Enstrophy Growth in the Viscous Burgers Equation

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- Burgers equation ($\mathbb{T} = [0,1]$, $u \in \mathbb{R}$)

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \qquad x \in \mathbb{T}, \quad t \in \mathbb{R}_+$$

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- Local solutions exist for all $u|_{t=0} \in H^s_{\rm per}(\mathbb{T})$ with $s > -\frac{1}{2}$ (Dix, 1996). Global existence holds for all $u|_{t=0} \in H^1_{\rm per}(\mathbb{T})$.

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- Hopf–Cole transformation

$$u(x,t) = -\frac{\partial}{\partial x} \log \psi(x,t) \quad \Rightarrow \quad \frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2},$$

provided $\psi(x,t) > 0$ for all (x,t).

- Enstrophy $E(u) = \frac{1}{2}\int_{\mathbb{T}} u_x^2 \, dx$ satisfies

$$\frac{dE(u)}{dt} = R(u) := -\int_{\mathbb{T}} (u_{xx}^2 + u_x^3) \, dx,$$

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- Using Young's inequality and the elementary bound

$$\|u_x\|_{L^\infty} \leq \|u_x\|_{L^2}^{1/2} \|u_{xx}\|_{L^2}^{1/2},$$

one can estimate

$$|R(u)| \leq -\|u_{xx}\|_{L^2}^2 + \|u_x\|_{L^2}^{5/2} \|u_{xx}\|_{L^2}^{1/2} \leq \frac{3}{4^{4/3}} \|u_x\|_{L^2}^{10/3} \equiv \frac{3}{2} E^{5/3}(u).$$

Lu and Doering (2008) considered the maximization problem

$$\max_{u \in H^2_{\rm per}(\mathbb{T})} R(u) \quad \text{subject to} \quad E(u) = \mathcal{E},$$

where $\mathcal{E} > 0$ is given.

Solutions were found analytically in terms of Jacobi's elliptic functions, and it was shown that

$$R(u) = \mathcal{O}(\mathcal{E}^{5/3}) \quad \text{as} \quad \mathcal{E} \to \infty.$$

This instantaneous bound is not related to the time evolution of the Burgers equation.

- Using energy balance

$$K(u) = \frac{1}{2}\int_{\mathbb{T}} u^2 dx \quad \Rightarrow \quad \frac{dK(u)}{dt} = -2E(u),$$

one can estimate

$$E^{1/3}(u(T)) - E^{1/3}(u_0) \leq \frac{1}{2}\int_0^T E(u(t))dt = \frac{1}{4}(K(u_0) - K(u(T)))$$

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- Using Poincaré's inequality for mean-zero periodic functions,

$$K(u_0) \leq \frac{1}{4\pi^2} E(u_0),$$

we obtain

$$E(u(T)) \leq \left(E^{1/3}(u_0) + \frac{1}{16\pi^2} E(u_0) \right)^3.$$

Bounds on the enstrophy growth

Ayala & Protas (2011) considered the finite-time maximization:

$$\max_{u_0 \in H^1_{\rm per}(\mathbb{T})} E(u(T)) \quad \text{subject to} \quad E(u_0) = \mathcal{E},$$

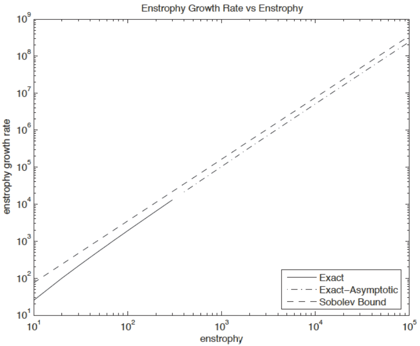
and showed that

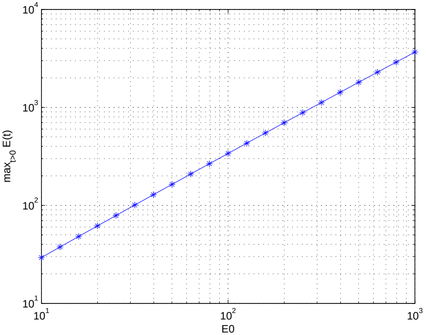
$$E(u(T_*)) = \mathcal{O}(\mathcal{E}^{1.5}), \quad T_* = \mathcal{O}(\mathcal{E}^{-0.5}), \quad \text{as} \quad \mathcal{E} \to \infty,$$

