Rogue waves arising on the standing periodic waves of the Ablowitz–Ladik equation

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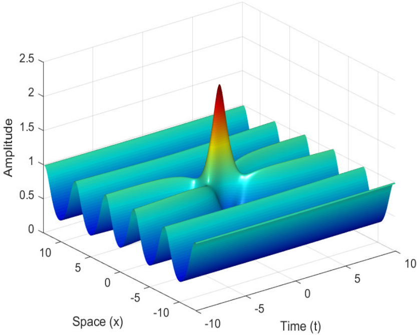
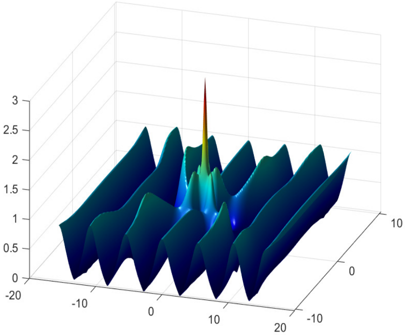
Rogue waves on the standing periodic waves

J. Chen, D. Pelinovsky, Proceedings A **474** (2018) 20170814
J. Chen, D. Pelinovsky, R. White, Physica D **405** (2020) 132378

$$(NLS) \quad i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$





Other examples of integrable Hamiltonian systems

- Modified Korteweg–de Vries equation

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

Dnoidal periodic waves are modulationally stable (no rogue waves).
Cnoidal periodic waves are modulationally unstable (rogue waves).
J. Chen & D. Pelinovsky, Nonlinearity **31** (2018) 1955–1980

- Sine–Gordon equation

$$u_{tt} - u_{xx} + \sin(u) = 0$$

Same conclusion.
D. Pelinovsky & R. White, Proceedings A **476** (2020) 20200490

- Derivative NLS equation

$$i\psi_t + \psi_{xx} + i(|\psi|^2\psi)_x = 0.$$

There exist modulationally stable periodic waves (no rogue waves).
J. Chen, D. Pelinovsky, & J. Upsal, J. Nonlinear Science **31** (2021) 58

Method of constructing the rogue waves

- Algebraic characterization of the periodic waves and the associate Lax spectrum (the so-called nonlinearization method).
- Relation between the squared periodic eigenfunctions at the end points of the Lax spectrum and the standing periodic waves.
- Analytical construction of the second (linearly growing) solutions of the Lax equations as integrals of the squared periodic eigenfunctions.
- Darboux transformation with the second (linearly growing) solutions to construct new solutions of the nonlinear PDE.

The Ablowitz–Ladik equation

The Ablowitz–Ladik (AL) equation as an integrable semi-discretization of NLS:

$$i\dot{u}_n + (1 + |u_n|^2)(u_{n+1} + u_{n-1}) = 0, \quad n \in \mathbb{Z}.$$

In the continuum limit, long standing waves of small amplitudes,

$$u_n(t) = \varepsilon \mathfrak{u}(\varepsilon n, \varepsilon^2 t) e^{2it},$$

satisfy the continuous NLS equation

$$i\mathfrak{u}_T + \mathfrak{u}_{XX} + 2|\mathfrak{u}|^2\mathfrak{u} = 0,$$

where $\mathfrak{u} = \mathfrak{u}(X, T)$ with $X := \varepsilon n$ and $T := \varepsilon^2 t$, and ε is small parameter.

Two standing periodic waves of the trivial phase exist

$$\mathfrak{u} = \mathrm{dn}(X; k) e^{i(2-k^2)T}, \quad \mathfrak{u} = k\mathrm{cn}(X; k) e^{i(2k^2-1)T}, \quad k \in (0, 1),$$

as well as standing periodic waves of the nontrivial phase.

