NLS models with competing nonlinearities: kinks, solitons, and their stability problems

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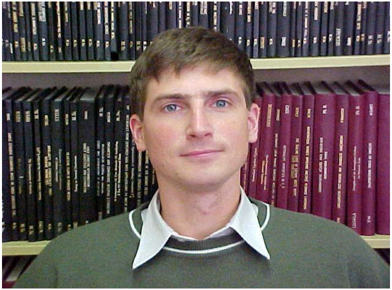
Workshop "Order and function in complex systems"
University of Minnesota, August 10–14, 2026

In honor of Arnd Scheel

We met long ago in Puerto Rico...



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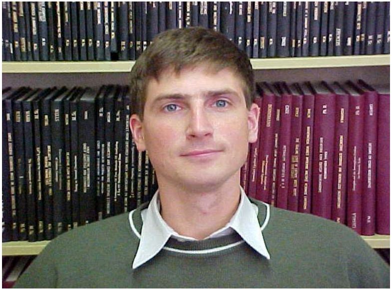


... and continue to discuss things nearly every year ...

My talk is about the Hamiltonian NLS model with competing focusing and defocusing nonlinearities:

$$i\psi_t = \psi_{xx} + 4|\psi|^2\psi - 3|\psi|^4\psi.$$

The main message is:

- there exists a kink (heteroclinic orbit) separating bright and dark solitons (homoclinic orbits);
- the kink is a degenerate minimizer of the conserved energy E,
- nevertheless, after the right modulation, weighted metric, and energy splitting,

$$E(\psi(t)) - E(\phi) \gtrsim \rho_R^2\big(e^{i\alpha(t)}\psi(t, \cdot + \beta(t)), \phi)],$$

  which gives orbital stability of the kink with the profile ϕ.
- the kink persists with respect to external potentials.

Joint work with J.Holmer (Brown) and P.Kevrekidis (Amherst), as well as with C. Lustri (Sydney), and G. Alfimov (Moscow).

The cubic–quintic model appears in two overlapping stories.

Optics. The quintic term is a higher-order correction to the nonlinear refractive index; bright and dark solitons have been studied both experimentally and analytically, for example in work of Akhmediev–Afanasjev (1995), Falcão-Filho et al. (2013), Reyna et al. (2020).

Quantum droplets. Lee–Huang–Yang corrections can compete with mean-field nonlinearities; Petrov (2015) and Petrov–Astrakharchik (2016) derived these corrections, and there is an active literature on droplets including Luo–Pang–Liu–Li–Malomed (2021), Katsimiga–Kevrekidis et al. (2023), Alfimov–Abdulaev (2025).

More general models with competing focusing and defocusing nonlinearities are used for similar applications.

The kink appears at the unit frequency, for which we use the normalized equation

$$i\psi_t = \psi_{xx} - \psi + 4|\psi|^2\psi - 3|\psi|^4\psi.$$

It is Hamiltonian with conserved energy

$$E(\psi) = \int_{\mathbb{R}} \left(|\psi_x|^2 + |\psi|^2(1 - |\psi|^2)^2\right) dx.$$

Critical points of E satisfy

$$\phi'' - \phi + 4|\phi|^2\phi - 3|\phi|^4\phi = 0.$$

For real profiles, the ODE is a classical mechanical system

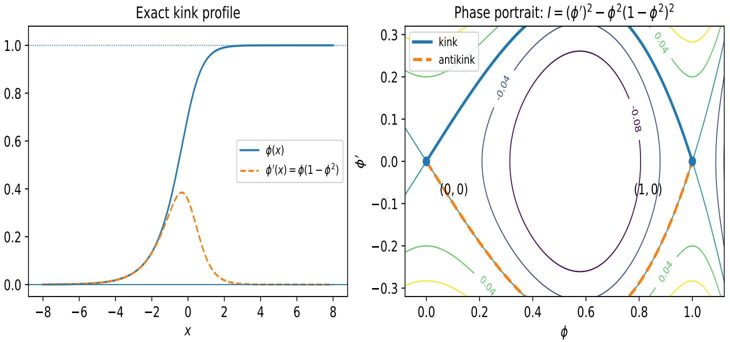
$$\phi'' + U'(\phi) = 0, \quad U'(\phi) = -\phi + 4\phi^3 - 3\phi^5, \quad U(\phi) = -\frac{1}{2}\phi^2(1-\phi^2)^2.$$

