Coupled-mode equations and gap solitons in two dimensions

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ICIAM-07, Zurich, July 16-20, 2007

Motivations

Gap solitons are localized stationary solutions of nonlinear PDEs with space-periodic coefficients which reside in the spectral gaps of associated linear operators.

Examples: Complex-valued Maxwell equation

$$\nabla^2 E - E_{tt} + \left(V(x) + \sigma |E|^2\right) E_{tt} = 0$$

and the Gross–Pitaevskii equation

$$iE_t = -\nabla^2 E + V(x)E + \sigma |E|^2 E,$$

where $E(x,t) : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{C}$, $V(x) = V(x + 2\pi e_j) : \mathbb{R}^N \mapsto \mathbb{R}$, and $\sigma = \pm 1$.

Existence of stationary solutions

Stationary solutions $E(x,t) = U(x)e^{-i\omega t}$ with $\omega \in \mathbb{R}$ satisfy a nonlinear elliptic problem with a periodic potential

$$\nabla^2 U + \omega U = V(x)U + \sigma |U|^2 U$$

Theorem: [Pankov, 2005] Let $V(x)$ be a real-valued bounded periodic potential. Let ω be in a finite gap of the spectrum of $L = -\nabla^2 + V(x)$. There exists a non-trivial weak solution $U(x) \in H^1(\mathbb{R}^N)$, which is (i) real-valued, (ii) continuous on $x \in \mathbb{R}^N$ and (iii) decays exponentially as $|x| \to \infty$.

Remark: Additionally, there exists a localized solution $U(x) \in H^1(\mathbb{R}^N)$ in the semi-infinite gap for $\sigma = -1$ (NLS soliton).

Asymptotic reductions in 1D

The nonlinear elliptic problem with a periodic potential can be reduced asymptotically for $N = 1$ to the following problems:

- Coupled-mode (Dirac) equations for small potentials

$$\begin{cases} ia'(x) + \Omega a + \alpha b = \sigma(|a|^2 + 2|b|^2)a \\ -ib'(x) + \Omega b + \alpha a = \sigma(2|a|^2 + |b|^2)b \end{cases}$$

- Envelope (NLS) equations for finite potentials near band edges

$$a''(x) + \Omega a + \sigma |a|^2 a = 0$$

- Lattice (dNLS) equations for large potentials

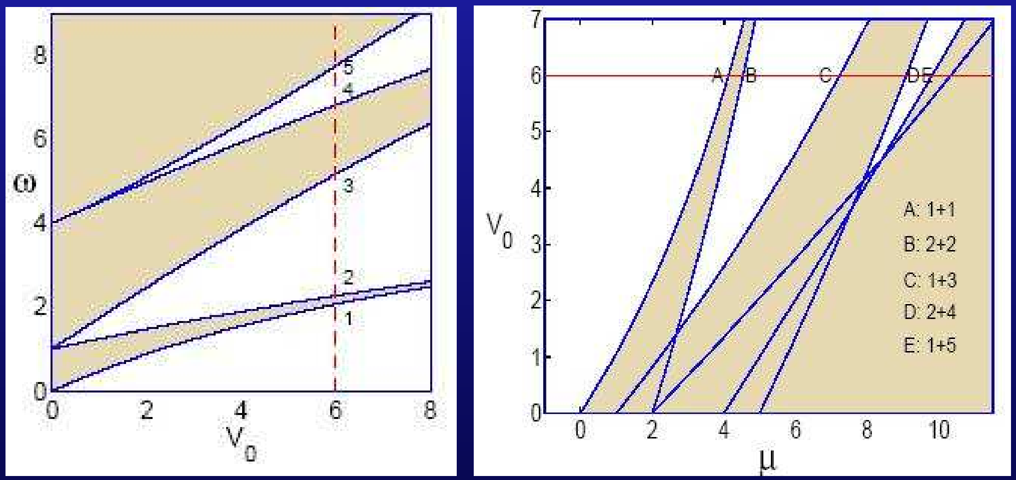
$$\alpha \left(a_{n+1} + a_{n-1} \right) + \Omega a_n + \sigma |a_n|^2 a_n = 0.$$

Localized solutions of reduced equations exist in the analytic form.

Bifurcation of gap solitons in 2D

Let $N = 2$ and $V(x) = \eta\,[W(x_1) + W(x_2)]$ be a separable potential. The band surface is given by $\omega = \rho(k_1) + \rho(k_2)$, while the eigenfunction is $\psi(x_1, x_2) = u(x_1)u(x_2)$, where

$$\begin{cases} -u''(x) + \eta W(x)u(x) = \rho u(x), \quad 0 \le x \le 2\pi, \\ u(2\pi) = e^{i2\pi k}u(0), \end{cases}$$



Left: spectrum of $L = -\partial_x^2 + \eta W(x)$ versus η.