Lax equations

The AL equation is a compatibility condition of the linear Lax system

$$\varphi_{n+1} = \frac{1}{\sqrt{1+|u_n|^2}} \begin{pmatrix} \lambda & u_n \\ -\bar{u}_n & \lambda^{-1} \end{pmatrix} \varphi_n$$

and

$$\dot{\varphi}_n = i \begin{pmatrix} \frac{1}{2}\left(\lambda^2 + \lambda^{-2} + u_n \bar{u}_{n-1} + \bar{u}_n u_{n-1}\right) & \lambda u_n - \lambda^{-1} u_{n-1} \\ -\lambda \bar{u}_{n-1} + \lambda^{-1} \bar{u}_n & -\frac{1}{2}\left(\lambda^2 + \lambda^{-2} + u_n \bar{u}_{n-1} + \bar{u}_n u_{n-1}\right) \end{pmatrix}$$

There exists a simpler Lax system representation:

$$\varphi_{n+1} = \begin{pmatrix} \lambda & u_n \\ -\bar{u}_n & \lambda^{-1} \end{pmatrix} \varphi_n$$

and

$$\dot{\varphi}_n = i \begin{pmatrix} \frac{1}{2}\left(\lambda^2 + \lambda^{-2}\right) + u_n \bar{u}_{n-1} & \lambda u_n - \lambda^{-1} u_{n-1} \\ -\lambda \bar{u}_{n-1} + \lambda^{-1} \bar{u}_n & -\frac{1}{2}\left(\lambda^2 + \lambda^{-2}\right) - \bar{u}_n u_{n-1} \end{pmatrix} \varphi_n.$$

However, the second Lax system does not characterize the stability spectrum for the standing periodic waves or the squared eigenfunction relation.

The standing periodic waves

If $u_n(t) = U_n e^{2i\omega t}$, then U_n satisfies

$$(1 + |U_n|^2)(U_{n+1} + U_{n-1}) = 2\omega U_n, \quad n \in \mathbb{Z} \tag{*}$$

associated with two conserved quantities:

$$F_0 := i(U_n \bar{U}_{n-1} - \bar{U}_n U_{n-1})$$

and

$$F_1 := \omega(U_n \bar{U}_{n-1} + \bar{U}_n U_{n-1}) - |U_n|^2 - |U_{n-1}|^2 - |U_n|^2 |U_{n-1}|^2$$

Non-trivial twist: We have previously obtained real solutions of (*) with $F_0 = 0$ in J. Chen, D. Pelinovsky, Physica D **445** (2023) 133652
by using an algebraic nonlinearization method for the discrete mKdV equation

$$\dot{u}_n = (1 + u_n^2)(u_{n+1} - u_{n-1}).$$

A similar nonlinearization method for the AL equation does not produce (*).

Dnoidal and cnoidal periodic waves

Dnoidal solutions have the form

$$U_n = \frac{\mathrm{sn}(\alpha; k)}{\mathrm{cn}(\alpha; k)} \mathrm{dn}(\alpha n; k), \qquad \omega = \frac{\mathrm{dn}(\alpha; k)}{\mathrm{cn}^2(\alpha; k)},$$

where $\alpha \in (0, K(k))$ and $k \in (0, 1)$ are arbitrary parameters.

Cnoidal solutions have the form

$$U_n = \frac{k\,\mathrm{sn}(\alpha; k)}{\mathrm{dn}(\alpha; k)} \mathrm{cn}(\alpha n; k), \qquad \omega = \frac{\mathrm{cn}(\alpha; k)}{\mathrm{dn}^2(\alpha; k)},$$

where $\alpha \in (0, 2K(k))$ and $k \in (0, 1)$ are arbitrary parameters.

There must exist periodic waves with a nontrivial phase, if $F_0 \neq 0$, but we did not consider such solutions.

