambda(\lambda^2 u_n - u_{n-1})}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_1^{-2})} \\ -\frac{\lambda(\lambda^2 u_{n-1} - u_n)}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_1^{-2})} & -\frac{1}{2} + \frac{\lambda^2 (F_1 - u_n u_{n-1})}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_1^{-2})} \end{pmatrix},$$

which gives

$$\det W(p_n, q_n, \lambda) = -\frac{P(\lambda)}{4(\lambda^2 - \lambda_1^2)^2 (\lambda^2 - \lambda_1^{-2})^2},$$

where

$$P(\lambda) := \lambda^8 - 2\omega\lambda^6 + (2 + \omega^2 - 4F_1^2)\lambda^4 - 2\omega\lambda^2 + 1.$$

Thus, λ_1 is selected from two quadruplets of $P(\lambda)$:

$$P(\lambda) = (\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_1^{-2})(\lambda^2 - \lambda_2^2)(\lambda^2 - \lambda_2^{-2}).$$

Dnoidal periodic waves

These are solutions of the form

$$u_n(t) = \frac{\mathrm{sn}(\alpha;k)}{\mathrm{cn}(\alpha;k)}\mathrm{dn}(\alpha n + ct;k), \quad c = \frac{2\mathrm{sn}(\alpha;k)}{\mathrm{cn}(\alpha;k)}, \quad \omega = \frac{2\mathrm{dn}(\alpha;k)}{\mathrm{cn}^2(\alpha;k)},$$

where $\alpha \in (0, K(k))$ and $k \in (0,1)$ are arbitrary parameters.

We can find explicitly

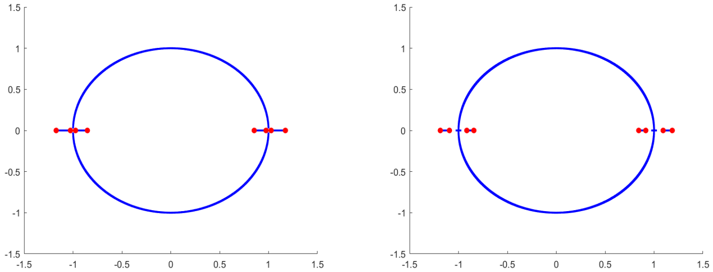
$$F_1 = \pm\sqrt{1-k^2}\frac{\mathrm{sn}^2(\alpha;k)}{\mathrm{cn}^2(\alpha;k)},$$

and

$$\lambda_1 = \frac{1}{\mathrm{cn}(\alpha;k)}\sqrt{(1-\mathrm{sn}(\alpha;k))\left(\mathrm{dn}(\alpha;k) - \sqrt{1-k^2}\mathrm{sn}(\alpha;k)\right)},$$
$$\lambda_2 = \frac{1}{\mathrm{cn}(\alpha;k)}\sqrt{(1-\mathrm{sn}(\alpha;k))\left(\mathrm{dn}(\alpha;k) + \sqrt{1-k^2}\mathrm{sn}(\alpha;k)\right)},$$

satisfying $0 < \lambda_1 < \lambda_2 < 1 < \lambda_2^{-1} < \lambda_1^{-1}$.

Lax spectrum for dnoidal waves



The spectrum is found numerically for $\alpha = K(k)/M$ with $u_{n+2M} = u_n$:

$$\left\{ \begin{array}{l} \sqrt{1+u_n^2}p_{n+1} + \sqrt{1+u_{n-1}^2}p_{n-1} - (u_n - u_{n-1})q_n = zp_n, \\ (u_n - u_{n-1})p_n + \sqrt{1+u_n^2}q_{n+1} + \sqrt{1+u_{n-1}^2}q_{n-1} = zq_n, \end{array} \right.$$

where $z := \lambda + \lambda^{-1}$ and (p_n, q_n) is the eigenvector satisfying

$$p_n = \hat{p}_n(\theta)e^{i\theta n}, \quad q_n = \hat{q}_n(\theta)e^{i\theta n}, \quad p_{n+2M} = p_n, \quad q_{n+2M} = q_n,$$