Right: spectrum of $L = -\partial_{x_1}^2 - \partial_{x_1}^2 + \eta W(x_1) + \eta W(x_2)$ versus η.

Resonant Bloch modes at the bifurcation

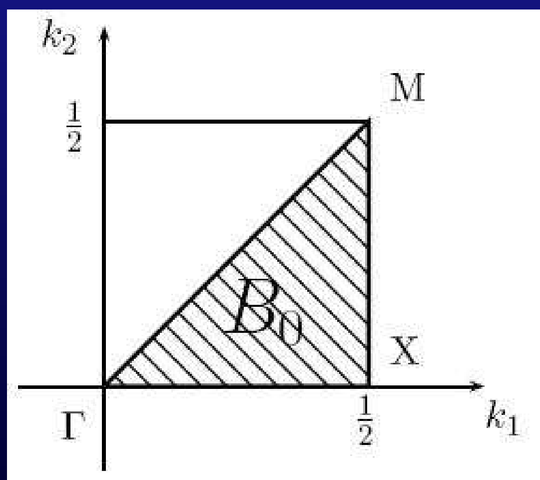
The first band gap opens up at $\eta = \eta_0 \approx 0.1747$, where three Bloch modes are in resonance

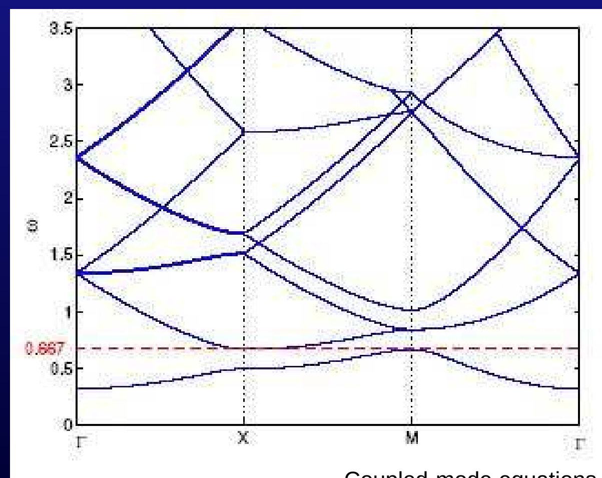
$$\phi_1 = \psi_1(x_1)\varphi_2(x_2), \; \phi_2 = \varphi_2(x_1)\psi_1(x_2), \; \phi_3 = \varphi_1(x_1)\varphi_1(x_2)$$

for corresponding eigenvalues

$$\omega = \lambda_1 + \mu_2 = \mu_2 + \lambda_1 = 2\mu_1.$$

Here $\psi_n(x)$ is a 2π-periodic function for eigenvalue λ_n and $\varphi_n(x)$ is a 2π-antiperiodic function for eigenvalue μ_n.





Derivation of coupled-mode equations

Let $\epsilon = \eta - \eta_0$, $\omega = \omega_0 + \epsilon\Omega$, and

$$U = \sqrt{\epsilon}\left[A_1\phi_1 + A_2\phi_2 + A_3\phi_3 + \epsilon\Phi(x_1, x_2)\right],$$

where $A_{1,2,3}$ are functions of $X = \sqrt{\epsilon}x$ and $\phi_{1,2,3}$ are functions of x. The projection algorithm leads to three coupled NLS equations:

$$(\Omega - \beta_1)A_1 + \left(\alpha_1\partial^2_{X_1} + \alpha_2\partial^2_{X_2}\right)A_1$$

$$= \sigma\left[\gamma_1|A_1|^2A_1 + \gamma_2(2|A_3|^2A_1 + A_3^2\bar{A}_1) + \gamma_3(2|A_2|^2A_1 + A_2^2\bar{A}_1)\right],$$

$$\cdots$$

$$(\Omega - \beta_2)A_3 + \alpha_3\left(\partial^2_{y_1} + \partial^2_{y_2}\right)A_3$$

$$= \sigma\left[\gamma_4|A_3|^2A_3 + 2\gamma_3(|A_1|^2 + |A_2|^2)A_3 + \gamma_3(A_1^2 + A_2^2)\bar{A}_3\right],$$

Remark: No first-order derivative terms occur in the coupled-mode system. Similar coupled NLS equations are derived near band edges by Z.Shi and J.Yang, PRE 75, 056602 (2007).