Stability spectrum for the standing periodic waves

Decomposition $u_n(t) = e^{2i\omega t}[U_n + v_n(t)]$ yields the linearized AL equation:

$$i\dot{v}_n - 2\omega v_n + (1 + |U_n|^2)(v_{n+1} + v_{n-1}) + (U_{n+1} + U_{n-1})(U_n \bar{v}_n + \bar{U}_n v_n) = 0$$

Separation of variables with $v_n(t) = V_n e^{\Lambda t}$ and $\bar{v}_n(t) = \tilde{V}_n e^{\Lambda t}$ gives the spectral stability problem:

$$\begin{aligned} i\Lambda V_n - 2\omega V_n + (1 + |U_n|^2)(V_{n+1} + V_{n-1}) + (U_{n+1} + U_{n-1})(U_n \tilde{V}_n + \bar{U}_n V_n) &= 0, \\ -i\Lambda \tilde{V}_n - 2\omega \tilde{V}_n + (1 + |U_n|^2)(\tilde{V}_{n+1} + \tilde{V}_{n-1}) + (\bar{U}_{n+1} + \bar{U}_{n-1})(\bar{U}_n V_n + U_n \tilde{V}_n) &= 0. \end{aligned}$$

The stability spectrum is obtained from the Lax spectrum:
Deconinck–Segal, 2017 (NLS); Deconinck–Upsal, 2020 (AKNS).

Relation to squared eigenfunctions of Lax system

Using $\varphi_n = [p_n(t)e^{i\omega t}, q_n(t)e^{-i\omega t}]^T$ for eigenfunctions of the Lax system yields the linear system:

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \frac{1}{\sqrt{1+|U_n|^2}} \begin{pmatrix} \lambda & U_n \\ -\bar{U}_n & \lambda^{-1} \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix},$$

and

$$\frac{d}{dt}\begin{pmatrix} p_n \\ q_n \end{pmatrix} = i \begin{pmatrix} W_n - \omega & \lambda U_n - \lambda^{-1} U_{n-1} \\ -\lambda \bar{U}_{n-1} + \lambda^{-1}\bar{U}_n & \omega - W_n \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix},$$

where $W_n := \frac{1}{2}\left(\lambda^2 + \lambda^{-2} + U_n \bar{U}_{n-1} + \bar{U}_n U_{n-1}\right)$.

We have obtained the squared eigenfunction relation by brutal computations:

$$v_n = \lambda p_n^2 - \bar{\lambda}^{-1}\bar{q}_n^2 + U_n(p_n q_n + \bar{p}_n \bar{q}_n).$$

If $p_n(t) = P_n e^{\Omega t}$, $q_n(t) = Q_n e^{\Omega t}$ with spectral parameter Ω, then

$$V_n = \lambda P_n^2 + U_n P_n Q_n, \quad \tilde{V}_n = -\lambda^{-1} Q_n^2 + \bar{U}_n P_n Q_n, \quad \Lambda = 2\Omega$$

is a solution of the spectral stability problem.

Polynomial for the standing periodic waves

After the separation of variables, the time-evolution problem is

$$\Omega \left(\begin{array}{c} P_n \\ Q_n \end{array} \right) = i \left(\begin{array}{cc} W_n - \omega & \lambda U_n - \lambda^{-1} U_{n-1} \\ -\lambda \bar{U}_{n-1} + \lambda^{-1} \bar{U}_n & \omega - W_n \end{array} \right) \left(\begin{array}{c} P_n \\ Q_n \end{array} \right),$$

Nonzero solutions exists if and only if

$$\left| \begin{array}{cc} W_n - \omega + i\Omega & \lambda U_n - \lambda^{-1} U_{n-1} \\ -\lambda \bar{U}_{n-1} + \lambda^{-1} \bar{U}_n & i\Omega + \omega - W_n \end{array} \right| = 0.$$

which yields the relation $\Omega^2 + P(\lambda) = 0$ with

$$P(\lambda) := \frac{1}{4}\left(\lambda^2 + \lambda^{-2}\right)^2 - \omega\left(\lambda^2 + \lambda^{-2}\right) + \omega^2 + \frac{i}{2} F_0 \left(\lambda^2 - \lambda^{-2}\right) - \frac{1}{4} F_0^2 - F_1.$$