Cnoidal periodic waves

These are solutions of the form

$$u_n(t) = \frac{k\,\mathrm{sn}(\alpha;k)}{\mathrm{dn}(\alpha;k)}\mathrm{cn}(\alpha n + ct;k), \quad c = \frac{2\mathrm{sn}(\alpha;k)}{\mathrm{dn}(\alpha;k)}, \quad \omega = \frac{2\mathrm{cn}(\alpha;k)}{\mathrm{dn}^2(\alpha;k)},$$

where $\alpha \in (0, K(k))$ and $k \in (0,1)$ are arbitrary parameters.

We can find explicitly

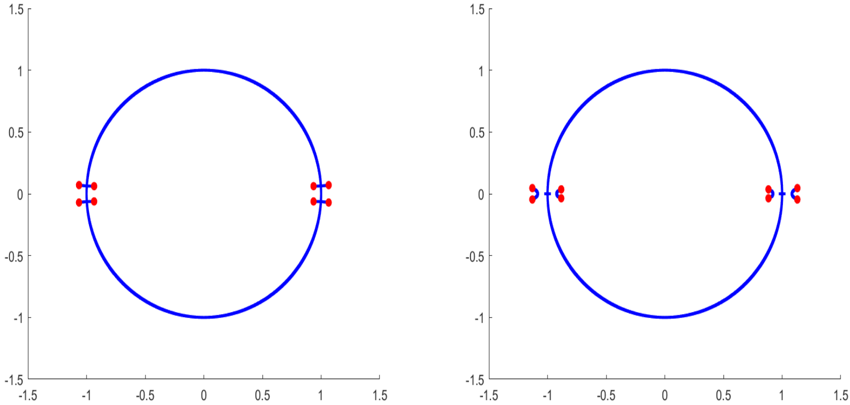
$$F_1 = \pm ik\sqrt{1-k^2}\frac{\mathrm{sn}^2(\alpha;k)}{\mathrm{dn}^2(\alpha;k)}$$

and

$$\lambda_1 = \frac{\sqrt{(1-k\,\mathrm{sn}(\alpha;k))(\mathrm{cn}(\alpha;k) + i\sqrt{1-k^2}\mathrm{sn}(\alpha;k))}}{\mathrm{dn}(\alpha;k)},$$

satisfying $|\lambda_1| < 1 < |\lambda_1|^{-1}$.

Lax spectrum for cnoidal waves



Continuous modified KdV equation

Let $\mathfrak{u}(x,t) = \phi(x - ct)$ be a traveling periodic wave of the mKdV equation

$$\mathfrak{u}_t + 6\mathfrak{u}^2 \mathfrak{u}_x + \mathfrak{u}_{xxx} = 0.$$

Let $\mathfrak{v}$ be a perturbation of $\mathfrak{u}$ satisfying the linearized mKdV equation

$$\mathfrak{v}_t + 6(\mathfrak{u}^2 \mathfrak{v})_x + \mathfrak{v}_{xxx} = 0,$$

which is obtained from mKdV after substituting $\mathfrak{u} + \mathfrak{v}$ to and neglecting $\mathfrak{v}^2$, $\mathfrak{v}^3$.

Separating variables $\mathfrak{v}(x,t) = \mathfrak{w}(x - ct)e^{\gamma t}$, we obtain the spectral problem

$$\gamma \mathfrak{w} = \partial_x \left[-\partial_x^2 + c - 6\mathfrak{u}^2 \right] \mathfrak{w}.$$

If $\mathfrak{u}(x+L) = \mathfrak{u}(x)$ is periodic, then by Floquet theory, $\mathfrak{w}(x+L) = \mathfrak{w}(x)e^{i\theta L}$, where $\theta \in [-\pi/L, \pi/L]$.