The mechanical Hamiltonian is

$$H_{\mathrm{mech}} = \frac{1}{2}(\phi')^2 + U(\phi).$$

The kink is the zero-energy heteroclinic orbit from $\phi = 0$ to $\phi = 1$. Equivalently, $H_{\mathrm{mech}} = 0$ yields $(\phi')^2 = \phi^2(1-\phi^2)^2$, or

$$\phi(x) = \left(\frac{1}{2}(1+\tanh x)\right)^{1/2}, \qquad \phi(-\infty) = 0, \quad \phi(+\infty) = 1.$$



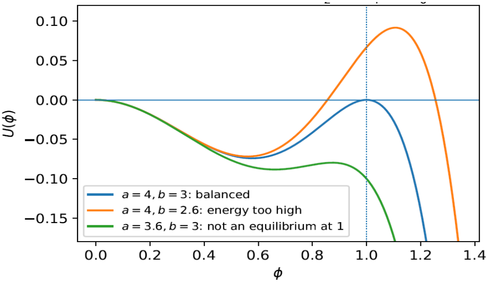
The kink exists because of a delicate coefficient balance. For the more general ODE

$$\phi'' - \phi + a\phi^3 - b\phi^5 = 0,$$

the mechanical Hamiltonian is

$$H_{\mathrm{mech}} = \frac{1}{2}(\phi')^2 + U(\phi), \quad U(\phi) = -\frac{1}{2}\phi^2 + \frac{a}{4}\phi^4 - \frac{b}{6}\phi^6.$$

The soliton solutions include bright solitons, black solitons, and bubbles if $(a, b) \neq (4, 3)$. Only kinks exist for $(a, b) = (4, 3)$.



For stability, we linearize about the real kink with $\psi = \phi + u + iv$,

$$\begin{cases} u_t = -\mathcal{L}_- v, \\ v_t = \mathcal{L}_+ u, \end{cases}$$

where

$$\mathcal{L}_- = -\partial_x^2 + 1 - 4\phi^2 + 3\phi^4, \qquad \mathcal{L}_+ = -\partial_x^2 + 1 - 12\phi^2 + 15\phi^4.$$

Sturm theory gives nonnegativity:

$$\mathcal{L}_-\phi = 0, \qquad \mathcal{L}_+\phi' = 0, \qquad \phi > 0, \quad \phi' > 0.$$

By the standard theory, the kink is spectrally stable.

- But $\mathcal{L}_-$ is not coercive in $H^1(\mathbb{R})$ since 0 is the end point of the continuous spectrum in $L^2(\mathbb{R})$.
- At the same time, $\mathcal{L}_+$ is coercive in $H^1(\mathbb{R})$ since 0 is an isolated eigenvalue from the continuous spectrum in $L^2(\mathbb{R})$.

The main issue is to turn spectral stability into nonlinear stability.

Theorem (Holmer–Kevrekidis–Pelinovsky, Proc AMS (2026))

Fix $R > R_0$. For every sufficiently small $\epsilon > 0$, there exists $\delta > 0$ such that if

$$\psi_0 \in \mathcal{E} \subset L^\infty(\mathbb{R}), \qquad \rho_R(\psi_0, \phi) < \delta,$$

then the corresponding global solution $\psi \in C^0(\mathbb{R}, \mathcal{E})$ satisfies

$$\inf_{\alpha,\beta \in \mathbb{R}} \rho_R\left(e^{i\alpha}\psi(t, \cdot + \beta), \phi\right) < \epsilon, \quad \forall t \in \mathbb{R}.$$

Here α is the phase modulation, β is the translation modulation, $\mathcal{E}$ is the energy space,

$$\mathcal{E} = \{\psi \in H^1_{\rm loc}(\mathbb{R}) : \; \psi_x \in L^2, \; \psi(1 - |\psi|^2) \in L^2\}.$$

and $\rho_R(\cdot, \cdot)$ is the distance in $H^1(\mathbb{R})$ with the exponential weight on $[R, \infty)$, where $\phi(x) \to 1$ as $x \to +\infty$.

The Lyapunov functional is the conserved energy difference

$$\Lambda(\psi) := E(\psi) - E(\phi),$$

but coercivity of $\Lambda(\psi)$ at ϕ requires a modified split of the energy.