where T_* is the value of T for which $\max_{u_0 \in H^1_{\rm per}(\mathbb{T})} E(u(T))$ is maximal over $T \in \mathbb{R}_+$.

In addition, they showed that $K(u(T_*)) = \mathcal{O}(\mathcal{E}^{1.0})$.

Numerical results: Lu & Doering (2008)





$$\max_{t\in[0,T]} R(u(t)) = 0.2476\mathcal{E}^{5/3}$$

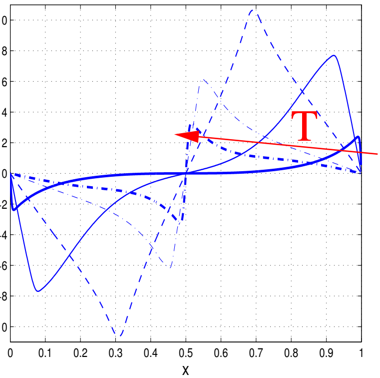
$$\max_{t\in[0,T]} E(u(t)) \leq C\mathcal{E}^{1.048}$$

Instantaneous growth

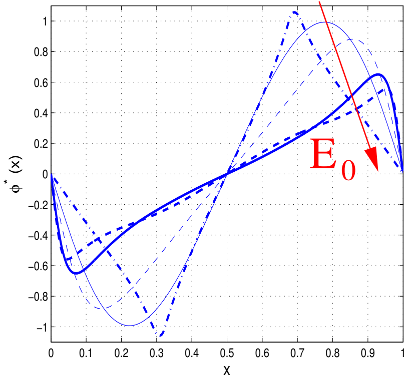
Finite–time growth

Numerical results: Ayala & Protas (2011)

Maximizers of the finite-time optimization problem



Fixed $\mathcal{E} = 10^3$, different T



Fixed $T = 0.0316$, different $\mathcal{E}$

Numerical results indicate that for all $u_0 \in H^1_{\rm per}(\mathbb{T})$, there is $C > 0$:

$$\sup_{t \in \mathbb{R}_+} E(u(t)) \leq C \mathcal{E}^{3/2}, \quad \mathcal{E} = E(u_0).$$

and this bound is sharp as $\mathcal{E} \to \infty$.

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The integral bound

$$E^{1/3}(u(T_*)) \leq \mathcal{E}^{1/3} + \frac{1}{4}(K(u_0) - K(u(T_*)))$$

is sharp if

$$K(u_0) - K(u(T_*)) = \mathcal{O}(\mathcal{E}^{1/2}), \quad {\rm as} \quad \mathcal{E} \to \infty,$$

but numerical results have low accuracy to justify this estimate.

Theorem 1

Consider the instantaneous maximization problem,

$$\max_{u \in H^2_{\rm per}(\mathbb{T})} R(u) \quad \text{subject to} \quad E(u) = \mathcal{E}.$$

There exists a unique odd function $u_* \in H^2_{\rm per}(\mathbb{T})$ *with* $u_*'(0) < 0$ *that solves the maximization problem and satisfies*

$$u_*(x) = 4k(2x - \tanh(kx)) + \mathcal{O}_{L^\infty}(k^2 e^{-k}), \quad \text{as} \quad k \to \infty,$$

where k *determines the leading order expansions,*

$$K(u_*) = \frac{8}{3}k^2 + \mathcal{O}(k), E(u_*) = \frac{32}{3}k^3 + \mathcal{O}(k^2), R(u_*) = \frac{256}{5}k^5 + \mathcal{O}(k^4).$$