If there exists γ with $\mathrm{Re}(\gamma) > 0$ for some $\theta \in [-\pi/L, \pi/L]$, then the standing periodic wave is spectrally unstable. It is modulationally unstable if the band with $\mathrm{Re}(\gamma) > 0$ intersects $\gamma = 0$ as $\theta \to 0$.

Relation to squared eigenfunctions

Recall the linear Lax system:

$$\varphi_x = \begin{pmatrix} \lambda & \mathfrak{u} \\ -\mathfrak{u} & -\lambda \end{pmatrix} \varphi$$

and

$$\varphi_t = \begin{pmatrix} 4\lambda^3 + 2\lambda\mathfrak{u}^2 & 4\lambda^2\mathfrak{u} + 2\lambda\mathfrak{u}_x + 2\mathfrak{u}^3 + \mathfrak{u}_{xx} \\ -4\lambda^2\mathfrak{u} + 2\lambda\mathfrak{u}_x - 2\mathfrak{u}^3 - \mathfrak{u}_{xx} & -4\lambda^3 - 2\lambda\mathfrak{u}^2 \end{pmatrix} \varphi.$$

If $\varphi = (p, q)^T$ is a solution of the Lax system, then $\mathfrak{v} = p^2 - q^2$ is a solution of the linearized mKdV equation. By the same Floquet theory, the solution of the Lax system has the form

$$\varphi(x,t) = \psi(x - ct)e^{\Omega t},$$

where $\psi(x + L) = \psi(x)e^{\frac{1}{2}i\theta L}$. The squared eigenfunctions ψ determines the eigenfunction $\mathfrak{w}$ of the spectral stability problem with $\gamma = 2\Omega$.

Relation to squared eigenfunctions

Theorem

Let λ belongs to the Lax spectrum of

$$\varphi_x = \begin{pmatrix} \lambda & \mathfrak{u} \\ -\mathfrak{u} & -\lambda \end{pmatrix} \varphi.$$

Then, $\Omega = \pm\sqrt{P(\lambda)}$, where $P(\lambda)$ is the characteristic polynomial for the traveling periodic waves:

$$P(\lambda) = 16\lambda^6 - 8c\lambda^4 + (4d + c^2)\lambda^2 - b^2.$$

Consequently, $\Lambda = 2\Omega = \pm 2\sqrt{P(\lambda)}$.

The proof follows from separation of variables for $\varphi_t = V(\lambda, \mathfrak{u})\varphi$:

$$\begin{vmatrix} 4\lambda^3 + 2\lambda\phi^2 - c\lambda - \Omega & 4\lambda^2\phi + 2\lambda\phi' + 2\phi^3 + \phi'' - c\phi \\ -4\lambda^2\phi + 2\lambda\phi' - 2\phi^3 - \phi'' + c\phi & -4\lambda^3 - 2\lambda\phi^2 + c\lambda - \Omega \end{vmatrix} = 0.$$

Stability of the dnoidal periodic waves

$$\mathfrak{u}(x,t) = \mathrm{dn}(x - ct; k), \quad c = 2 - k^2, \quad L = 2K(k).$$

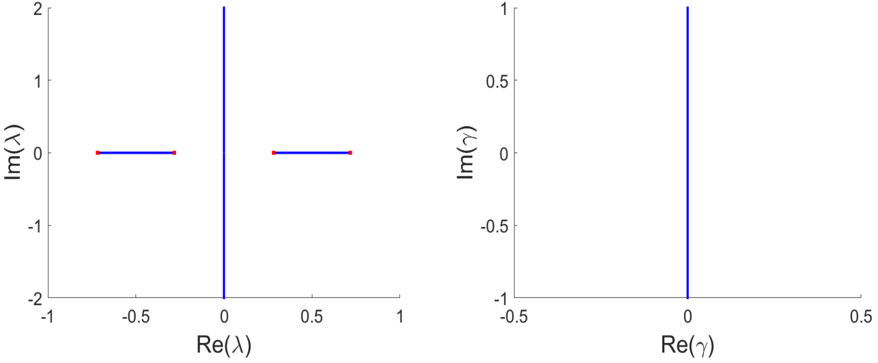


Figure: Lax spectrum (left) and stability spectrum (right) for $k = 0.9$ obtained with $\gamma = 2\Omega = \pm 2\sqrt{P(\lambda)}$.

