Stability of Dirac solitons

(the massive Thirring model)

Dmitry Pelinovsky and Yusuke Shimabukuro
(McMaster University, Canada)

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The model

The nonlinear Dirac equations in one spatial dimension,

$$\begin{cases} i(u_t + u_x) + v = \partial_{\bar{u}} W(u, v), \\ i(v_t - v_x) + u = \partial_{\bar{v}} W(u, v), \end{cases}$$

where $W(u, v) : \mathbb{C}^2 \to \mathbb{R}$ satisfies the following three conditions:

- symmetry $W(u, v) = W(v, u)$;
- gauge invariance $W(e^{i\theta} u, e^{i\theta} v) = W(u, v)$ for any $\theta \in \mathbb{R}$;
- quartic polynomial in (u, v) and $(\bar{u}, \bar{v})$.

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Examples of nonlinear potentials:

- Coupled-mode system: $W = |u|^4 + 4|u|^2|v|^2 + |v|^4$.
- Gross–Neveu model: $W = (\bar{u}v + u\bar{v})^2$.
- Massive Thirring model: $W = |u|^2|v|^2$

Massive Thirring Model (MTM)

The MTM in laboratory coordinates

$$\begin{cases} i(u_t + u_x) + v = 2|v|^2 u, \\ i(v_t - v_x) + u = 2|u|^2 v, \end{cases}$$

First three conserved quantities are

$$Q = \int_{\mathbb{R}} \left(|u|^2 + |v|^2\right) dx,$$

$$P = \frac{i}{2} \int_{\mathbb{R}} \left(u\bar{u}_x - u_x\bar{u} + v\bar{v}_x - v_x\bar{v}\right) dx,$$

$$H = \frac{i}{2} \int_{\mathbb{R}} \left(u\bar{u}_x - u_x\bar{u} - v\bar{v}_x + v_x\bar{v}\right) dx + \int_{\mathbb{R}} \left(-v\bar{u} - u\bar{v} + 2|u|^2|v|^2\right) dx.$$

An infinite set of conserved quantities is available thanks to the integrability of the MTM.

A physical context of the MTM system

Dynamics of nonlinear waves in the Gross–Pitaevskii equation with a one-dimensional (stripe) periodic potential

$$i\psi_t = -\psi_{xx} - \psi_{yy} + 2\epsilon\cos(x)\psi + |\psi|^2\psi, \quad \epsilon \ll 1,$$

can be described by the slowly varying decomposition

$$\psi(x,y,t) \approx \sqrt{\epsilon}\left[u(\epsilon x, \sqrt{\epsilon}y, \epsilon t)e^{\frac{i}{2}x - \frac{i}{4}t} + v(\epsilon x, \sqrt{\epsilon}y, \epsilon t)e^{-\frac{i}{2}x - \frac{i}{4}t}\right].$$

The amplitude u and v in slow variables X, Y, and T satisfy the perturbed MTM equations

$$\begin{cases} i(u_T + u_X) + v + u_{YY} = (|u|^2 + 2|v|^2)u, \\ i(v_T - v_X) + u + v_{YY} = (2|u|^2 + |v|^2)v. \end{cases}$$

Reference: T.Dohnal & A.B. Aceves (2005).

Transverse stability mystery

- J.Yang *et al.* [Opt. Lett. **37** (2012), 1571] - predicted no transverse instability of gap solitons in stripe (one-dimensional) periodic potentials.

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- Using the opposite limit of small-amplitude periodic potential, we clarify the mystery and show that gap solitons are indeed transversely unstable for all parameters.

Questions for MTM

- Existence of local and global solutions in $H^1(\mathbb{R})$ or $L^2(\mathbb{R})$
- Orbital stability of gap solitons in $H^1(\mathbb{R})$ or $L^2(\mathbb{R})$
- Transverse instability of gap solitons in two dimensions

Local and global existence

Theorem

Assume $\mathbf{u}_0 \in H^1(\mathbb{R})$*. There exists* $T > 0$ *such that the nonlinear Dirac equations admit a unique solution*

$$\mathbf{u}(t) \in C([0,T], H^1(\mathbb{R})) \cap C^1([0,T], L^2(\mathbb{R})) : \quad \mathbf{u}(0) = \mathbf{u}_0,$$

which depends continuously on the initial data.

