

Bäcklund Transformation and L^2 -stability of NLS Solitons

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Ground states of an L^2 -subcritical focusing nonlinear Schrödinger (NLS) equation are known to be orbitally stable in the energy class $H^1(\mathbb{R})$ thanks to its variational characterization. In this paper, we will show L^2 -stability of 1-solitons to a one-dimensional cubic NLS equation in the sense that for any initial data which are sufficiently close to a 1-soliton in $L^2(\mathbb{R})$, the solution remains in an L^2 -neighborhood of a nearby 1-soliton for all the time. The proof relies on the Bäcklund transformation between zero and soliton solutions of this integrable equation.

1 Introduction

In this paper, we study the nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + 2|u|^2u = 0, \quad (\text{NLS})$$

where $u(t, x) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$. The NLS equation arises in various areas to describe quasi-monochromatic waves such as laser beams or capillary gravity waves. It is well known that the NLS equation is well posed in L^2 [21, 28] and in H^k for any $k \in \mathbb{N}$ [10, 18]. Moreover, solutions of the NLS equation satisfy conservation laws for the charge N and

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Hamiltonian H ,

$$N(u(t, \cdot)) := \|u(t, \cdot)\|_{L^2}^2 = N(u(0, \cdot)), \tag{1}$$

$$H(u(t, \cdot)) := \|\partial_x u(t, \cdot)\|_{L^2}^2 - \|u(t, \cdot)\|_{L^4}^4 = H(u(0, \cdot)), \tag{2}$$

from which global existence follows in L^2 or H^1 . Note that the NLS equation has actually an infinite set of conserved quantities that resemble norms in H^k for any $k \in \mathbb{N}$ [30] and these quantities give global existence in H^k for any $k \in \mathbb{N}$.

The NLS equation has a family of solitary waves (called 1-solitons) that are written as

$$u(t, x) = Q_{k,v}(t - t_0, x - x_0), \quad Q_{k,v}(t, x) := Q_k(x - vt) e^{ivx/2 + i(k^2 - v^2/4)t}, \tag{3}$$

where $Q_k(x) = k \operatorname{sech}(kx)$ and $(k, v, x_0, t_0) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ are arbitrary parameters.

These 1-solitons play an important role to describe the long-time behavior of solutions of the NLS equation. Since Q_k is a minimizer of the functional $H(u)$ restricted on a manifold $M = \{u \in H^1(\mathbb{R}) : \|u\|_{L^2} = \|Q_k\|_{L^2}\}$, the 1-soliton (3) is stable in H^1 up to translations in space and time variables (see, e.g., [3, 12, 29]). As for orbital stability of 1-solitons to rougher perturbations, Colliander et al. [6] show that the H^s -norm ($0 < s < 1$) of a perturbation to a soliton grows at most polynomially in time if the initial data are close to the soliton in $H^s(\mathbb{R})$ ($0 < s < 1$) but not necessarily in $H^1(\mathbb{R})$. The result of [6] suggests that even for rough initial data for which the Hamiltonian is not well defined, the 1-soliton (3) could be stable.

In this paper, we aim to show the Lyapunov stability of 1-solitons in the L^2 class. Our idea is to use the Bäcklund transformation to define an isomorphism which maps solutions in an L^2 -neighborhood of the zero solution to those in an L^2 -neighborhood of a 1-soliton and utilize the L^2 -stability of the zero solution.

The integrability via the inverse scattering transform method has been exploited in many details for analysis of spectral stability of solitary and periodic wave solutions [17, 20]. It was also used to analyze orbital stability of dark solitons in the defocusing version of the NLS equation [9] and to analyze the long-time asymptotics of solutions of the NLS equation [8]. However, L^2 -stability of 1-solitons of the NLS equation using the Bäcklund transformation have not been addressed in literature. In particular, the solvability of the Lax equations to generate the Bäcklund transformation in the L^2 -framework is beyond the standard formalism of the inverse scattering of the NLS equation which requires the initial data to be in L^1 , see [1, Lemma 2.1].

This is not the first time that the integrability is used to prove stability of solitary waves in the context of other nonlinear evolution equations. Merle and Vega [23] used the Miura transformation and proved that 1-solitons of the Korteweg–de Vries (KdV) equation are stable to L^2 -perturbations. The idea was recently applied by Mizumachi and Tzvetkov [26] to prove L^2 -stability of line solitons of the Kadomtsev–Petviashvili (KP-II) equation. The Miura transformation is one of the Bäcklund transformations which connects solutions of the KdV and the modified KdV equations. The Bäcklund transformation seems to give a simplified local coordinate frame which facilitates to observe stability of solitons. In particular, Mizumachi and Pego [25] proved asymptotic stability of Toda lattice solitons by using the Bäcklund transformation to show the equivalence of linear stability of solitons and that of the zero solution. Our use of the Bäcklund transformation for the L^2 -stability of NLS solitons is expected to be applicable to other nonlinear evolution equations associated to the AKNS scheme of inverse scattering.