Instability of the cnoidal periodic waves

$$\mathfrak{u}(x,t) = k\,\mathrm{cn}(x - ct; k), \quad c = 2k^2 - 1, \quad L = 4K(k).$$

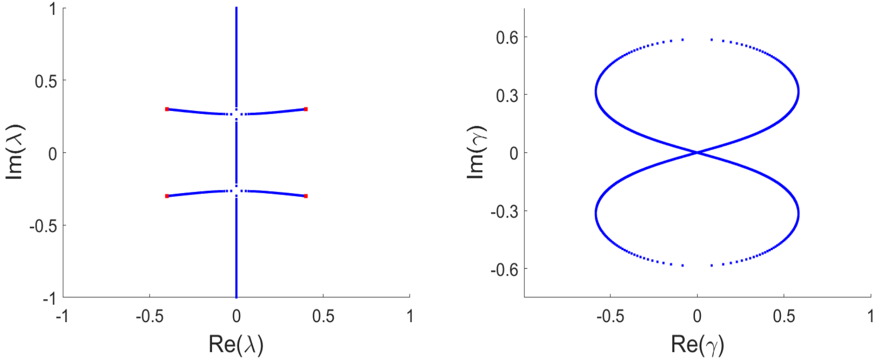


Figure: Lax spectrum (left) and stability spectrum (right) for the cnoidal wave with $k = 0.8$ obtained with $\gamma = 2\Omega = \pm 2\sqrt{P(\lambda)}$.

Instability of the cnoidal periodic waves

$$\mathfrak{u}(x,t) = k\,\mathrm{cn}(x-ct;k), \quad c = 2k^2 - 1, \quad L = 4K(k).$$

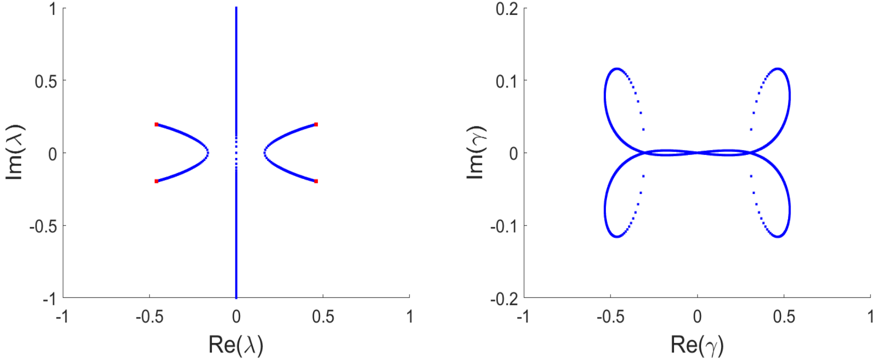


Figure: Lax spectrum (left) and stability spectrum (right) for the cnoidal wave with $k = 0.92$ obtained with $\gamma = 2\Omega = \pm 2\sqrt{P(\lambda)}$.

Discrete modified KdV equation

Let $u_n(t) = \phi(\alpha n + ct)$ be a traveling periodic wave of the dsicrete mKdV equation

$$\dot{u}_n = (1 + u_n^2)(u_{n+1} - u_{n-1}), \quad n \in \mathbb{Z}.$$

Let $\{v_n(t)\}_{n \in \mathbb{Z}}$ be a perturbation of $\{v_n(t)\}_{n \in \mathbb{Z}}$ satisfying the linearized mKdV equation

$$\dot{v}_n = (1 + u_n^2)(v_{n+1} - v_{n-1}) + 2u_n(u_{n+1} - u_{n-1})v_n, \quad n \in \mathbb{Z}.$$

We still have the squared eigenfunction relation:

$$v_n = \lambda p_n^2 - \lambda^{-1} q_n^2 + 2u_n p_n q_n,$$

where $\varphi_n = (p_n, q_n)^T$ is a solution of the linear Lax system.