Theorem

Assume that W *is a polynomial in variables* $|u|^2$ *and* $|v|^2$*. A local solution in* H^1 *is extended globally as* $\mathbf{u}(t) \in C(\mathbb{R}_+, H^1(\mathbb{R}))$*.*

References: Delgado (1978); Goodman-Weinstein-Holmes (2001); Selberg-Tesfahun (2010); Huh (2011); Zhang (2013).

Quick proof of global well-posedness in $H^1(\mathbb{R})$

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$$\partial_t \left(|u|^{2p+2} + |v|^{2p+2}\right) + \partial_x \left(|u|^{2p+2} - |v|^{2p+2}\right) = i(p+1)(v\bar{u} - \bar{v}u)(|u|^{2p} - |v|^{2p}).$$

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- By Gronwall's inequality, we have

$$\|\mathbf{u}(t)\|_{L^{2p+2}} \leq e^{2|t|}\|\mathbf{u}(0)\|_{L^{2p+2}}, \quad t \in [0,T],$$

which holds for any $p \geq 0$ including $p \to \infty$.

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  which holds for any $p \geq 0$ including $p \to \infty$.
- This allows to control

$$\frac{d}{dt}\|\partial_x \mathbf{u}(t)\|^2_{L^2} \leq C_W e^{4(N-1)|t|}\|\partial_x \mathbf{u}(t)\|^2_{L^2},$$

  where N is the degree of W in variables $|u|^2$ and $|v|^2$.

Local and global well-posedness in $L^2(\mathbb{R})$

Theorem

For any $(u_0, v_0) \in L^2(\mathbb{R})$, there exists a unique solution of the MTM $(u, v) \in C(\mathbb{R}, L^2(\mathbb{R}))$:

$$\|u(\cdot,t)\|_{L^2}^2 + \|v(\cdot,t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2.$$

References: T. Candy (2011); Y. Zhang & Q. Zhao (2015).

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Existence of solitary waves

Time-periodic space-localized solutions

$$u(x,t) = U_\omega(x) e^{-i\omega t}, \quad v(x,t) = V_\omega(x) e^{-i\omega t}$$

satisfy a system of stationary Dirac equations. They are known in the closed analytic form

$$\begin{cases} u(x,t) = i \sin(\gamma) \text{ sech} \left[x \sin\gamma - i \frac{\gamma}{2} \right] \; e^{-it\cos\gamma}, \\ v(x,t) = -i \sin(\gamma) \text{ sech} \left[x \sin\gamma + i \frac{\gamma}{2} \right] \; e^{-it\cos\gamma}, \end{cases}$$

where $\omega = \cos(\gamma)$.

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where $\omega = \cos(\gamma)$.

- Translations in x and t can be added as free parameters.
- Constraint $\omega = \cos\gamma \in (-1,1)$ exists because spectrum of linear waves is located for $(-\infty,-1] \cup [1,\infty)$.
- Moving solitons can be obtained from the stationary solitons with the Lorentz transformation.

Orbital stability of solitary waves

Definition

We say that the solitary wave $e^{-i\omega t}\mathbf{U}_\omega(x)$ is orbitally stable if for any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$, such that if

$$\|\mathbf{u}(\cdot, 0) - \mathbf{U}_\omega(\cdot)\|_{H^1} \leq \delta(\epsilon)$$

then

$$\inf_{\theta, a \in \mathbb{R}} \|\mathbf{u}(\cdot, t) - e^{-i\theta}\mathbf{U}_\omega(\cdot + a)\|_{H^1} \leq \epsilon,$$

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- Spectral stability of Dirac solitons was mainly studied numerically, e.g., by I. Barashenkov (1998), G. Gottwald (2005), M. Chugunova (2006), A. Comech (2012), A. Saxena (2014), P. Kevrekidis (2014), ...
- Asymptotic stability of Dirac solitons was proved for quintic nonlinearities by D.P. & A. Stefanov (2012).