Define the weighted space $\mathcal{H}_R$ by

$$\langle f, g \rangle_{\mathcal{H}_R} = \int_{-\infty}^{R} f \bar{g}\, dx + \int_{R}^{\infty} \frac{1 - \phi^2(x)}{1 - \phi^2(R)} f \bar{g}\, dx.$$

The orbital distance is

$$\rho_R(\psi, \phi) = \sqrt{\|\psi' - \phi'\|_{L^2}^2 + \|\psi - \phi\|_{\mathcal{H}_R}^2 + \||\psi|^2 - |\phi|^2\|_{L^2(R,\infty)}^2}.$$

This metric sees the derivative globally, gives a weighted control of the complex perturbation on the nonzero background, and controls the density defect on (R, ∞).

Following Gravejat–Smets (2015), and later variants by Alejo–Corcho (2024), Contreras–Pelinovsky–Plum (2018), and Pelinovsky–Plum (2024), we introduce

$$\eta = |\phi + u + iv|^2 - \phi^2 = 2\phi u + u^2 + v^2.$$

A direct expansion gives

$$E(\phi + u + iv) - E(\phi) =$$
$$\int_{\mathbb{R}} \left[(u')^2 + (v')^2 + (1-\phi^2)(1-3\phi^2)(u^2+v^2) + (3\phi^2-2)\eta^2 + \eta^3\right] dx.$$

Equivalently,

$$E(\phi+u+iv) - E(\phi) = Q_-(u) + Q_-(v) + \int_{\mathbb{R}} \left[(3\phi^2-2)\eta^2 + \eta^3\right] dx,$$

where $Q_-(u)$ is a quadratic form related to $\mathcal{L}_-$ which is not coercive in $L^2(\mathbb{R})$ but is coercive in $\mathcal{H}_R \supset L^2(\mathbb{R})$.

The problem: $3\phi^2 - 2 < 0$ on the left and $3\phi^2 - 2 > 0$ on the right.

Choose R with $\phi(R) > \sqrt{2/3}$. On $(-\infty, R)$ use the $\mathcal{L}_+$ quadratic density for u; on (R, ∞) use the $\mathcal{L}_-$ density and the good η^2 term.

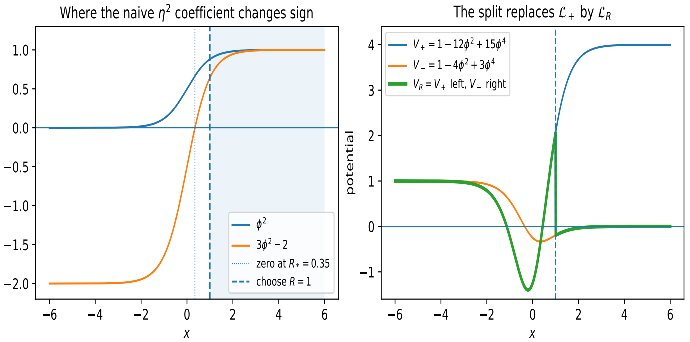
$$E(\phi + u + iv) - E(\phi) = Q_R(u) + Q_-(v) + \int_R^{+\infty} \left[(3\phi^2 - 2)\eta^2 + \eta^3\right] dx$$
$$+ \int_{-\infty}^{R} \left[(3\phi^2 - 2)(u^2 + v^2)(4\phi u + u^2 + v^2) + (2\phi u + u^2 + v^2)^3\right] dx$$

This produces a piecewisely defined Schrödinger operator

$$\mathcal{L}_R = -\partial_x^2 + V_R,$$

where

$$V_R(x) = \begin{cases} 1 - 12\phi^2(x) + 15\phi^4(x), & x < R, \\ 1 - 4\phi^2(x) + 3\phi^4(x), & x > R. \end{cases}$$



Where the naive η^2 coefficient changes sign
ϕ^2
$3\phi^2 - 2$
zero at $R_* = 0.35$
choose $R = 1$
x
The split replaces $\mathcal{L}_+$ by $\mathcal{L}_R$
$V_+ = 1 - 12\phi^2 + 15\phi^4$
$V_- = 1 - 4\phi^2 + 3\phi^4$
$V_R = V_+$ left, V_- right
potential
x

The key coercivity estimates are:

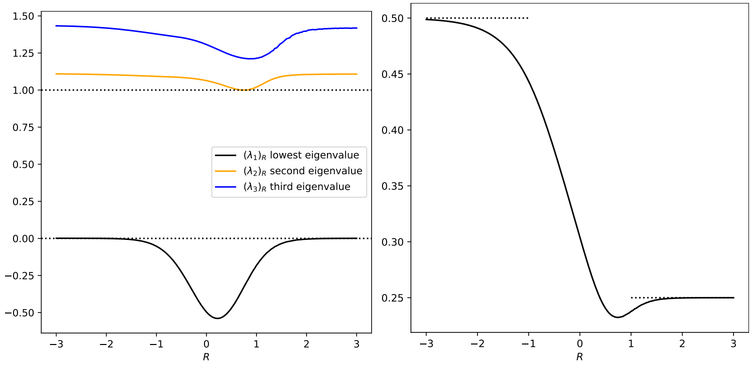
$$Q_-(v) \geq C_- \left(\|v'\|_{L^2}^2 + \|v\|_{\mathcal{H}_R}^2 \right), \qquad \langle v, \phi \rangle_{\mathcal{H}_R} = 0$$

and

$$Q_R(u) \geq C_R \left(\|u'\|_{L^2}^2 + \|u\|_{\mathcal{H}_R}^2 \right), \qquad \langle u, \phi' \rangle_{L^2} = 0.$$

The coercivity estimate for $Q_-(v)$ follows from the fact that $\phi \in \mathcal{H}_R$, for which $\mathcal{L}_- \phi = 0$, and the continuous spectrum of $\mathcal{L}_-$ in $\mathcal{H}_R$ is $[1, \infty)$.

The coercivity estimate for $Q_R(u)$ is more subtle since $\mathcal{L}_R$ has a small negative eigenvalue before imposing the constraint.



Imposing the constraint $\langle u, \phi' \rangle_{L^2} = 0$ yields coercivity of Q_R if and only if

$$\langle \mathcal{L}_R^{-1} \phi', \phi' \rangle_{L^2} < 0.$$

This can be proven for sufficiently large $R > 0$ or numerically for every R, see the right panel which shows $\lambda_1 \langle \mathcal{L}_R^{-1} \phi', \phi' \rangle_{L^2} > 0$.

The two orthogonality conditions are enforced by modulations of the kink orbit.

If the solution remains close to the orbit, the implicit function theorem gives unique continuous parameters $\alpha(t)$ and $\beta(t)$ so that

$$e^{i\alpha(t)}\psi(t, \cdot + \beta(t)) = \phi + u(t) + iv(t),$$

with

$$\langle u(t), \phi' \rangle_{L^2} = 0, \qquad \langle v(t), \phi \rangle_{\mathcal{H}_R} = 0.$$

The phase parameter removes the neutral direction of $\mathcal{L}_-$; the translation parameter removes the negative direction of $\mathcal{L}_R$.

With the modulation in place, conservation of E gives the Lyapunov estimate for small perturbations:

$$\begin{aligned} E(\psi_0) - E(\phi) &= E(\phi + u + iv) - E(\phi) \\ &\geq C\left(\|u' + iv'\|_{L^2}^2 + \|u + iv\|_{\mathcal{H}_R}^2 + \|\eta\|_{L^2(R,\infty)}^2\right) \\ &\quad - \text{higher order terms.} \end{aligned}$$

The higher order terms are controlled by:

- Sobolev control on $(-\infty, R)$,
- the density variable η on (R, ∞),
- a bootstrap bound $\|\eta\|_{L^\infty(R,\infty)} \lesssim \epsilon_0$.

Thus $E(\psi_0) - E(\phi) \gtrsim \rho_R^2(\phi + u + iv, \phi)$ and the theorem is proven.

The same argument extends beyond cubic–quintic powers.

For $2 \le p < q$, the normalized competing-power model is

$$i\psi_t = \psi_{xx} + \frac{q(p+2)}{2(q-p)}|\psi|^p\psi - \frac{p(q+2)}{2(q-p)}|\psi|^q\psi.$$

The kink appears again at the unit frequency and at the zero level curve

$$(\phi')^2 = \phi^2 \frac{q(1-\phi^p) - p(1-\phi^q)}{q-p}.$$

The same structure survives:

- weighted phase coercivity,
- constrained $\mathcal{L}_R$ coercivity,
- modulation by phase and translation,
- energy control in the ρ_R metric.

The kink also persists in a small bounded potential.