Now let us introduce our main result on L^2 -stability of 1-solitons.

Theorem 1.1. Let $k > 0$ and let $u(t, x)$ be a solution of the NLS equation in the class

$$u \in C(\mathbb{R}; L^2(\mathbb{R})) \cap L_{\text{loc}}^8(\mathbb{R}; L^4(\mathbb{R})). \quad (4)$$

There exist positive constants C and ε depending only on k such that if $\|u(0, \cdot) - Q_k\|_{L^2} \leq \varepsilon$, then there exist real constants k_0, v_0, t_0 , and x_0 such that

$$\sup_{t \in \mathbb{R}} \|u(t + t_0, \cdot + x_0) - Q_{k_0, v_0}\|_{L^2} + |k_0 - k| + |v_0| + |t_0| + |x_0| \leq C \|u(0, \cdot) - Q_k\|_{L^2}. \quad (5)$$

□

Remark 1.1. Theorem 1.1 tells us that solutions of the NLS equation which are close initially to a 1-soliton in the L^2 -norm remain close to a nearby 1-soliton solution for all the time and the speed, phase, gauge, and amplitude parameters of a nearby 1-soliton are almost the same as those of the original 1-soliton. This makes a contrast with the result of Martel and Merle [22] for the KdV equation that shows that perturbations of 1-solitons in $H^1(\mathbb{R})$ can cause a logarithmic growth of the phase shift thanks to collisions with infinitely many small solitary waves. To the best of our knowledge, this is the first result for the cubic NLS equation in the L^2 (or $H^k, k \in \mathbb{N}$) framework which shows that a solution remains close to a neighborhood of a 1-soliton for all the time. □

Remark 1.2. Asymptotic stability of solitary waves to a generalized nonlinear Schrödinger equation with a bounded potential in one dimension,

$$iu_t + u_{xx} = V(x)u - |u|^{2p}u, \tag{6}$$

has been studied by using dispersive decay estimates for solutions to the linearized equation around solitary waves (see [2] for $p \geq 4$ and [7, 24] for $p \geq 2$). However, the PDE approach has not yet resolved the asymptotic stability of solitary waves in the NLS equation (6) with $p = 1$. The difficulty comes from the slow decay of solutions in the L^∞ norm which makes it difficult to show convergence of modulation parameters of solitary waves in time. □

The article is organized as follows. Section 2 reviews the Bäcklund transformation for the NLS equation. In Section 3, we pull back initial data around a 1-soliton to data around the zero solution by solving the Bäcklund transformation at $t = 0$. When we solve the Bäcklund transformation around a 1-soliton solution at $t = 0$, the parameters which describe the amplitude, the velocity, and the phase shifts of the time and space variables of the largest soliton in the solution are uniquely determined. This shows one of the difference between our approach and the method based on the modulation theory (see, e.g. [2, 7, 24]), where convergence of varying parameters in time is achieved using the decay estimates of the dispersive part of the solution.

In Section 4, we prove that the Bäcklund transformation defines a continuous mapping from an L^2 -neighborhood of the origin to an L^2 -neighborhood of a 1-soliton and that the Bäcklund transformation connects solutions around 1-solitons and solutions around the zero solution for all the time if initial data are smooth. Thanks to the L^2 -conservation law of the NLS equation, the zero solution is stable in L^2 and we conclude that if a perturbation to initial data is small in L^2 , then a solution stays in L^2 -neighborhood of the 1-soliton. Section 5 concludes the article with discussion of open problems.

2 Bäcklund Transformation for the NLS Equation

We recall the Bäcklund transformation between two different solutions $q(t, x)$ and $Q(t, x)$ of the NLS equation. This transformation was found in two different but equivalent forms [5, 19].

The NLS equation is a solvability condition of the Lax operator system

$$\partial_x \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \eta & q \\ -\bar{q} & -\eta \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{7}$$

and

$$\partial_t \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = i \begin{bmatrix} 2\eta^2 + |q|^2 & \partial_x q + 2\eta q \\ \partial_x \bar{q} - 2\eta \bar{q} & -2\eta^2 - |q|^2 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad (8)$$

where parameter η is (t, x) -independent.