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Orbital stability of MTM solitons in H^1

Theorem

There is $\omega_0 \in (0,1]$ such that for any fixed $\omega = \cos\gamma \in (-\omega_0, \omega_0)$, the MTM soliton is a local non-degenerate minimizer of R in $H^1(\mathbb{R}, \mathbb{C}^2)$ under the constraints of fixed values of Q and P.

The higher-order Hamiltonian R is

$$R = \int_{\mathbb{R}} \Big[|u_x|^2 + |v_x|^2 - \frac{i}{2}(u_x \overline{u} - \overline{u}_x u)(|u|^2 + 2|v|^2) + \frac{i}{2}(v_x \overline{v} - \overline{v}_x v)(2|u|^2 + |v|^2)$$

$$-(u\overline{v} + \overline{u}v)(|u|^2 + |v|^2) + 2|u|^2|v|^2(|u|^2 + |v|^2) \Big] \, dx.$$

R is a conserved quantity of the MTM in addition to the standard Hamiltonian H, the charge Q, and the momentum P.

The energy functionals

- Critical points of $H + \omega Q$ for a fixed $\omega \in (-1,1)$ satisfy the stationary MTM equations. After the reduction $(u,v) = (U,\overline{U})$, we obtain the first-order equation

$$i\frac{dU}{dx} - \omega U + \overline{U} = 2|U|^2 U,$$

which is satisfied by the MTM soliton $U = U_\omega$.

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- Critical points of $R + \Omega Q$ for some fixed $\Omega \in \mathbb{R}$ satisfy another system of equations. After the reduction $(u,v) = (U, \overline{U})$, we obtain the second-order equation

$$\frac{d^2 U}{dx^2} + 6i|U|^2\frac{dU}{dx} - 6|U|^4 U + 3|U|^2\bar{U} + U^3 = \Omega U.$$

  $U = U_\omega$ satisfies this equation if $\Omega = 1 - \omega^2$.

The Lyapunov functional for MTM solitons

There is no chance for the standard energy functional

$$\Lambda_\omega := H + \omega Q$$

to become a Lyapunov functional for MTM solitons.

However, the higher-order energy functional

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... the second variation of $\tilde{\Lambda}_\omega$ at U_ω is proved to have exactly one negative eigenvalue for small $\omega \neq 0$ in addition to the double zero eigenvalue. (For $\omega = 0$, no negative eigenvalues exist but the zero eigenvalue is quadruple.)

Constrained Hilbert spaces

Assume that $(u, v) \in L^2(\mathbb{R}; \mathbb{C}^2)$ satisfies the constraints:

$$\int_{\mathbb{R}} \left(\bar{U}_\omega u + U_\omega v\right) dx = 0, \qquad (1)$$

$$\int_{\mathbb{R}} \left(\bar{U}'_\omega u + U'_\omega v\right) dx = 0. \qquad (2)$$

- Real part of Eq (1) corresponds to fixed Q (charge).
- Imaginary part of Eq. (2) corresponds to fixed P (momentum).
- Imaginary part of Eq. (1) corresponds to orthogonality to the gauge translation $u \mapsto ue^{i\alpha}$, $v \mapsto ve^{i\alpha}$.
- Real part of Eq. (2) corresponds to orthogonality to the space translation $u(x) \mapsto u(x + x_0)$, $v(x) \mapsto v(x + x_0)$.

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The constraints (1)–(2) remove the negative and zero eigenvalues of the second variation of $\tilde{\Lambda}_\omega$.