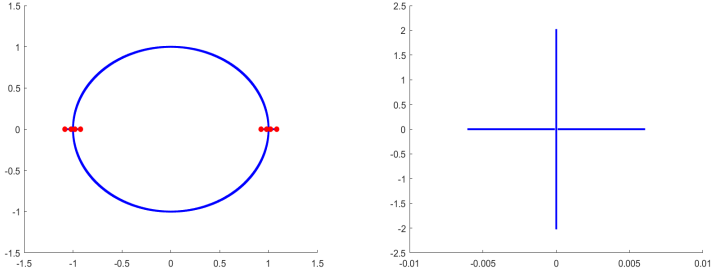
For the discrete mKdV, $P(\lambda)$ is obtained from the nonlinearization method as the separation of variables does not work. For the AL, the separation of variables works, but the nonlinearization method is not applicable.

Stability of the dnoidal periodic waves

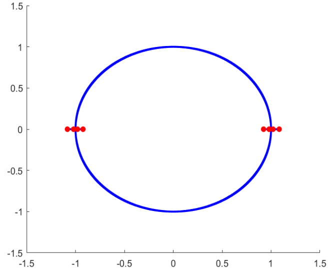
Dnoidal solutions have the form

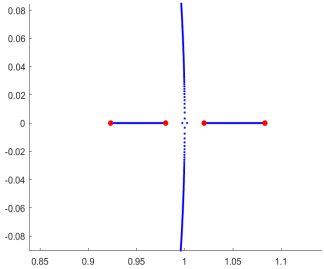
$$U_n = \frac{\mathrm{sn}(\alpha; k)}{\mathrm{cn}(\alpha; k)} \mathrm{dn}(\alpha n; k), \qquad \omega = \frac{\mathrm{dn}(\alpha; k)}{\mathrm{cn}^2(\alpha; k)}.$$

The Lax and stability spectra for $k = 0.8$ and $\alpha = K(k)/20$:



Lax spectrum for dnoidal waves





The spectrum is found numerically for $\alpha = K(k)/M$ with $u_{n+2M} = u_n$:

$$\left\{ \begin{array}{l} \sqrt{1+U_n^2} P_{n+1} + \sqrt{1+U_{n-1}^2} P_{n-1} - (U_n - U_{n-1}) Q_n = z P_n, \\ (U_n - U_{n-1}) P_n + \sqrt{1+U_n^2} Q_{n+1} + \sqrt{1+U_{n-1}^2} Q_{n-1} = z Q_n, \end{array} \right.$$

where $z := \lambda + \lambda^{-1}$ and

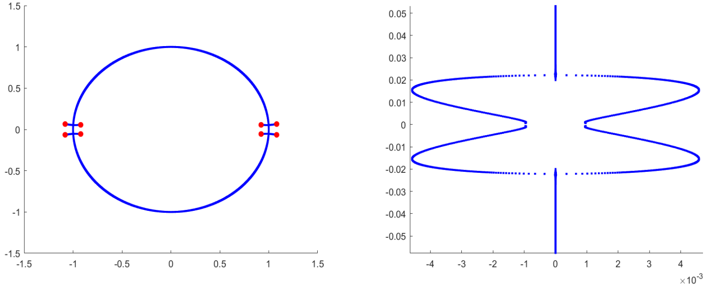
$$P_n = \hat{P}_n(\theta) e^{i\theta n}, \quad Q_n = \hat{Q}_n(\theta) e^{i\theta n}, \quad \hat{P}_{n+2M}(\theta) = \hat{P}_n(\theta), \quad \hat{Q}_{n+2M}(\theta) = \hat{Q}_n(\theta),$$