Main theorem I

Theorem: Let $W(x)$ be a bounded, piecewise-continuous, even and 2π-periodic function on $x \in \mathbb{R}$. Let $\frac{1}{4} < r < \frac{1}{2}$. The nonlinear elliptic problem has a continuous and decaying solution $U(x)$ for sufficiently small $|\epsilon| < \epsilon_0$ if there exists a non-trivial solution of

$$\left\{ \begin{array}{rcl} (\Omega - \beta_1 - \alpha_1 p_1^2 - \alpha_2 p_2^2)\, \hat{B}_1(p) - \sigma \hat{Q}_1(p) & = & \epsilon^{\tilde{r}} \hat{R}_1(p), \\ & \dots & \\ (\Omega - \beta_2 - \alpha_3 p_1^2 - \alpha_3 p_2^2)\, \hat{B}_3(p) - \sigma \hat{Q}_3(p) & = & \epsilon^{\tilde{r}} \hat{R}_3(p), \end{array} \right.$$

where $\tilde{r} = \min(4r-1, 1-2r)$, $\hat{B}_{1,2,3}(p)$ are compactly supported on the disk $D_\epsilon = \{p \in \mathbb{R}^2 : \ |p| < \epsilon^{r-\frac{1}{2}}\} \subset \mathbb{R}^2$, $\hat{Q}_{1,2,3}(p)$ denote the cubic nonlinear terms of the coupled-mode system, and

$$\|\hat{R}_{1,2,3}\|_{L^1(D_\epsilon)} \leq C_{1,2,3} \left(\|\hat{B}_1\|_{L^1(D_\epsilon)} + \|\hat{B}_2\|_{L^1(D_\epsilon)} + \|\hat{B}_3\|_{L^1(D_\epsilon)} \right).$$

Bloch-Fourier transform in 1D

Lemma: There exists a unitary transformation $\mathcal{T} : \phi \in L^2(\mathbb{R}) \mapsto \hat{\phi} \in l^2(\mathbb{N}, L^2(\mathbb{T}))$ given by

$$\forall \phi \in L^2(\mathbb{R}) : \quad \hat{\phi}_n(k) = \int_{\mathbb{R}} \bar{u}_n(y;k)\phi(y)dy$$

with the inverse transformation $\mathcal{T}^{-1}$:

$$\forall \hat{\phi} \in l^2(\mathbb{N}, L^2(\mathbb{T})) : \quad \phi(x) = \sum_{n \in \mathbb{N}} \int_{\mathbb{T}} \hat{\phi}_n(k) u_n(x;k)dk.$$

Lemma: If $\hat{\phi} \in l^1_s(\mathbb{N}, L^1(\mathbb{T}))$ for $s > \frac{1}{2}$ with the norm $\|\hat{\phi}\|_{l^1_s(\mathbb{N}, L^1(\mathbb{T}))} = \sum_{n \in \mathbb{N}} (1+n)^s \int_{\mathbb{T}} |\hat{\phi}_n(k)| dk < \infty$, then $\phi(x)$ is a continuous and decaying function on $x \in \mathbb{R}$.

Nonlinear problem in the Bloch space

With the Bloch-Fourier transform in 2D, the elliptic problem is reduced to the form

$$\left[\rho_{n_1}(k_1) + \rho_{n_2}(k_2) - \omega_0 - \epsilon\Omega\right] \hat{\Phi}_n(k) =$$
$$-\epsilon\sigma \sum_{(m,i,j)\in\mathbb{N}^6} \int_{\mathbb{T}^6} M_{n,m,i,j}(k,l,\kappa,\lambda)\hat{\Phi}_m(l)\bar{\hat{\Phi}}_i(\kappa)\hat{\Phi}_j(\lambda) dl d\kappa d\lambda,$$

where

$$M_{n,m,i,j}(k,l,\kappa,\lambda) = \langle u_n(\cdot;k)u_i(\cdot;\kappa), u_m(\cdot;l)u_j(\cdot;\lambda)\rangle_{\mathbb{R}^2}.$$

Lemma: The nonlinear vector field (in 1D) is closed in space $l^1_s(\mathbb{N}, L^1(\mathbb{T}))$ for $s < 1$, such that

$$\|\hat{\phi} \star \hat{\varphi}\|_{l^1_s(\mathbb{N},L^1(\mathbb{T}))} \leq C\|\hat{\phi}\|_{l^1_s(\mathbb{N},L^1(\mathbb{T}))}\|\hat{\varphi}\|_{l^1_s(\mathbb{N},L^1(\mathbb{T}))},$$

for some $C > 0$. The same is true in 2D for separable potentials.