Using the variable

$$\gamma = \frac{\psi_1}{\psi_2},$$

we obtain the Riccati equations for the NLS equation

$$\begin{aligned} \partial_x \gamma &= 2\eta\gamma + q + \bar{q}\gamma^2, \\ \partial_t \gamma &= i(4\eta^2 + 2|q|^2)\gamma + i(\partial_x q + 2\eta q) - i(\partial_x \bar{q} - 2\eta \bar{q})\gamma^2. \end{aligned} \quad (9)$$

A new solution $Q(t, x)$ of the same equation NLS is obtained from the old solution $q(t, x)$ and the solution $\gamma(t, x)$ of the Riccati equations (9) (or equivalently, from the solution $\psi_1(t, x)$ and $\psi_2(t, x)$ of the Lax equations (7) and (8)) by

$$Q + q = \frac{-4\Re(\eta)\gamma}{1 + |\gamma|^2} = \frac{-4\Re(\eta)\psi_1\bar{\psi}_2}{|\psi_1|^2 + |\psi_2|^2}. \quad (10)$$

The new solution Q appears as the potential in the same Riccati equations (9) for Γ and in the same Lax equations (7) and (8) for Ψ_1 and Ψ_2 if

$$\Gamma = \frac{1}{\gamma}, \quad \Psi_1 = \frac{\bar{\psi}_2}{|\psi_1|^2 + |\psi_2|^2}, \quad \Psi_2 = \frac{\bar{\psi}_1}{|\psi_1|^2 + |\psi_2|^2}. \quad (11)$$

As a simple example, we can start from the zero solution $q(x, t) \equiv 0$ and assume that $k = 2\eta$ is a real positive number. Equations (7)–(10) give a soliton solution

$$Q(t, x) = Q_k(x) e^{ik^2 t}, \quad Q_k(x) := k \operatorname{sech}(kx), \quad (12)$$

if

$$\psi_1 = e^{(kx + ik^2 t)/2}, \quad \psi_2 = -e^{-(kx + ik^2 t)/2}, \quad \gamma = -e^{kx + ik^2 t} \quad (13)$$

or equivalently,

$$\Psi_1 = -\frac{e^{-(kx - ik^2 t)/2}}{2 \cosh(kx)}, \quad \Psi_2 = \frac{e^{(kx - ik^2 t)/2}}{2 \cosh(kx)}, \quad \Gamma = -e^{-kx + ik^2 t}. \quad (14)$$

Compared to a general family of 1-solitons (3), solution (12) is centered at $x = 0$ and has zero velocity and zero phase.

Remark 2.1. If we eliminate the variable γ from Equation (10) and close the system of Equations (9) for the new and old solutions Q and q , then γ satisfies a quadratic equation that has two roots

$$\gamma = -\frac{k \pm \sqrt{k^2 - |Q + q|^2}}{\bar{Q} + \bar{q}}. \tag{15}$$

This form of the Bäcklund transformation was considered in [5, 19]. Unfortunately, the explicit solution (12) and (13) show that the upper root in (15) is taken for $x > 0$ and the lower root in (15) is taken for $x < 0$ with a weak singularity at $x = 0$. □

Remark 2.2. General solutions of the Lax equations (7) and (8) for $q = 0$ and $\eta = (k + iv)/2$ with $(k, v) \in \mathbb{R}^2$ are given by

$$\psi_1(t + t_0, x + x_0) = e^{(k(x-2vt)+i\omega t+ivx)/2}, \quad \psi_2(t + t_0, x + x_0) = -e^{-(k(x-2vt)+i\omega t+ivx)/2},$$

where $(x_0, t_0) \in \mathbb{R}^2$ are arbitrary parameters for the soliton position and phase, and $\omega = k^2 - v^2$. This solution gives a family of 1-solitons (3). □

3 From a 1-soliton to the Zero Solution at $t = 0$

In this section, we will pull back solutions around a 1-soliton to those around the zero solution by using the Bäcklund transformation at time $t = 0$.