Orbital stability result

- Strict positivity (coercivity) of the second variation implies

$$\tilde{\Lambda}_\omega(\mathbf{U}_\omega + \mathbf{u}) - \tilde{\Lambda}_\omega(\mathbf{U}_\omega) \geq C\|\mathbf{u}\|_{H^1}^2 + \mathcal{O}(3),$$

  for all $\mathbf{u} \in H^1(\mathbb{R}; \mathbb{C}^2)$ in the constrained space.

- R, Q, and P are constant in time t and so is $\tilde{\Lambda}_\omega$.

- Then, we obtain the global lower bound for the solution $\mathbf{u}$:

$$\tilde{\Lambda}_\omega(\mathbf{u}) - \tilde{\Lambda}_\omega(\mathbf{U}_\omega) \geq \inf_{\theta, x_0} \|\mathbf{u}(\cdot, t) - e^{i\theta}\mathbf{U}_\omega(\cdot + x_0)\|_{H^1}^2$$

  for every $t \in \mathbb{R}$.

- This yields orbital stability in $H^1(\mathbb{R})$ for $\omega \in (-\omega_0, \omega_0)$.

Orbital stability of MTM solitons in L^2

Theorem

Let $(u, v) \in C(\mathbb{R}; L^2(\mathbb{R}))$ be a solution of the MTM system and λ_0 be a complex non-zero number. There exist a real positive constant ϵ such that if the initial value $(u_0, v_0) \in L^2(\mathbb{R})$ satisfies

$$\|u_0 - u_{\lambda_0}(\cdot, 0)\|_{L^2} + \|v_0 - v_{\lambda_0}(\cdot, 0)\|_{L^2} \leq \epsilon,$$

then for every $t \in \mathbb{R}$, there exists $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_0| \leq C\epsilon$,

$$\inf_{a,\theta \in \mathbb{R}} (\|u(\cdot + a, t) - e^{-i\theta} u_\lambda(\cdot, t)\|_{L^2} + \|v(\cdot + a, t) - e^{-i\theta} v_\lambda(\cdot, t)\|_{L^2}) \leq C\epsilon,$$

where the constant C is independent of ϵ and t.

Lax operators for the MTM

The MTM is obtained from the compatibility condition of the linear system

$$\vec{\phi}_x = L\vec{\phi} \quad \text{and} \quad \vec{\phi}_t = A\vec{\phi},$$

where

$$L = \frac{i}{2}(|v|^2-|u|^2)\sigma_3 - \frac{i\lambda}{\sqrt{2}}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{\sqrt{2}\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\frac{1}{\lambda^2} - \lambda^2\right)\sigma_3$$

and

$$A = -\frac{i}{4}(|u|^2+|v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\sigma_3$$

References:
Kaup–Newell (1977); Kuznetsov–Mikhailov (1977).

Bäcklund transformation for the MTM

- Let (u, v) be a C^1 solution of the MTM system.
- Let $\vec{\phi} = (\phi_1, \phi_2)^t$ be a C^2 nonzero solution of the linear system associated with (u, v) and $\lambda = \delta e^{i\gamma/2}$.

A new C^1 solution of the MTM system is given by

$$\mathbf{u} = -u \frac{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2} + \frac{2i\delta^{-1}\sin\gamma\bar{\phi}_1\phi_2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2}$$

$$\mathbf{v} = -v \frac{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2} - \frac{2i\delta\sin\gamma\bar{\phi}_1\phi_2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2},$$

A new C^2 nonzero solution $\vec{\psi} = (\psi_1, \psi_2)^t$ of the linear system associated with $(\mathbf{u}, \mathbf{v})$ and same λ is given by

$$\psi_1 = \frac{\bar{\phi}_2}{|e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2|}, \quad \psi_2 = \frac{\bar{\phi}_1}{|e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2|}.$$

Bäcklund transformation $0 \leftrightarrow 1$ soliton

Let $(u, v) = (0, 0)$ and define

$$\begin{cases} \phi_1 = e^{\frac{i}{4}(\lambda^2 - \lambda^{-2})x + \frac{i}{4}(\lambda^2 + \lambda^{-2})t}, \\ \phi_2 = e^{-\frac{i}{4}(\lambda^2 - \lambda^{-2})x - \frac{i}{4}(\lambda^2 + \lambda^{-2})t}. \end{cases}$$

Then, $(\mathbf{u}, \mathbf{v}) = (u_\lambda, v_\lambda)$.