Stability of the cnoidal periodic waves

Cnoidal solutions have the form

$$U_n = \frac{k\,\mathrm{sn}(\alpha; k)}{\mathrm{dn}(\alpha; k)} \mathrm{cn}(\alpha n; k), \qquad \omega = \frac{\mathrm{cn}(\alpha; k)}{\mathrm{dn}^2(\alpha; k)},$$

The Lax and stability spectra for $k = 0.8$ and $\alpha = K(k)/20$:

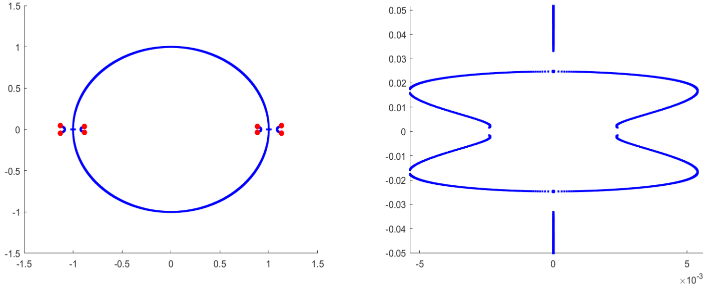


Stability of the cnoidal periodic waves

Cnoidal solutions have the form

$$U_n = \frac{k\,\mathrm{sn}(\alpha; k)}{\mathrm{dn}(\alpha; k)} \mathrm{cn}(\alpha n; k), \qquad \omega = \frac{\mathrm{cn}(\alpha; k)}{\mathrm{dn}^2(\alpha; k)},$$

The Lax and stability spectra for $k = 0.95$ and $\alpha = K(k)/20$:



Polynomial for the standing periodic waves

Recall the polynomial from $\Omega^2 + P(\lambda) = 0$, where

$$P(\lambda) := \frac{1}{4\lambda^4}\left[\lambda^8 - 4\omega\lambda^6 + 2(1 + 2\omega^2 - 2F_1)\lambda^4 - 4\omega\lambda^2 + 1\right].$$

- For the dnoidal waves, the roots are ordered by $\{\pm\lambda_1, \pm\lambda_1^{-1}, \pm\lambda_2, \pm\lambda_2^{-1}\}$.
- For the cnoidal waves, the roots are ordered by $\{\pm\lambda_1, \pm\lambda_1^{-1}, \pm\bar{\lambda}_1, \pm\bar{\lambda}_1^{-1}\}$.

Let λ_1 be any root of $P(\lambda)$. Then there exists a sign of $\sigma_1 = \{+1, -1\}$ such that $\omega = \frac{1}{2}(\lambda_1^2 + \lambda_1^{-2}) + \sigma_1\sqrt{F_1}$ and

$$\begin{aligned} P_n^2 &= \lambda_1 U_n - \lambda_1^{-1} U_{n-1}, \\ Q_n^2 &= \lambda_1 \bar{U}_{n-1} - \lambda_1^{-1}\bar{U}_n, \\ P_n Q_n &= \sigma_1\sqrt{F_1} - \frac{1}{2}(U_n\bar{U}_{n-1} + \bar{U}_n U_{n-1}). \end{aligned}$$

The corresponding squared eigenfunctions are periodic with the period of U_n.

Nonperodic eigenfunctions for the same eigenvalue λ_1

Such solutions can be constructed for any $\lambda_1 \in \mathbb{C}$:

$$\hat{P}_n(t) = P_n(t)\theta_n(t) - \frac{\bar{Q}_n(t)}{|P_n(t)|^2 + |Q_n(t)|^2}, \hat{Q}_n(t) = Q_n(t)\theta_n(t) + \frac{\bar{P}_n(t)}{|P_n(t)|^2 + |Q_n(t)|^2},$$