Corollary

When k is expressed in terms of $\mathcal{E}$, we obtain

$$\left.\begin{array}{l} K(u_*) = \frac{1}{6^{1/3}}\mathcal{E}^{2/3} + \mathcal{O}(\mathcal{E}^{1/3}), \\ R(u_*) = \frac{3^{5/3}}{5\cdot 2^{1/3}}\mathcal{E}^{5/3} + \mathcal{O}(\mathcal{E}^{4/3}), \end{array}\right\} \quad \text{as} \quad \mathcal{E} \to \infty.$$

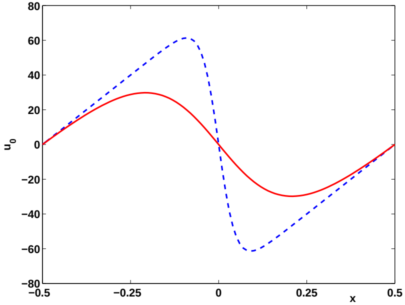
- Poincaré's inequality is not saturated by u_*.
- Instantaneous bound $R(u) \leq C\mathcal{E}^{5/3}$ is sharp up to a choice of the numerical constant

$$C = \frac{3^{5/3}}{5 \cdot 2^{1/3}} < \frac{3}{2}.$$

Initial conditions for the Burgers equation

$$u_0(x) = 4k(2x - f(x)), \quad f(x) = \frac{\tanh(lx)}{\tanh(l/2)}, \quad x \in \mathbb{T},$$

where $k > 0$ is a free parameter and either $l = k$ (maximizer) or $l = \mathcal{O}(1)$ as $k \to \infty$.

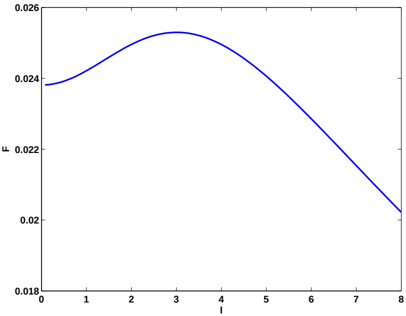


Initial data for $k = 20$ and either $l = 20$ (dashed) or $l = 5$ (solid).

Poincare's inequality

For the initial condition, we compute

$$K(u_0) = k^2 \tilde{K}(l), \quad E(u_0) = k^2 \tilde{E}(l).$$



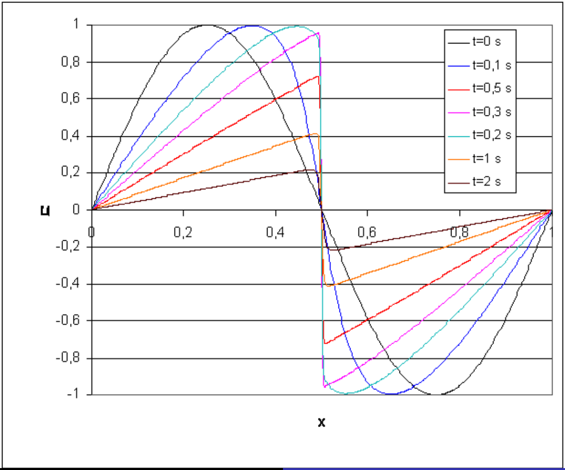
The maximum of $\tilde{K}(l)/\tilde{E}(l)$ occurs for $l = l_0 \approx 3.0$, where

$$F(l_0) \approx 0.025297 < \frac{1}{4\pi^2},$$

which is 99.9% close to the Poincaré constant.