Relation to squared eigenfunctions

Thus, $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ is a solution of the linear Lax system:

$$\varphi_{n+1} = \frac{1}{\sqrt{1+u_n^2}} \begin{pmatrix} \lambda & u_n \\ -u_n & \lambda^{-1} \end{pmatrix} \varphi_n$$

and

$$\dot{\varphi}_n = \begin{pmatrix} \frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) & \lambda u_n + \lambda^{-1} u_{n-1} \\ -\lambda u_{n-1} - \lambda^{-1} u_n & -\frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) \end{pmatrix} \varphi_n,$$

with the obvious decomposition since $u_n(t) = \phi(\alpha n + ct)$:

$$\varphi_n(t) = \psi(\alpha n + ct) e^{\Omega t}.$$

However, the explicit relation between Ω and $P(\lambda)$ is missing!

$$P(\lambda) := \lambda^8 - 2\omega\lambda^6 + (2 + \omega^2 - 4F_1^2)\lambda^4 - 2\omega\lambda^2 + 1.$$

Stability of the dnoidal periodic waves

$$u_n(t) = \frac{\mathrm{sn}(\alpha;k)}{\mathrm{cn}(\alpha;k)} \mathrm{dn}(\alpha n + ct; k), \quad c = \frac{2\mathrm{sn}(\alpha;k)}{\mathrm{cn}(\alpha;k)},$$

where $\alpha \in (0, K(k))$ and $k \in (0,1)$ are arbitrary parameters.

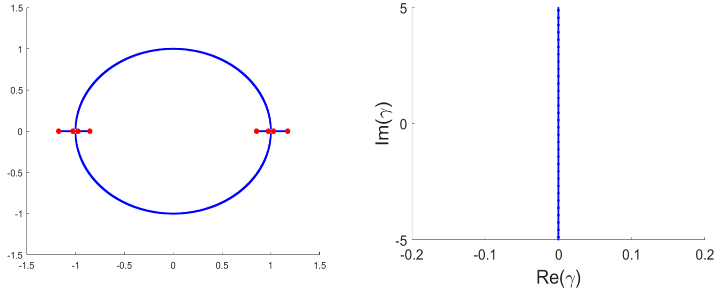


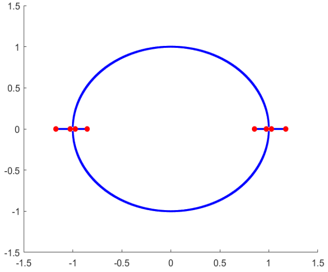
Figure: Lax spectrum (left) and stability spectrum (right) for $k = 0.7$ obtained numerically.

1-fold Darboux transformation

We can use eigenvalues found in the nonlinearization method to define a new solution to the discrete mKdV equation:

$$\hat{u}_n = -\frac{p_n^2 + \lambda_1^2 q_n^2}{\lambda_1^2 p_n^2 + q_n^2} u_n + \frac{(1-\lambda_1^4) p_n q_n}{\lambda_1(\lambda_1^2 p_n^2 + q_n^2)}$$

where $\varphi_n = (p_n, q_n)^T$ is a solution of Lax equations with $\lambda = \lambda_1$.



Trivial new solution

$$\varphi_{n+1} = \frac{1}{\sqrt{1+u_n^2}} \left(\begin{array}{cc} \lambda & u_n \\ -u_n & \lambda^{-1} \end{array} \right) \varphi_n$$

and

$$\dot{\varphi}_n = \left(\begin{array}{cc} \frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) & \lambda u_n + \lambda^{-1} u_{n-1} \\ -\lambda u_{n-1} - \lambda^{-1} u_n & -\frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) \end{array} \right) \varphi_n.$$