where $\theta_n(t)$ is a solution of the linear equations

$$\theta_{n+1} - \theta_n = \frac{(|\lambda_1|^2 - 1)(\bar{\lambda}_1 U_n \bar{P}_n^2 - \lambda_1 \bar{U}_n \bar{Q}_n^2 - (1 + |\lambda_1|^2)\bar{P}_n \bar{Q}_n)}{(|P_n|^2 + |Q_n|^2)\Delta_n}$$

and

$$\dot{\theta}_n = \frac{i(|\lambda_1|^2 - 1)\Sigma_n}{|\lambda_1|^2(|P_n|^2 + |Q_n|^2)^2}$$

where

$$\begin{aligned}\Delta_n := {} & |\lambda_1|^4 |P_n|^2 + |Q_n|^2 + |\lambda_1|^2 |U_n|^2 (|P_n|^2 + |Q_n|^2) \\ & + (|\lambda_1|^2 - 1)(\bar{\lambda}_1 U_n \bar{P}_n Q_n + \lambda_1 \bar{U}_n P_n \bar{Q}_n), \\ \Sigma_n := {} & (\lambda_1 U_n + \bar{\lambda}_1 U_{n-1})\bar{P}_n^2 + (\bar{\lambda}_1 \bar{U}_n + \lambda_1 \bar{U}_{n-1})\bar{Q}_n^2 - (1 + |\lambda_1|^{-2})(\lambda_1^2 - \bar{\lambda}_1^2)\bar{P}_n \bar{Q}_n.\end{aligned}$$

Nonperodic eigenfunctions for the same eigenvalue λ_1

If λ_1 is a root of the polynomial $P(\lambda)$ and the squared eigenfunction relations are used, then $\theta_n(t) = \Theta_n + it$ and $\{\Theta_n\}_{n\in\mathbb{Z}}$ is a time-independent solution of

$$\theta_{n+1} - \theta_n = \frac{(\lambda_1 + \lambda_1^{-1})(|U_n|^2 - \sigma_1\sqrt{F_1})}{(\lambda_1 - \lambda_1^{-1})(F_1 + 2(1+\omega)|U_n|^2 + |U_n|^4)}$$

or

$$\theta_{n+1} - \theta_n = \frac{(|\lambda_1|^2 - 1)(\bar{\lambda}_1\lambda_1^{-1} + \bar{\lambda}_1^{-2})|U_n|^2 + \sqrt{F_1}(|\lambda_1|^2 - |\lambda_1|^{-2})}{\Gamma_n},$$

for the dnoidal and cnoidal periodic waves respectively, where

$$\Gamma_n = |U_n|^2\left(|\lambda_1|^4 + |\lambda_1|^{-4} + 2|U_{n-1}|^2 - 2|U_n|^2\right) + 2|U_{n-1}|^2$$
$$+ (|\lambda_1|^2 + |\lambda_1|^{-2})\left(|U_n|^4 - F_1 - \bar{\lambda}_1\lambda_1^{-1}\bar{U}_n U_{n-1} - \lambda_1\bar{\lambda}_1^{-1} U_n \bar{U}_{n-1}\right).$$

1-fold Darboux transformation

The 1-fold DT was found by brutal computations:

$$\hat{u}_n = -\frac{\lambda_1(|p_n|^2 + |\lambda_1|^2|q_n|^2)}{\bar{\lambda}_1(|\lambda_1|^2|p_n|^2 + |q_n|^2)} u_n + \frac{\lambda_1(1-|\lambda_1|^4)p_n\bar{q}_n}{\bar{\lambda}_1^2(|\lambda_1|^2|p_n|^2 + |q_n|^2)}$$

where $\varphi_n = (p_n, q_n)^T$ is a solution of Lax equations for a given $\lambda = \lambda_1 \in \mathbb{C}$.

Similar computations for the semi-discrete equations can be found in:
Tao Xu, D. Pelinovsky, Phys Lett A **383** (2019) 125948
J. Chen, D. Pelinovsky, Physica D **445** (2023) 133652
All constraints of the Darboux equations have been checked.