Numerical simulations of the initial-value problem

$$\begin{cases} u_t + 2uu_x = u_{xx} & x \in \mathbb{T}, \quad t \in \mathbb{R}_+, \\ u|_{t=0} = k\sin(2\pi x), & x \in \mathbb{T}. \end{cases}$$



Self-similar transformation for Burgers equation

Let us consider the initial-value problem for the Burgers equation:

$$\begin{cases} u_t + 2uu_x = u_{xx} & x \in \mathbb{T}, \quad t \in \mathbb{R}_+, \\ u|_{t=0} = 4k(2x - f(x)), & x \in \mathbb{T}. \end{cases}$$

The unique solution $u \in C(\mathbb{R}_+, H^1_{\rm per}(\mathbb{T}))$ is given by

$$u(x,t) = p(t)\left(2x - w(\xi(x,t),\tau(t))\right), \quad x \in \mathbb{T}, \quad t \in \mathbb{R}_+,$$

where

$$p(t) = \frac{4k}{1+16kt}, \quad \xi(x,t) = \frac{4kx}{1+16kt}, \quad \tau(t) = \frac{16k^2 t}{1+16kt}.$$

Self-similar transformation for Burgers equation

The function $w(\xi, \tau)$ satisfies the rescaled Burgers equation,

$$\begin{cases} w_\tau = 2ww_\xi + w_{\xi\xi}, & |\xi| < 2(k-\tau), \quad \tau \in (0,k), \\ w|_{\tau=0} = f(\xi/4k), & |\xi| \leq 2k, \end{cases}$$

subject to the boundary conditions

$$w(\xi,\tau) = \pm 1, \quad \xi = \pm 2(k-\tau), \quad \tau \in [0,k).$$

The stationary viscous kink on the infinite line is

$$w_\infty(\xi) = \tanh(\xi), \quad \xi \in \mathbb{R}.$$

Metastable state for Burgers equation

We shall prove that u is close to u_∞, where

$$u_\infty(x,t) = p(t)\left(2x - \tanh(p(t)x)\right), \quad p(t) = \frac{4k}{1+16kt} = \mathcal{O}(k),$$

in the inertial range $C_- < kt < C_+$ for some $0 < C_- < C_+ < \infty$ as $k \to \infty$.

Now we have $k = \mathcal{O}(\mathcal{E}^{1/2})$ as $\mathcal{E} \to \infty$ and

$$K(u_\infty) = \mathcal{O}(p^2) = \mathcal{O}(\mathcal{E}), \quad E(u_\infty) = \mathcal{O}(p^3) = \mathcal{O}(\mathcal{E}^{3/2}),$$

and the maximum of $E(u)$ occurs in the inertial range, where $t = \mathcal{O}(k^{-1}) = \mathcal{O}(\mathcal{E}^{-1/2})$.

Theorem 2

Consider the initial-value problem for the Burgers equation:

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \qquad x \in \mathbb{T}, \quad t \in \mathbb{R}_+.$$

There exists $T_* > 0$ such that the enstrophy $E(u)$ achieves its maximum at $u_* = u(\cdot, T_*)$. If $l = \mathcal{O}(k)$ as $k \to \infty$, then

$$T_* = \mathcal{O}(\mathcal{E}^{-2/3}\log(\mathcal{E})), \quad E(u_*) = \mathcal{O}(\mathcal{E}), \quad K(u_*) = \mathcal{O}(\mathcal{E}^{2/3}),$$

whereas if $l = \mathcal{O}(\log(k))$, then $T_* = \mathcal{O}(\mathcal{E}^{-1/2}\log^{1/2}(\mathcal{E}))$,

$$E(u_*) = \mathcal{O}(\mathcal{E}^{3/2}\log^{-3/2}(\mathcal{E})), \quad K(u_*) = \mathcal{O}(\mathcal{E}\log^{-1}(\mathcal{E})).$$

Remarks

The goal is to consider the case $l = \mathcal{O}(1)$ as $k \to \infty$ and show

$$T_* = \mathcal{O}(\mathcal{E}^{-1/2}), \quad E(u_*) = \mathcal{O}(\mathcal{E}^{3/2}), \quad K(u_*) = \mathcal{O}(\mathcal{E}),$$

and

$$K(u_0) - K(u(T_*)) = \mathcal{O}(\mathcal{E}^{1/2}), \quad \text{as} \quad \mathcal{E} \to \infty.$$

This goal is not achieved yet because our technique relies on good decay of the shock solution near $x = \pm\frac{1}{2}$ and on the separation of the temporal scales for the dynamics of the viscous shock and the dynamics of the rarefactive wave.