Decomposition in the Bloch space

Resonant Bloch modes correspond to k and n in the sets

$$k \in \left\{ \left(0, \frac{1}{2}\right); \left(\frac{1}{2}, 0\right); \left(\frac{1}{2}, \frac{1}{2}\right) \right\} \subset \mathbb{T}^2$$

and

$$n \in \{(1,3); (3,1); (2,2)\} \in \mathbb{N}^2$$

The decomposition is

$$\hat{\Phi}(k) = \hat{U}_1(k)\chi_{D_1}(k)e_{1,3} + \hat{U}_2(k)\chi_{D_2}(k)e_{3,1} + \hat{U}_3(k)\chi_{D_3}(k)e_{2,2} + \hat{\Psi}(k),$$

where $\{e_{1,3}, e_{3,1}, e_{2,2}\}$ are unit vectors on $\mathbb{N}^2$, $D_{1,2,3}$ are disks of radius ϵ^r centered at the points k of the resonant set, $\chi_D(k)$ is a characteristic function on $k \in \mathbb{T}^2$, and $\hat{\Psi}(k)$ is zero identically on $k \in D_{1,2,3}$ for the corresponding values of n.

Projection to the coupled-mode system

The diagonal multiplication operator can be inverted since

$$\min_{k \in \mathrm{supp}(\hat{\Psi})} |\rho_{n_1}(k_1)|_{\eta=\eta_0} + \rho_{n_2}(k_2)|_{\eta=\eta_0} - \omega_0| \geq C\epsilon^{2r}.$$

If $2r < 1$, the lower bound is still larger than the perturbation terms of order ϵ. By the Implicit Function Theorem in the space $l^1_s(\mathbb{N}^2, L^1(\mathbb{T}^2))$ for any $\frac{1}{2} < s < 1$, there exists a unique map $\hat{\Psi}_\epsilon(\hat{U}_1, \hat{U}_2, \hat{U}_3) : L^1(D_1) \times L^1(D_2) \times L^1(D_3) \mapsto l^1_s(\mathbb{N}^2, L^1(\mathbb{T}^2))$ for sufficiently small ϵ, such that

$$\|\hat{\Psi}_\epsilon\|_{l^1_s(\mathbb{N}^2, L^1(\mathbb{T}^2))} \leq \epsilon^{1-2r} C \left(\|\hat{U}_1\|_{L^1(D_1)} + \|\hat{U}_2\|_{L^1(D_2)} + \|\hat{U}_3\|_{L^1(D_3)} \right),$$

for some constant $C > 0$ uniformly in $|\epsilon| < \epsilon_0$.

Extended coupled-mode system

Using the scaling transformation

$$\hat{B}_j(p) = \epsilon \hat{U}_j \left(\frac{k - k_0}{\epsilon^{1/2}} \right), \quad \forall k \in D_j \subset \mathbb{T}^2, \quad j = 1, 2, 3,$$

we map all disks $D_{1,2,3}$ to the disk $D_\epsilon = \{p \in \mathbb{R}^2 : \ |p| < \epsilon^{r-\frac{1}{2}}\}$, which covers the entire plane $p \in \mathbb{R}^2$ as $\epsilon \to 0$ if $2r < 1$. Note that $\|\hat{U}_j\|_{L^1(D_j)} = \|\hat{B}_j\|_{L^1(D_\epsilon)}$ for any $j = 1, 2, 3$. The remainder terms are due to three sources:

- The component $\hat{\Psi} = \hat{\Psi}_\epsilon(\hat{U}_1, \hat{U}_2, \hat{U}_3)$ is eliminated and it has the order of ϵ^{1-2r}.
- The perturbation terms in powers of ϵ occur at the order of ϵ^1.
- The expansion of all coefficients in powers of $k - k_0$ has the order of ϵ^{4r-1}.

End of the proof

The last property is due to the bound

$$\epsilon \| |p|^4 \hat{B}_j \|_{L^1(D_\epsilon)} = \epsilon \int_{D_\epsilon} |p|^4 \left| \hat{B}_j(p) \right| dp \leq \epsilon^{4r-1} \| \hat{B}_j \|_{L^1(D_\epsilon)}.$$

The theorem is proved if $\frac{1}{4} < r < \frac{1}{2}$ with $\tilde{r} = \min(4r - 1, 1 - 2r)$.