1-fold DT with periodic eigenfunctions

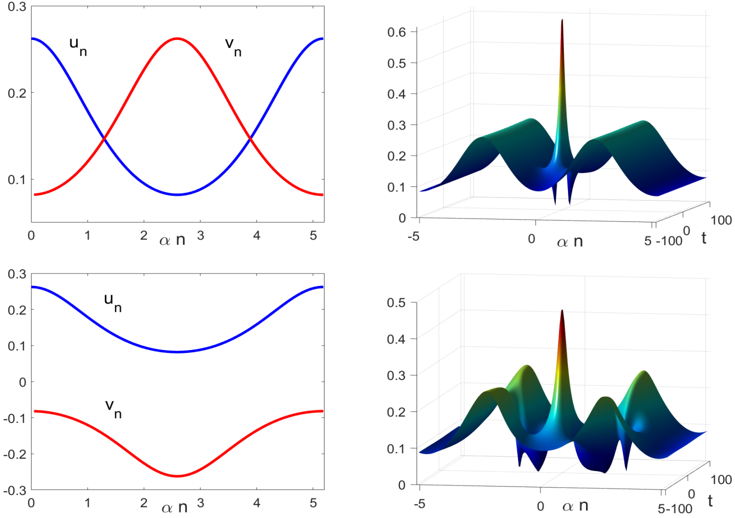
Let λ_1 be a root of the polynomial $P(\lambda)$ and the squared eigenfunction relation are used. Then,

$$\begin{aligned}
\hat{u}_n &= \left[-\frac{|P_n|^2 + \lambda_1^2 |Q_n|^2}{\lambda_1^2 |P_n|^2 + |Q_n|^2} U_n - \frac{(\lambda_1^3 - \lambda_1^{-1}) P_n \bar{Q}_n}{\lambda_1^2 |P_n|^2 + |Q_n|^2} \right] e^{2i\omega t} \\
&= -\frac{\sigma_1 \sqrt{F_1}}{U_n} e^{2i\omega t} \\
&= -\sigma_1 \frac{\mathrm{sn}(\alpha; k)}{\mathrm{cn}(\alpha; k)} \frac{\sqrt{1-k^2}}{\mathrm{dn}(\alpha n; k)} e^{2i\omega t} \\
&= -\sigma_1 \frac{\mathrm{sn}(\alpha; k)}{\mathrm{cn}(\alpha; k)} \mathrm{dn}(\alpha n + K(k); k) e^{2i\omega t} \\
&= -\sigma_1 u_n(\alpha n + K(k))
\end{aligned}$$

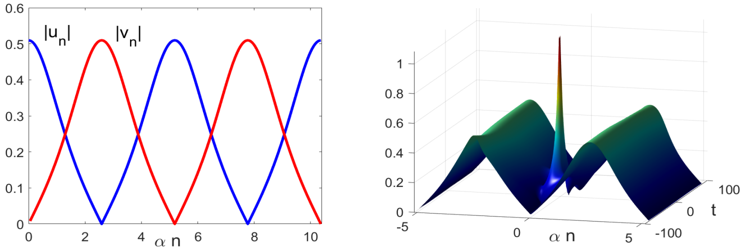
for the dnoidal waves. This a half-period translation and a possible sign-flip.

Similarly, the cnoidal waves are translated in space by a quarter-period and multiplied by a complex phase factor.