Proof of Theorem 1

Consider the instantaneous maximization problem,

$$\max_{u \in H^2_{\mathrm{per}}(\mathbb{T})} R(u) \quad \text{subject to} \quad E(u) = \mathcal{E}.$$

Set $v = u_x$ and look for critical points $v \in H^1_{\mathrm{per}}(\mathbb{T})$ of the functional (Lu & Doering, 2008),

$$J(v) = \int_{\mathbb{T}} \left(v_x^2 + v^3 + \lambda v^2 + \mu v \right) dx,$$

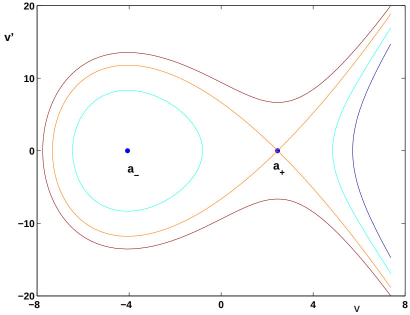
subject to

$$\frac{1}{2} \int_{\mathbb{T}} v^2(x) dx = \mathcal{E}, \quad \int_{\mathbb{T}} v(x) dx = 0.$$

Euler–Lagrange equations give the stationary KdV equation,

$$\frac{d^2 v}{dx^2} = \frac{3}{2} v^2 + \lambda v - 3\mathcal{E},$$

where $\lambda \to \infty$ as $\mathcal{E} \to \infty$ and $(a_\pm, 0)$ are equilibrium states with $a_- < 0 < a_+$. We are looking for a 1-periodic solution $v(x)$.



We can write

$$v(x) = a_+ - 4k^2 y(\xi), \quad \xi = kx,$$

where y is k-periodic and $k = \frac{1}{2}\sqrt[4]{\lambda^2 + 18\mathcal{E}} \to \infty$.

The rescaled differential equation is

$$\frac{d^2 y}{d\xi^2} - 4y + 6y^2 = 0$$

and we are looking for a k-periodic solution $y(\xi)$.

Lemma

$$\sup_{\xi \in [-k/2,k/2]} \left| y(\xi) - \mathrm{sech}^2(\xi) \right| \leq C e^{-k} \quad \text{as} \quad k \gg 1.$$

Hence, we obtain $a_+ = 8k(1 + \mathcal{O}(ke^{-k}))$ and then

$$k = \left(\frac{3}{32} \mathcal{E} \right)^{1/3} + 1 + \mathcal{O}(\mathcal{E}^{-1/3}), \quad \lambda = \left(\frac{3}{4} \mathcal{E} \right)^{2/3} + \mathcal{O}(\mathcal{E}^{1/3}).$$

The Burgers equation,

$$\begin{cases} u_t + 2uu_x = u_{xx} & x \in \mathbb{T}, \quad t \in \mathbb{R}_+, \\ u|_{t=0} = 4k(2x - f(x)), & x \in \mathbb{T}, \end{cases}$$

is transformed to the rescaled form

$$\begin{cases} w_\tau = 2ww_\xi + w_{\xi\xi}, & |\xi| < 2(k-\tau), \quad \tau \in (0,k), \\ w|_{\tau=0} = f(\xi/4k), & |\xi| \leq 2k, \end{cases}$$

after the self-similar transformation:

$$u(x,t) = p(t)\left(2x - w(\xi(x,t),\tau(t))\right), \quad x \in \mathbb{T}, \quad t \in \mathbb{R}_+,$$

where

$$p(t) = \frac{4k}{1+16kt}, \quad \xi(x,t) = \frac{4kx}{1+16kt}, \quad \tau(t) = \frac{16k^2 t}{1+16kt}.$$