If $\varphi_n = (p_n, q_n)^T$ is obtained from $u_n = \lambda_1 p_n^2 + \lambda_1^{-1} q_n^2$ and λ_1 is a root of $P(\lambda)$, then new solution $\hat{u}_n$ is a half-period translation of the dnoidal wave:

$$\begin{aligned} \hat{u}_n &= -F_1 u_n^{-1} \\ &= -\frac{\sigma_1 \mathrm{sn}(\alpha; k)}{\mathrm{cn}(\alpha; k)} \frac{\sqrt{1-k^2}}{\mathrm{dn}(\xi; k)} \\ &= -\frac{\sigma_1 \mathrm{sn}(\alpha; k)}{\mathrm{cn}(\alpha; k)} \mathrm{dn}(\xi + K(k); k) \\ &= -\sigma_1 u_n(t + c^{-1} K(k)). \end{aligned}$$

Nontrivial new solution

The second, linearly independent solution can be found in the form:

$$\hat{p}_n = p_n \theta_n - \frac{q_n}{p_n^2 + q_n^2}, \quad \hat{q}_n = q_n \theta_n + \frac{p_n}{p_n^2 + q_n^2},$$

where

$$\theta_{n+1} - \theta_n = \frac{(\lambda_1 + \lambda_1^{-1})^2 (u_n^2 - F_1)}{(u_n + u_{n-1})(u_n + u_{n+1})(1 + u_n^2)}$$

and

$$\dot{\theta}_n = \frac{(\lambda_1 + \lambda_1^{-1})^2 (u_n^2 + u_{n-1}^2 - 2F_1)}{(u_n + u_{n-1})^2}.$$

If $u_n(t) = \phi(\alpha n + ct)$ is the traveling wave with periodic ϕ, then $\theta_n(t) = an + bt + \chi(\alpha n + ct)$ with periodic χ and uniquely computed parameters a and b.

Algebraic soliton propagating on the dnoidal wave

The new solution is now nontrivial:

$$\hat{u}_n = -\frac{\hat{p}_n^2 + \lambda_1^2 \hat{q}_n^2}{\lambda_1^2 \hat{p}_n^2 + \hat{q}_n^2} u_n + \frac{(1 - \lambda_1^4)\hat{p}_n \hat{q}_n}{\lambda_1(\lambda_1^2 \hat{p}_n^2 + \hat{q}_n^2)}$$

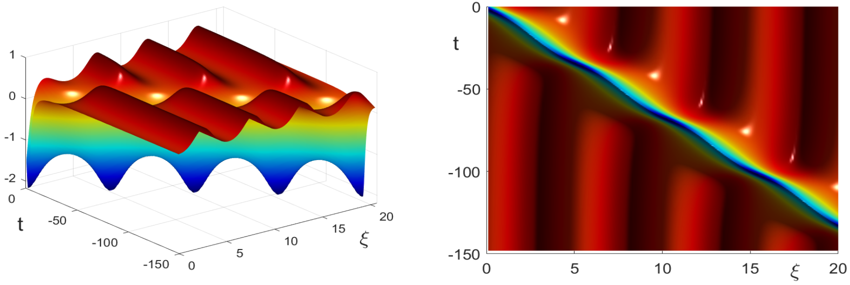


Figure: The solution surface (left: sideview, right: topview) for λ_1.

Algebraic soliton propagating on the dnoidal wave

The new solution is now nontrivial:

$$\hat{u}_n = -\frac{\hat{p}_n^2 + \lambda_1^2 \hat{q}_n^2}{\lambda_1^2 \hat{p}_n^2 + \hat{q}_n^2} u_n + \frac{(1 - \lambda_1^4)\hat{p}_n \hat{q}_n}{\lambda_1(\lambda_1^2 \hat{p}_n^2 + \hat{q}_n^2)}$$

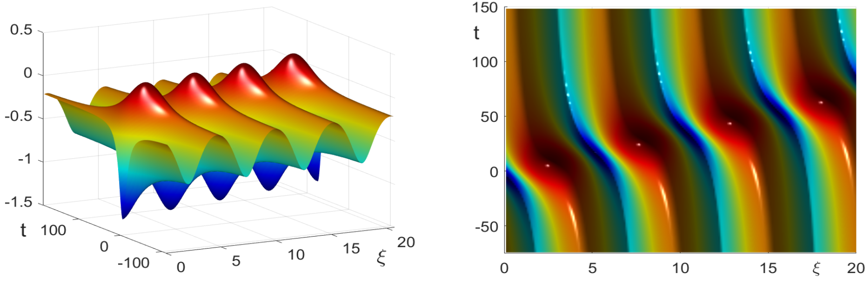


Figure: The solution surface (left: sideview, right: topview) for λ_2.