If $\lambda = e^{i\gamma/2}$ (stationary case), the vector $\vec{\psi}$ is given by

$$\begin{cases} \psi_1 = e^{\frac{1}{2}x \sin\gamma + \frac{i}{2}t \cos\gamma} \left| \operatorname{sech}\left(x \sin\gamma - i\frac{\gamma}{2}\right) \right|, \\ \psi_2 = e^{-\frac{1}{2}x \sin\gamma - \frac{i}{2}t \cos\gamma} \left| \operatorname{sech}\left(x \sin\gamma - i\frac{\gamma}{2}\right) \right|. \end{cases}$$

It decays exponentially as $|x| \to \infty$.

In the opposite direction, if $(u, v) = (u_\lambda, v_\lambda)$ and $\vec{\phi} = \vec{\psi}$, then $(\mathbf{u}, \mathbf{v}) = (0, 0)$.

Steps in the proof of the main result

- Step 1: From a perturbed one-soliton to a small solution at the initial time $t = 0$.
- Step 2: Time evolution of the small solution for $t \in \mathbb{R}$.
- Step 3: From the small solution to the perturbed one-soliton for every $t \in \mathbb{R}$.

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- Transverse instability of gap solitons in two dimensions

Transverse stability problem

The 2D version of the MTM:

$$\begin{cases} i(u_t + u_x) + v + u_{yy} = 2|v|^2 u, \\ i(v_t - v_x) + u + v_{yy} = 2|u|^2 v. \end{cases}$$

Using the Fourier decomposition like

$$u(x,y,t) = e^{i\omega t}[U_\omega(x) + u_1(x) e^{\lambda t + ipy}], \quad \omega \in (-1,1),$$

we obtain the spectral stability problem

$$i\lambda\sigma\mathbf{U} = (D_\omega + W_\omega + p^2 I)\mathbf{U},$$

where $\mathbf{U} \in \mathbb{C}^4$, $\sigma = \text{diag}(1,-1,1,-1)$, W_ω is a decaying potential, and

$$D_\omega = \begin{bmatrix} -i\partial_x + \omega & 0 & -1 & 0 \\ 0 & i\partial_x + \omega & 0 & -1 \\ -1 & 0 & i\partial_x + \omega & 0 \\ 0 & -1 & 0 & -i\partial_x + \omega \end{bmatrix}.$$

Properties of the spectral problem

- Continuous spectrum is located along the segments $\pm i\Lambda_1$ and $\pm i\Lambda_2$, where

$$\Lambda_1 \in [1 + \omega + p^2, \infty), \quad \Lambda_2 \in [1 - \omega - p^2, \infty).$$

The gap near $\lambda = 0$ exists for small p.

- If $p = 0$, there exist exactly two eigenvectors for $\lambda = 0$:

$$\mathbf{U}_t = \partial_x \mathbf{U}_\omega, \quad \mathbf{U}_g = i\sigma \mathbf{U}_\omega,$$

and exactly two generalized eigenvectors

$$\tilde{\mathbf{U}}_t = i\omega x \sigma \mathbf{U}_\omega - \frac{1}{2}\tilde{\sigma}\mathbf{U}_\omega, \quad \tilde{\mathbf{U}}_g = \partial_\omega \mathbf{U}_\omega.$$