The initial condition is now

$$f(x) = \frac{\tanh(lx)}{\tanh(l/2)} \quad \Rightarrow \quad w_0(\xi) = \frac{\tanh(\xi/a)}{\tanh(l/2)}, \quad a = \frac{4k}{l}.$$

- When $l = k$ (maximizer), $a = 4$.
- When $l = \mathcal{O}(\log(k))$, $a = \mathcal{O}(k/\log(k))$.
- When $l = \mathcal{O}(1)$, $a = \mathcal{O}(k)$.

The boundary conditions are

$$w(\xi,\tau) = \pm 1 \quad \text{for} \quad \xi = \pm 2(k-\tau), \quad \tau \in [0,k).$$

Steps to prove Theorem 1

1. Consider the Burgers equation on the infinite line,

$$\begin{cases} w_\tau = 2ww_\xi + w_{\xi\xi}, & \xi \in \mathbb{R}, \quad \tau \in \mathbb{R}_+, \\ w|_{\tau=0} = \tanh(\xi/a), & \xi \in \mathbb{R}, \\ w|_{\xi\to\pm\infty} = \pm 1, & \tau \in \mathbb{R}_+, \end{cases}$$

   and prove convergence of $w(\xi,\tau)$ to $w_\infty(\xi) = \tanh(\xi)$ in the H^1-norm as $\tau \to \infty$.

2. Control the approximation error for the Burgers equation in a bounded domain for large k from the smallness of $w(\xi,\tau) - w_\infty(\xi)$ for large values of ξ and all $\tau \geq 0$.

Approximate solution for $l = k$ $(a = 4)$

Approximate solution solves the Burgers equation on the line:

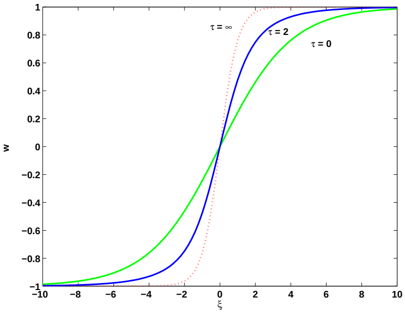
$$\begin{cases} w_\tau = 2ww_\xi + w_{\xi\xi}, & \xi \in \mathbb{R}, \quad \tau \in \mathbb{R}_+, \\ w|_{\tau=0} = \tanh(\xi/4), & \xi \in \mathbb{R}, \end{cases}$$

An exact solution is available via the Hopf–Cole transformation

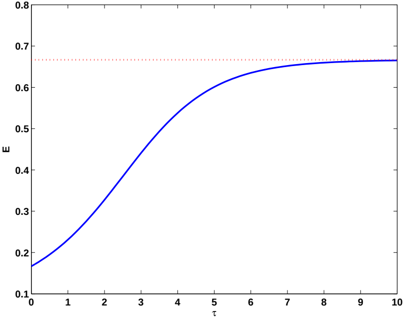
$$w(\xi,\tau) = \tanh(\xi) + \tilde{w}(\xi,\tau),$$

where

$$\tilde{w} = e^{-3\tau/4}\mathrm{sech}(\xi)\frac{2\sinh(\xi/2) - 4\cosh(\xi/2)\tanh(\xi) - 3\tanh(\xi)e^{-\tau/4}}{1 + 4\cosh(\xi/2)\mathrm{sech}(\xi)e^{-3\tau/4} + 3\mathrm{sech}(\xi)e^{-\tau}}.$$



Dynamics of the viscous shock.



Enstrophy E versus time τ.