Remark: If $r = \frac{1}{3}$, then $\tilde{r} = r = \frac{1}{3}$ and both remainder terms have the same order of $\epsilon^{1/3}$ which gives the smallest convergence rate for the approximation error.

Remark: The proof does not work if the potential is not separable (the range $\frac{1}{2} < s < 1$ may become empty), if the function $W(x)$ is not piecewise-continuous (analyticity of expansions in powers of k may be lost), or if the new band gap is not smallest (eigenvalues can be multiple and analyticity of expansions in ϵ may be lost).

Reversible solutions

Definition: A solution (A_1, A_2, A_3) of the coupled-mode system is called a reversible solution if it satisfies one of the constraints

$$A(y_1, y_2) = s_1 A(-y_1, y_2) = s_2 A(y_1, -y_2), \qquad \text{or}$$
$$A(y_1, y_2) = s_1 \bar{A}(-y_1, y_2) = s_2 \bar{A}(y_1, -y_2), \qquad \text{or}$$
$$A(y_1, y_2) = s_1 A(y_2, y_1) = s_2 A(-y_2, -y_1), \qquad \text{or}$$
$$A(y_1, y_2) = s_1 \bar{A}(y_2, y_1) = s_2 \bar{A}(-y_2, -y_1),$$

for each function (A_1, A_2, A_3), where $s_1, s_2 = \pm 1$.

Remark: The reversible constraints are inherited from the nonlinear elliptic problem with the symmetric potential function $V(x_1, x_2) = V(-x_1, x_2) = V(x_1, -x_2) = V(x_2, x_1)$ on $x \in \mathbb{R}^2$.

Main Theorem II

Theorem: Let (A_1, A_2, A_3) be a reversible solution of the differential coupled-mode system $\mathbf{F}(\mathbf{A}) = \mathbf{0}$ such that their Fourier transforms satisfy $\hat{\mathbf{A}} \in L^1_q(\mathbb{R}^2, \mathbb{C}^3)$ for some $q \geq 0$. Let Ω belong to the interior of the band gap of the coupled-mode system $\mathbf{F}(\mathbf{A}) = \mathbf{0}$. Assume that the Jacobian operator $D_{\mathbf{A}}\mathbf{F}(\mathbf{A})$ has a three-dimensional kernel with the eigenvectors $\{\partial_{y_1}\mathbf{A}, \partial_{y_2}\mathbf{A}, i\mathbf{A}\}$. Then, there exists a reversible solution of the extended coupled-mode system such that $(\hat{B}_1, \hat{B}_2, \hat{B}_3) \in L^1(D_\epsilon, \mathbb{C}^3)$ and

$$\forall |\epsilon| < \epsilon_0 : \quad \|\hat{B}_j - \hat{A}_j\|_{L^1(D_\epsilon)} \leq C_j \epsilon^{\tilde{r}}, \;\; \forall j = 1, 2, 3.$$

Corollary: The reversible solution $U(x)$ satisfies the bound

$$\|U - \epsilon^{1/2}\left(A_1\phi_1 - A_2\phi_2 - A_3\phi_3\right)\|_{C^0_b(\mathbb{R}^2)} \leq C\epsilon^{\tilde{r}+1/2},$$

where $\phi_{1,2,3}(x)$ are resonant Bloch modes.

Proof of Theorem 2

First, consider the extended system $\hat{\mathbf{F}}(\hat{\mathbf{B}}) = \epsilon^{\tilde{r}} \hat{\mathbf{R}}(\hat{\mathbf{B}})$ on $p \in \mathbb{R}^2$ and use $\hat{\mathbf{B}} = \hat{\mathbf{A}} + \hat{\mathbf{b}}$ to represent the system in the form $\hat{J}\hat{\mathbf{b}} = \hat{\mathbf{N}}(\hat{\mathbf{b}})$, where

$$\hat{J} = D_{\hat{\mathbf{A}}} \hat{\mathbf{F}}(\hat{\mathbf{A}}), \quad \hat{\mathbf{N}}(\hat{\mathbf{b}}) = \epsilon^{\tilde{r}} \hat{\mathbf{R}}(\hat{\mathbf{A}} + \hat{\mathbf{b}}) - \left[\hat{\mathbf{F}}(\hat{\mathbf{A}} + \hat{\mathbf{b}}) - \hat{J}\hat{\mathbf{b}} \right].$$