2-fold Darboux transformation

The 2-fold transformation uses two eigenvalues λ_1 and λ_2:

$$\hat{u}_n = \frac{\Upsilon_n}{\Delta_n} u_n - \frac{\Sigma_n}{\lambda_1 \lambda_2 \Delta_n},$$

where

$$\begin{aligned}
\Upsilon_n &= \lambda_2^2(q_{2n}^2 + \lambda_2^2 p_{2n}^2)(p_{1n}^2 + \lambda_1^6 q_{1n}^2) + \lambda_1^2(q_{1n}^2 + \lambda_1^2 p_{1n}^2)(p_{2n}^2 + \lambda_2^6 q_{2n}^2) \\
&\quad - 2\lambda_1^2\lambda_2^2(p_{1n}^2 + \lambda_1^2 q_{1n}^2)(p_{2n}^2 + \lambda_2^2 q_{2n}^2) - 2p_{1n}q_{1n}p_{2n}q_{2n}\lambda_1\lambda_2(\lambda_1^4 - 1)(\lambda_2^4 - 1), \\
\Sigma_n &= (\lambda_1^2 - \lambda_2^2)(\lambda_1^2\lambda_2^2 - 1)[\lambda_1(\lambda_2^4 - 1)p_{2n}q_{2n}(q_{1n}^2 + \lambda_1^2 p_{1n}^2) \\
&\quad - \lambda_2(\lambda_1^4 - 1)p_{1n}q_{1n}(q_{2n}^2 + \lambda_2^2 p_{2n}^2)], \\
\Delta_n &= (\lambda_1^2\lambda_2^2 - 1)^2(\lambda_1^2 p_{1n}^2 q_{2n}^2 + \lambda_2^2 p_{2n}^2 q_{1n}^2) + (\lambda_1^2 - \lambda_2^2)^2(\lambda_1^2\lambda_2^2 p_{1n}^2 p_{2n}^2 + q_{1n}^2 q_{2n}^2) \\
&\quad - 2p_{1n}q_{1n}p_{2n}q_{2n}\lambda_1\lambda_2(\lambda_1^4 - 1)(\lambda_2^4 - 1).
\end{aligned}$$