Lemma

For any integer $m \geq 0$, there is a $C_m > 0$ such that

$$\sup_{\xi \in \mathbb{R}} \left| e^{|\xi|/2} \partial_\xi^m \left(w(\xi,\tau) - \tanh(\xi) \right) \right| \leq C_m e^{-3\tau/4}, \quad \tau \in \mathbb{R}_+.$$

Fix $\delta > 0$. There exist $K > 0$ and $C > 0$ such that for all $k \geq K$, we have

$$\sup_{x \in \mathbb{R}} |u(x,t) - u_\infty(x)| \leq \frac{C}{k^\delta}, \quad \text{for all} \ \ t \geq T_* := \frac{(1+\delta)\log(k)}{12k^2}.$$

If $\mathcal{E} = E(u_0) = \mathcal{O}(k^3)$, then $k = \mathcal{O}(\mathcal{E}^{1/3})$ and

$$E(u_\infty) = \mathcal{O}(\mathcal{E}) \quad \text{and} \quad T_* = \mathcal{O}(\mathcal{E}^{-2/3}\log(\mathcal{E})) \quad \text{as} \quad \mathcal{E} \to \infty.$$

Error of the approximation for $l = k$ $(a = 4)$

The approximation error $\|w - w_{\rm app}\|_{H^1}$ is controlled by a priori energy estimates for the heat equation (via the Hopf–Cole transformation).

In new variables, the Hopf–Cole transformation

$$w(\xi,\tau) = \frac{\partial}{\partial x} \log \psi(\xi,\tau) \quad \Rightarrow$$

gives the rescaled heat equation,

$$\left\{ \begin{array}{lll} \psi_\tau = \psi_{\xi\xi}, & |\xi| < 2(k-\tau), & \tau \in (0,k), \\ \psi|_{\tau=0} = \psi_0(\xi), & |\xi| \leq 2k, & \\ \psi_\xi = \pm\psi, & \xi = \pm 2(k-\tau), & \tau \in (0,k). \end{array} \right.$$

Using the decomposition $\psi = \psi_{\rm app}(1+\Psi)$, we obtain

$$\left\{ \begin{array}{lll} \Psi_\tau = \Psi_{\xi\xi} + 2w_{\rm app}\Psi_\xi, & |\xi| < 2(k-\tau), & \tau \in (0,k), \\ \Psi|_{\tau=0} = \Psi_0(\xi), & |\xi| \leq 2k, & \\ \Psi_\xi = \pm\chi(\tau)(1+\Psi), & \xi = \pm 2(k-\tau), & \tau \in (0,k), \end{array} \right.$$

where $\chi(\tau) = 1 - w_{\rm app}(2(k-\tau),\tau)$ is small in C^2 norm and Ψ_0 is small in H^2-norm.

Lemma

Fix $C_0 \in (0,1)$. For sufficiently large k, there is a small C_k such that

$$\|w - w_{\rm app}\|^2_{H^1_{k,\tau}} \leq C_k, \quad \tau \in (0, C_0 k).$$

What goes wrong if $l = \mathcal{O}(1)$ $(a = \mathcal{O}(k))$

An approximate solution of the Burgers equation on the line starting with $w_0(\xi) = \tanh(\xi/a)$ satisfies the following bounds. For fixed $\delta > 0$ and large a, we have

$$\sup_{\xi \in \mathcal{R}} |w(\xi,\tau) - \tanh(\xi)| \leq \frac{C}{a^{3\delta}\tau^{1/2}} \text{ for all } \tau \geq \frac{1}{2}(1+\delta)^2 a \log(a)$$

and

$$|w(\xi,\tau) - \tanh(\xi)| \leq \frac{C}{a^{1+\delta}} \text{ for all } |\xi| \geq \frac{1}{2}(1+\delta)^2 a \log(a) \text{ and } \tau \geq 0.$$

If $a = \mathcal{O}(k)$, we lose control of the approximation error, because $\tau = \mathcal{O}(k\log(k)) \gg \mathcal{O}(k)$ and $\xi = \mathcal{O}(k\log(k)) \gg \mathcal{O}(k)$.