Perturbation theory result

Theorem

For every $\omega \in (-1,1)$, there exists $p_0 > 0$ such that for every p with $0 < |p| < p_0$, the spectral stability problem admits a pair of real eigenvalues λ with the eigenvectors $\mathbf{V} \in H^1(\mathbb{R})$ such that

$$\lambda = \pm p\Lambda_r(\omega) + \mathcal{O}(p^3), \quad \mathbf{V} = \mathbf{V}_t \pm p\Lambda_r(\omega)\tilde{\mathbf{V}}_t + \mathcal{O}_{H^1}(p^2) \quad \text{as} \quad p \to 0,$$

where $\Lambda_r = (1-\omega^2)^{-1/4}\|U'_\omega\|_{L^2} > 0$. Simultaneously, it admits a pair of purely imaginary eigenvalues λ with the eigenvector $\mathbf{V} \in H^1(\mathbb{R})$ such that

$$\lambda = \pm ip\Lambda_i(\omega) + \mathcal{O}(p^3), \quad \mathbf{V} = \mathbf{V}_g \pm ip\Lambda_i(\omega)\tilde{\mathbf{V}}_g + \mathcal{O}_{H^1}(p^2) \quad \text{as} \quad p \to 0,$$

where $\Lambda_i = \sqrt{2}(1-\omega^2)^{1/4}\|U_\omega\|_{L^2} > 0$.

Numerical method: Chebyshev interpolation

- Grid points at $x_j = L\tanh^{-1}(z_j)$, with $j = 0, 1, ..., N$, where $z_j = \cos(j\pi/N)$ is the Chebyshev node.
- Parameter L is at our disposal for better resolution of the fast change of the MTM soliton ($L = 10$).

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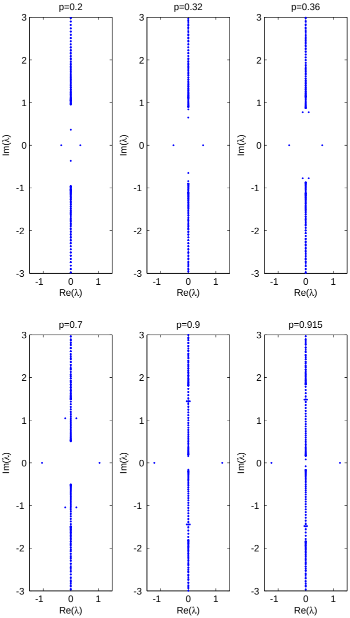
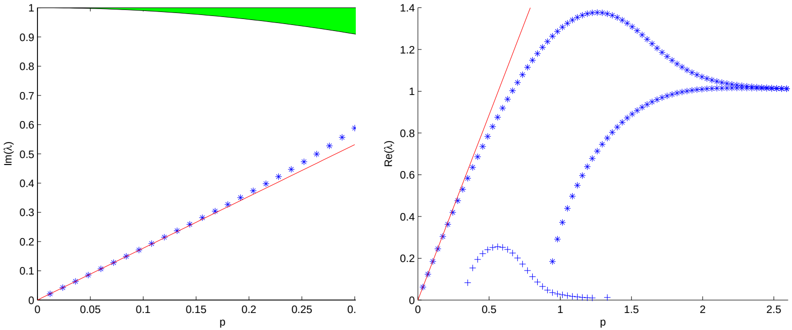
- Eigenvalues are found from $4(N+1) \times 4(N+1)$ matrices.

Reference: M. Chugunova & D.P. [SIAD **5** (2006), 55].

Numerical approximations of eigenvalues for $\omega = 0$



Isolated eigenvalues for $\omega = 0$



Accuracy of numerical computations

	$\omega = -0.5$	$\omega = 0$	$\omega = 0.5$
$N = 100$	1.96×10^{-1}	2.57×10^{-1}	1.16×10^{-1}
$N = 300$	1.36×10^{-4}	2.18×10^{-4}	7.02×10^{-5}
$N = 500$	2.22×10^{-7}	8.77×10^{-5}	6.56×10^{-8}

Table: $\max|\mathrm{Re}(\lambda)|$ along the continuous band for $p = 0$.

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- Transverse instability of gap solitons in two dimensions

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