Two algebraic solitons on the dnoidal wave

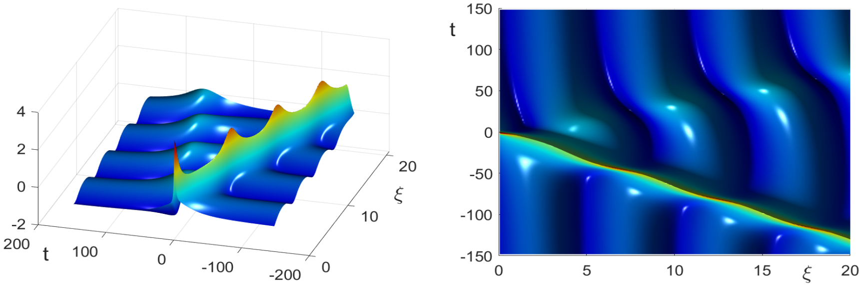


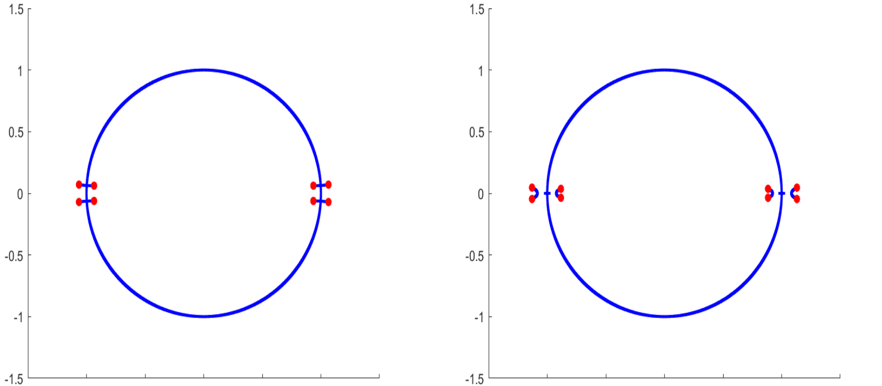
Figure: The solution surface (left: sideview, right: topview) for eigenvalues λ_1 and λ_2.

Similar new solutions for the cnoidal wave

For the cnoidal wave, the new solution after 2-fold transformation is real valued if $\lambda_2 = \bar{\lambda}_1$. However, $p_n^2 + q_n^2$ is not sign-definite and the representation

$$\hat{p}_n = p_n \theta_n - \frac{q_n}{p_n^2 + q_n^2}, \quad \hat{q}_n = q_n \theta_n + \frac{p_n}{p_n^2 + q_n^2}$$

cannot be used.



Another representation

The second, linearly independent solution can be found in the form:

$$\hat{p}_n = p_n \theta_n - \frac{1}{2q_n}, \quad \hat{q}_n = q_n \theta_n + \frac{1}{2p_n},$$

where

$$\theta_{n+1} - \theta_n = \frac{(\lambda_1^2 - \lambda_1^{-2})^2 u_n^2}{2(1 + u_n^2)(F_1 - u_n u_{n-1})(F_1 - u_{n+1} u_n)}$$

and

$$\dot{\theta}_n = \frac{(\lambda_1^2 - \lambda_1^{-2})^2 u_n u_{n-1}}{(F_1 - u_n u_{n-1})^2},$$

If $u_n(t) = \phi(\alpha n + ct)$ is the traveling wave with periodic ϕ, then $\theta_n(t) = an + bt + \chi(\alpha n + ct)$ with periodic χ and uniquely computed parameters a and b.

Rogue wave on the cnoidal wave

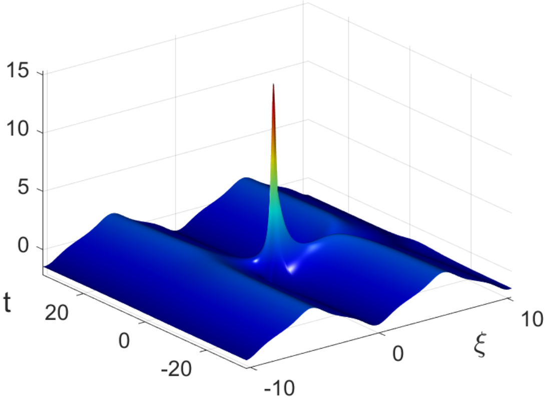


Figure: The solution surface for eigenvalues λ_1 and $\lambda_2 = \bar{\lambda}_1$.

Summary

- Traveling periodic waves are recovered from the nonlinearization method based on the constraint $u_n = \lambda_1 p_n^2 + \lambda_1^{-1} q_n^2$ with λ_1 being a root of $P(\lambda)$.
- Dnoidal waves are spectrally (modulationally) stable, whereas cnoidal waves are spectrally (modulationally) unstable.
- Only two distinct algebraic solitons exist on the background of dnoidal waves. A rogue wave exists on the background of cnoidal waves.
- Two open questions include
  1. relation between $(1+u_n^2)(u_{n+1}+u_{n-1}) = \omega u_n$ and $\dot{u}_n = (1+u_n^2)(u_{n+1}-u_{n-1})$
  2. connection between $P(\lambda)$ and the stability spectrum Ω.

Many thanks for your attention!