Assume that $V \in L^2(\mathbb{R})$ and $\varepsilon \in \mathbb{R}$ is small in

$$i\psi_t = \psi_{xx} + 4|\psi|^2\psi - 3|\psi|^4\psi + \varepsilon V(x)\psi.$$

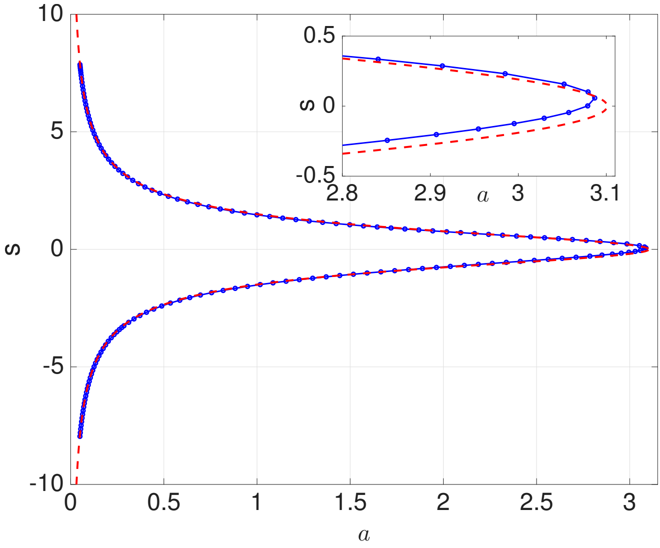
Theorem (Kevrekidis–Pelinovsky, in progress)

Let $s_0 \in \mathbb{R}$ be a simple root of the effective potential

$$V_{\rm eff}(s) = \frac{1}{4}\int V(x)\mathrm{sech}^2(x-s)dx.$$

There is $\varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$, there exists a kink with the profile ϕ_ε satisfying $\phi_\varepsilon(x) \to 0$ as $x \to -\infty$ and $\phi_\varepsilon(x) \to 1$ as $x \to +\infty$ pinned to $x = s_0$. It is a stable minimizer of energy for $V'_{\rm eff}(s_0) < 0$ and an unstable saddle point if $V'_{\rm eff}(s_0) > 0$.

Example: $V(x) = (1 - ax^2)\mathrm{sech}^2(x)$.



Does the kink persist in lattices?

Discrete competing models appear common in applications:

$$i\frac{du_n}{dt} + \varepsilon \Delta_2 u_n + |u_n|^p u_n - \gamma |u_n|^q u_n = 0, \quad n \in \mathbb{Z},$$

where $\gamma > 0$ is bifurcating parameter for standing waves of unit frequency $\mathrm{u}(t) = e^{it}\hat{\mathrm{u}}$.

- $\varepsilon \to 0$ is the anticontinuum limit, where many results can be derived with the implicit function theorem.
Alfimov–Korchagin–P, SIAM J. Appl. Dynam. Syst. (2026)
- $\varepsilon \to \infty$ is the continuous limit, where many results can be derived with the beyond-all-order asymptotics.
Lustri–Kevrekidis–Chapman, Quart. Appl. Math. (2025)
Lustri–Aniceto–Kevrekidis, SIAM J. Appl. Math. (2026)
Adriano–Hasmi–Kusdiantara–Susanto, Physica D (2025)

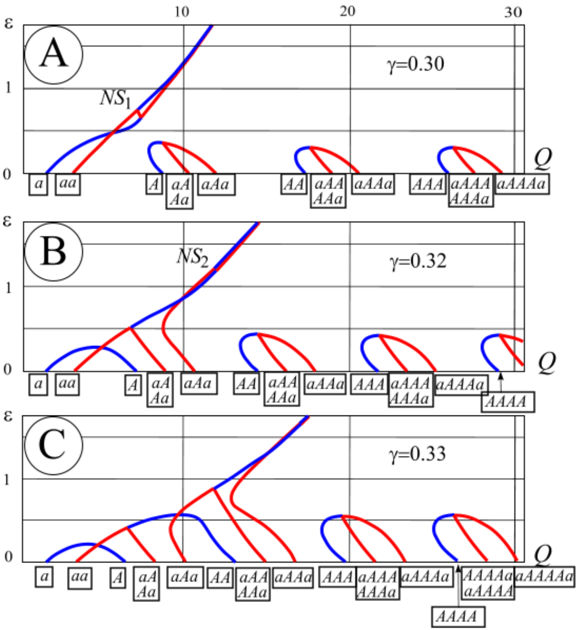
Theorem (Alfimov–Korchagin–Pelinovsky, SIAD (2026))