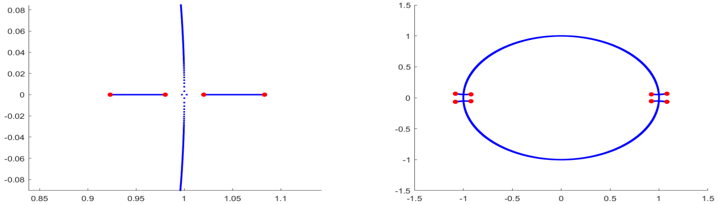
Rogue waves on the dnoidal standing waves



Rogue waves on the cnoidal standing waves



Two rogue waves exist for dnoidal waves and one rogue wave exists for cnoidal waves because of the Lax spectrum:



Magnification factor for the dnoidal rogue waves

To compute the magnification factor, we use

$$\hat{P}_n(t) = P_n(t)\theta_n(t) - \frac{\bar{Q}_n(t)}{|P_n(t)|^2 + |Q_n(t)|^2}, \quad \hat{Q}_n(t) = Q_n(t)\theta_n(t) + \frac{\bar{P}_n(t)}{|P_n(t)|^2 + |Q_n(t)|^2},$$

and

$$\hat{u}_n = -\frac{\lambda_1(|\hat{P}_n|^2 + |\lambda_1|^2|\hat{Q}_n|^2)}{\bar{\lambda}_1(|\lambda_1|^2|\hat{P}_n|^2 + |\hat{Q}_n|^2)} u_n + \frac{\lambda_1(1 - |\lambda_1|^4)\hat{P}_n\bar{\hat{Q}}_n}{\bar{\lambda}_1^2(|\lambda_1|^2|\hat{P}_n|^2 + |\hat{Q}_n|^2)}$$

with $\theta_n(t) = \Theta_n + it$. The maximal amplitude is attained at $(n,t) = (0,0)$, where $\theta_0(0) = 0$. This yields

$$\hat{U}_0(0) = \frac{\mathrm{sn}(\alpha;k)(1 + \mathrm{dn}(\alpha;k) - \sigma_1\sqrt{1-k^2})}{\mathrm{cn}(\alpha;k)\mathrm{dn}(\alpha;k)}.$$

Since $U_0(0) = \mathrm{sn}(\alpha;k)/\mathrm{cn}(\alpha;k)$, the magnification factor is exact:

$$M_{\mathrm{dn}}(\alpha,k) = 1 + \frac{1 - \sigma_1\sqrt{1-k^2}}{\mathrm{dn}(\alpha;k)} \to 2 - \sigma_1\sqrt{1-k^2} \;\; \text{as} \;\; \alpha \to 0.$$

Magnification factor for the cnoidal rogue waves

The maximal amplitude is attained close to $(n, t) = (0, 0)$ but not exactly at $(0, 0)$. As an approximation, we still compute it at $(n, t) = (0, 0)$:

$$\hat{U}_0(0) = \frac{k\mathrm{sn}(\alpha;k)(\mathrm{dn}(\alpha;k)+1)(\mathrm{cn}(\alpha;k)+i\sqrt{1-k^2}\mathrm{sn}(\alpha;k))}{\mathrm{dn}(\alpha;k)[k^2\mathrm{cn}^2(\alpha;k)+(1-k^2)]},$$

which yields

$$|\hat{U}_0(0)| = \frac{k\mathrm{sn}(\alpha;k)(\mathrm{dn}(\alpha;k)+1)}{\mathrm{dn}^2(\alpha;k)}.$$

Since $U_0(0) = k\mathrm{sn}(\alpha;k)/\mathrm{dn}(\alpha;k)$, the magnification factor is

$$M_{\mathrm{cn}}(\alpha, k) = 1 + \frac{1}{\mathrm{dn}(\alpha;k)} \to 2 \quad \text{as} \quad \alpha \to 0.$$

Summary

- Stability of the standing periodic waves in the AL lattice is obtained from the non-standard Lax pair.
- The polynomial $P(\lambda)$ characterizing end points of the spectral bands is obtained from the separation of variables in the time-dependent Lax pair.
- Spectral bands of the Lax spectrum are computed numerically and are connected to the stability spectrum. Both dnoidal and cnoidal waves are modulationally unstable.
- Two basic rogue waves exist on the background of dnoidal waves. A single rogue wave exists on the background of cnoidal waves. Rogue waves are obtained fully analytically.

Many thanks for your attention!