The desired bound follows by the Implicit Function Theorem in space $\hat{\mathbf{b}} \in L^1_q(\mathbb{R}^2, \mathbb{C}^3)$. Then, estimate the truncated terms on $p \in \mathbb{R}^2 \backslash D_\epsilon$. The largest truncated terms are bounded by

$$\|\hat{\mathbf{b}}\|_{L^1_{q+2}(D_\epsilon^\perp, \mathbb{C}^3)} \leq \|\hat{\mathbf{b}}\|_{L^1_{q+2}(\mathbb{R}^2, \mathbb{C}^3)} \leq C \|\hat{\mathbf{N}}(\hat{\mathbf{b}})\|_{L^1_q(\mathbb{R}^2, \mathbb{C}^3)} \leq \tilde{C} \epsilon^{\tilde{r}},$$

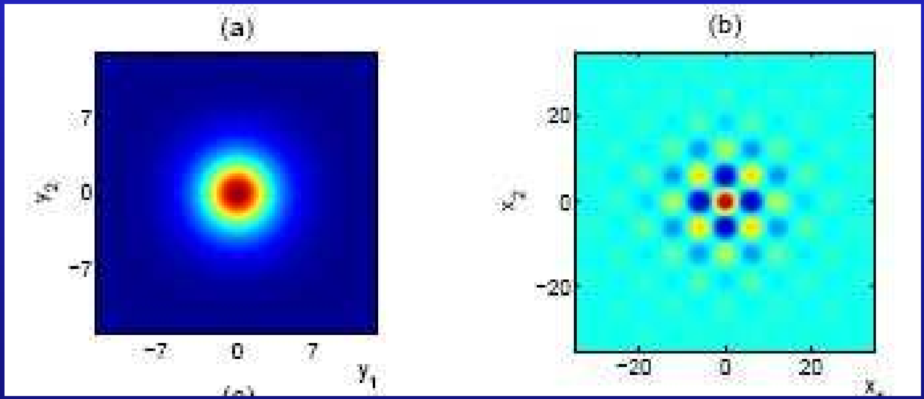
for any $\hat{\mathbf{b}} = \hat{J}^{-1} \hat{\mathbf{N}}(\hat{\mathbf{b}}) \in L^1_q(\mathbb{R}^2, \mathbb{C}^3)$. Therefore, the truncated terms are comparable with the residual terms of the extended coupled-mode system.

Numerical example 1

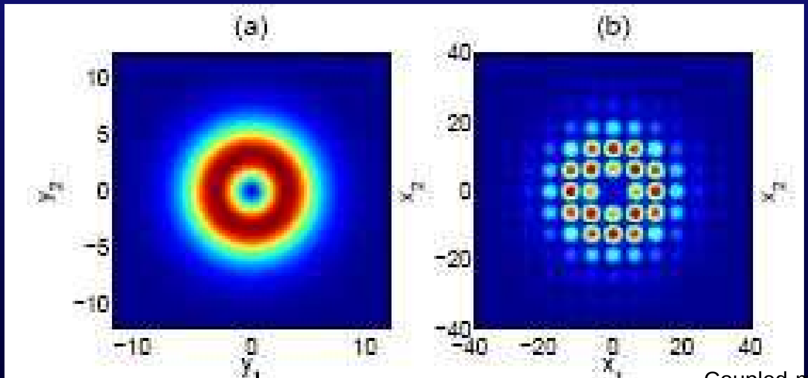
One-component gap solitons:

$$\sigma = 1: \quad A_1 = A_2 = 0, \; A_3 = R(r)e^{im\theta}$$

with $m = 0$ (radially symmetric positive soliton):



and $m = 1$ (vortex of charge one):