For every $1 \leq p < q$, *there exists* $\gamma_{p,q} > 0$ *such that for every* $\gamma \in (0, \gamma_{p,q})$ *and small* $\varepsilon \in (0, \varepsilon_0)$, *there exists a soliton* $\mathrm{u} \in \ell^2(\mathbb{Z})$ *satisfying*

$$\|\mathrm{u} - \mathrm{u}^{(0)}\|_{\ell^2(\mathbb{Z})} \leq C_0 \varepsilon,$$

where $\mathrm{u}^{(0)} = (\dots, 0, 0, \tilde{\mathrm{u}}, 0, 0, \dots)$ *with* $\tilde{\mathrm{u}} \in \mathbb{R}^N$ *of any length* $N \geq 1$ *defined by nonzero roots* $\pm a$ *and* $\pm A$ *with* $0 < a < A$ *of the function* $f(u) = u(1 - |u|^p + \gamma |u|^q)$.

Example: $(p, q) = (2, 3)$



The value $\gamma = \gamma_{p,q}$ is the bifurcation where $a = A$ and the kink appears instead of the soliton in the standing wave $\mathrm{u}(t) = e^{it}\hat{\mathrm{u}}$.

For the cubic–quintic case,

$$\varphi(u_{n+1} - 2u_n + u_{n-1}) = u_n(1 - u_n^2)(1 - 3u_n^2), \quad n \in \mathbb{Z},$$

two distinct kinks appear in the anti-continuum limit:

$$\varepsilon = 0 : \quad u_n^{(1)} = \begin{cases} 0, & n \leq 0, \\ 1, & n \geq 1, \end{cases} \qquad u_n^{(2)} = \begin{cases} 0, & n \leq -1, \\ \frac{1}{\sqrt{3}}, & n = 0, \\ 1, & n \geq 1, \end{cases}$$

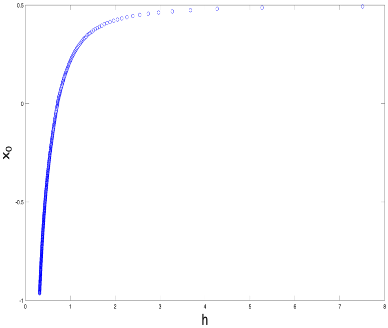
$\mathrm{u}^{(1)}$ is a stable minimizer of energy, $\mathrm{u}^{(2)}$ is an unstable saddle point. Adriano–Susanto, Stud. Appl. Math. (2026)

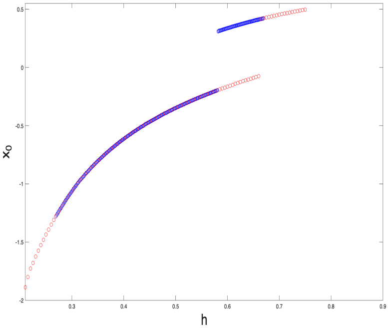
Do they persist in the continuum limit $\varepsilon \to \infty$?

Persistence of kinks is expected at the $\frac{1}{4}$ and $\frac{3}{4}$ spacing between the lattice sites in the beyond-all-orders asymptotics.

C. Lustri, P. Kevrekidis, D. Pelinovsky, in progress (2026)

However, preliminary numerical results did not show persistence of any of the two kinks as $\varepsilon = h^{-1}$ grows (h is the lattice spacing).





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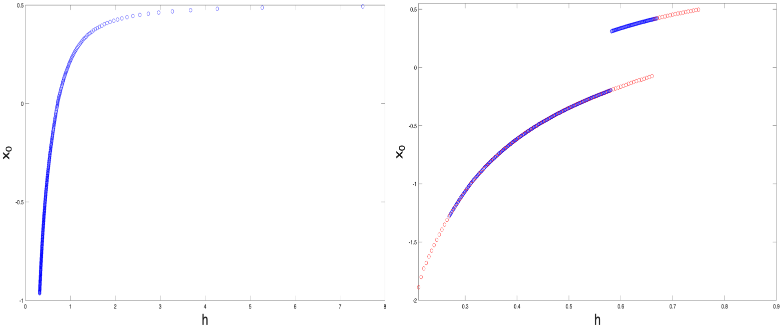

We need to ask Arnd for an advice... Happy birthday, Arnd!