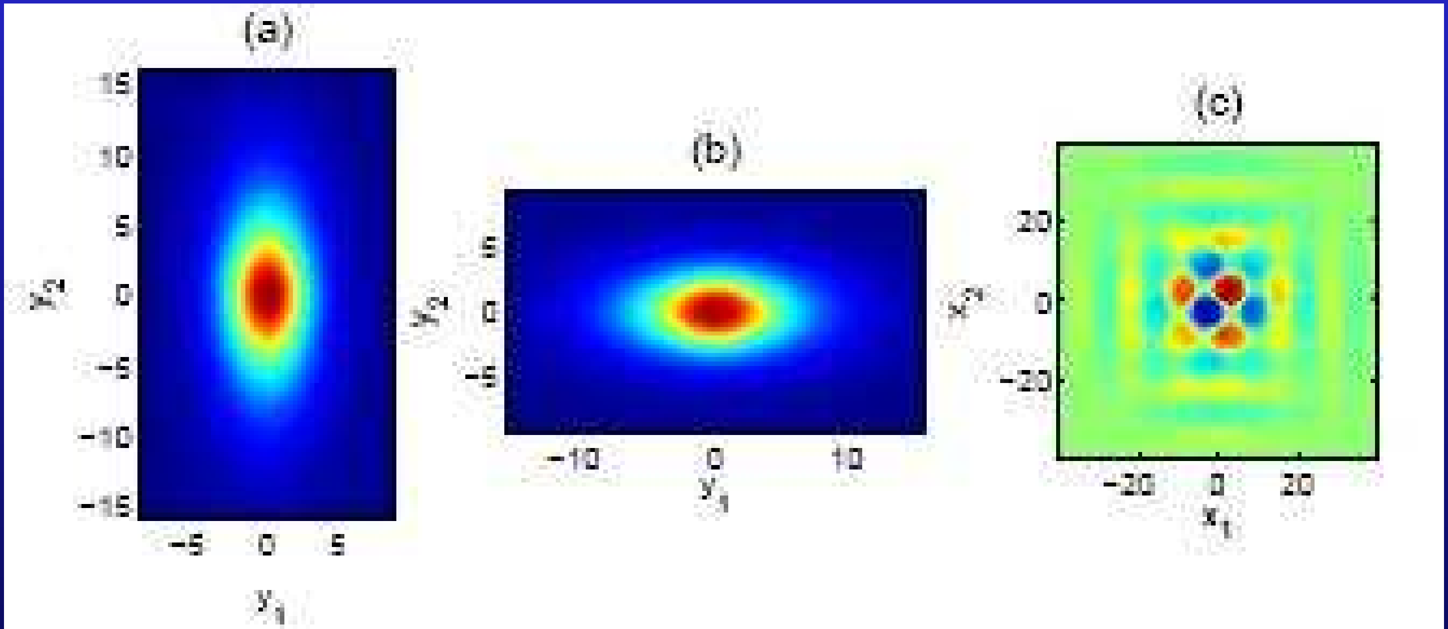


Numerical example 2

A symmetric coupled two-component gap soliton

$$\sigma = -1 : \quad A_1(y_1, y_2) = \pm A_2(y_2, y_1) \in \mathbb{R}, \; A_3 = 0$$

is shown here:

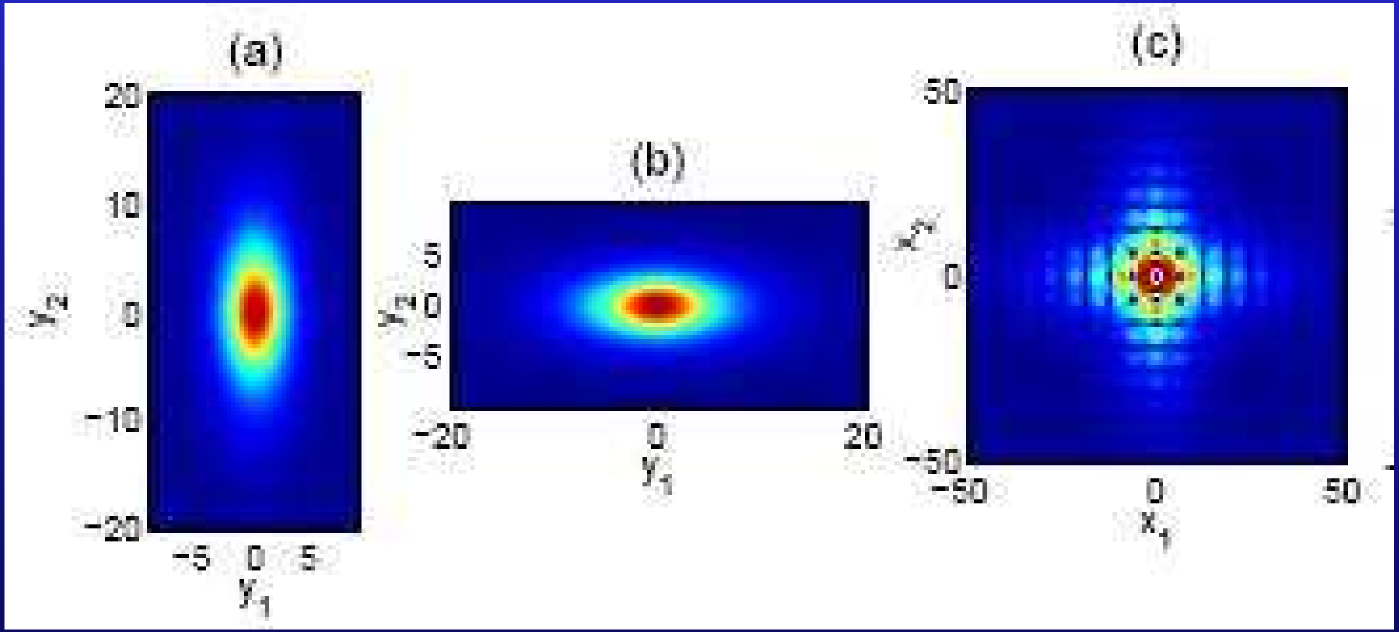


Numerical example 3

A $\pi/2$-phase delay coupled two-component gap solitons:

$$\sigma = -1 : \quad A_1(y_1, y_2) = \pm i A_2(y_2, y_1) \in i\mathbb{R},\ A_3 = 0$$

is shown here:

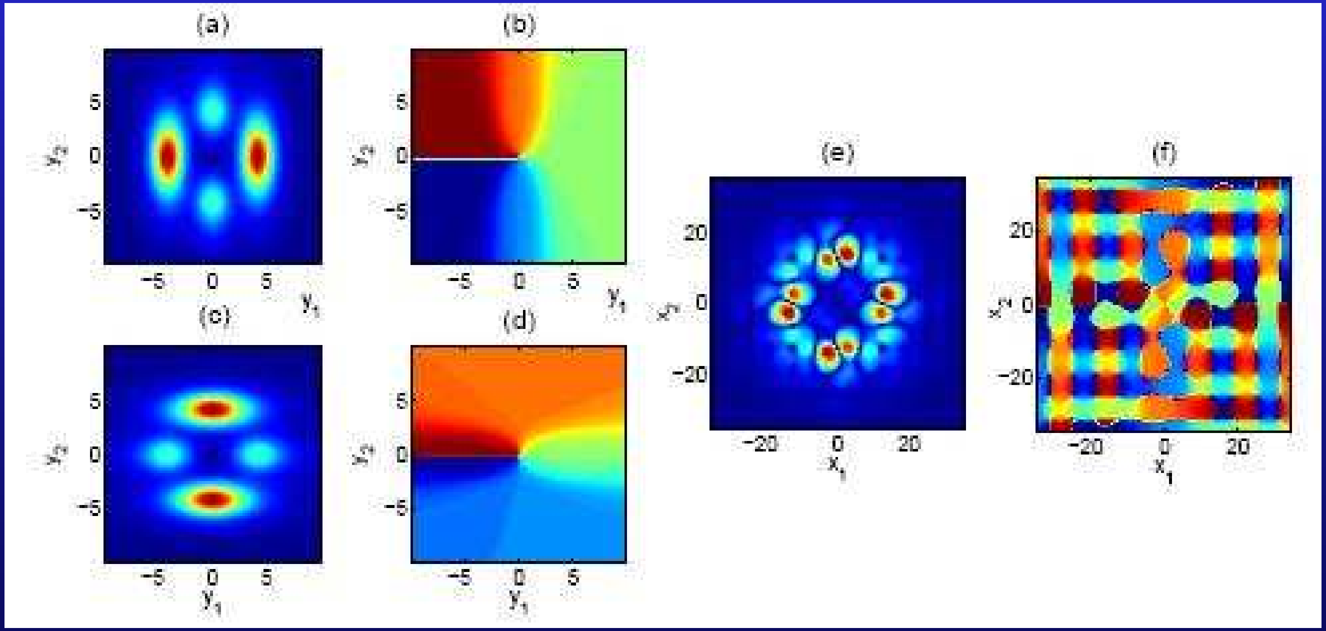


Numerical example 4

Two-component coupled vortex of charge one

$$\sigma = -1 : \quad A_1(y_1, y_2) = \pm i \bar{A}_2(y_2, y_1) \in \mathbb{C}, \; A_3 = 0$$

is shown here:



Similar bifurcation problems

Our technique can be extended with some modifications to the following bifurcation problems:

- Bifurcations from band edges
- Bifurcations in the higher-order band gaps
- Bifurcations in anisotropic separable potentials
- Bifurcations in finite-gap potentials
- Bifurcations in super-lattices with 4π-periodic perturbations
- Bifurcations in three-dimensional separable potentials.

Additionally, we can apply this technique to prove persistence of time-dependent solutions on a finite-time interval and to study convergence of the nonlinear elliptic problem with large potential functions to the nonlinear lattice equation.