For a given $Q(x)$, let us define $q(x)$ by the Bäcklund transformation

$$Q + q = \frac{-4\Re(\eta)\Psi_1\bar{\Psi}_2}{|\Psi_1|^2 + |\Psi_2|^2}, \tag{16}$$

associated to solutions of the Lax equation

$$\partial_x \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = \begin{bmatrix} \eta & Q \\ -\bar{Q} & -\eta \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}. \tag{17}$$

When $\eta = \frac{1}{2}$ and $Q(x) = Q_1(x) \equiv \operatorname{sech}(x)$, the spectral problem (17) has a fundamental system $\{\Psi_1(x), \Psi_2(x)\}$, where

$$\Psi_1(x) = \begin{bmatrix} -e^{-x/2} \\ e^{x/2} \end{bmatrix} \operatorname{sech}(x), \quad \Psi_2 = \begin{bmatrix} (e^x + 2(1+x)e^{-x})e^{x/2} \\ (e^{-x} - 2xe^x)e^{-x/2} \end{bmatrix} \operatorname{sech}(x). \quad (18)$$

We obtain $q = 0$ when the first solution Ψ_1 is used in the Bäcklund transformation (16) with $\eta = \frac{1}{2}$ and

$$q(x) = \frac{2xe^{2x} + (4x^2 + 4x - 1) - 2x(1+x)e^{-2x}}{\cosh(3x) + 4(1+x+x^2)\cosh(x)} - \operatorname{sech}(x) \quad (19)$$

when the second solution Ψ_2 is used in (16) with $\eta = \frac{1}{2}$. The latter solution corresponds to the weak (logarithmic in time) scattering of two nearly identical solitons. This interaction between two solitons was studied by Zakharov and Shabat [31] shortly after the integrability of the NLS equation was discovered by themselves [30]. We are interested in the decaying solution of the spectral problem (17), which corresponds to the eigenvector for a simple isolated eigenvalue $\eta = \frac{1}{2}$ associated to the potential $Q_1(x) = \operatorname{sech}(x)$.

Let us recall the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The spectral problem (17) is equivalent to an eigenvalue problem

$$(L - M(S))\Psi = \lambda\Psi, \quad (20)$$

where $\lambda = \eta - \frac{1}{2}$, $S = Q - Q_1$,

$$L := \begin{bmatrix} \partial_x - \frac{1}{2} & -Q_1 \\ -Q_1 & -\partial_x - \frac{1}{2} \end{bmatrix} = \sigma_3 \partial_x - \frac{1}{2}I - Q_1 \sigma_1 \equiv L_0 - Q_1 \sigma_1,$$

and

$$M(S) := \begin{bmatrix} 0 & S \\ \bar{S} & 0 \end{bmatrix} = \sigma_1 \Re(S) - \sigma_2 \Im(S).$$

We consider L as a closed operator on $L^2(\mathbb{R}; \mathbb{C}^2)$ whose domain is $H^1(\mathbb{R}; \mathbb{C}^2)$. If $S = 0$, then $\lambda = 0$ is an eigenvalue of (20) whose eigenspace is spanned by Ψ_1 . Since

$$\left(M(Q_1)L_0^{-1} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right) (x) = -Q_1(x) \begin{bmatrix} \int_x^x e^{-(x-y)/2} f_2(y) dy \\ \int_x^{-\infty} e^{(x-y)/2} f_1(y) dy \end{bmatrix},$$

we see that $M(Q_1)L_0^{-1}$ is Hilbert–Schmidt and thus a compact operator on $L^2(\mathbb{R}; \mathbb{C}^2)$. Thus, by Weyl’s essential spectrum theorem, we have $\sigma_c(L) = \sigma(L_0) = \{-\frac{1}{2} + ik, k \in \mathbb{R}\}$ and the zero eigenvalue is bounded away from the rest of the spectrum of L . Thus, for small S , we will see that the eigenvalue problem (20) has a simple eigenvalue near 0.

Lemma 3.1. There exist positive constants C and ε and real constants k and v such that if $\|Q - Q_1\|_{L^2} \leq \varepsilon$, then there exists a solution $\Psi = {}^t(\Psi_1, \Psi_2) \in H^1(\mathbb{R}; \mathbb{C}^2)$ of the system (17) with $\eta = (k + iv)/2$ such that

$$|k - 1| + |v| + \|\Psi - \Psi_1\|_{L^\infty} \leq C \|Q - Q_1\|_{L^2}. \tag{21}$$

□

Proof. We will prove Lemma 3.1 by the Lyapunov–Schmidt method. Let us write $Q = Q_1 + S$ and

$$\Psi = \Psi_1 + \Phi, \quad \langle \Psi_1, \Phi \rangle_{L^2} = 0. \tag{22}$$

Let P be a spectral projection associated with L on $L^2(\mathbb{R}; \mathbb{C}^2)$, or explicitly,

$$P\mathbf{u} = \mathbf{u} - \frac{1}{4} \langle \mathbf{u}, \Theta \rangle_{L^2} \Psi_1, \quad \Theta(x) = \begin{bmatrix} -e^{x/2} \\ e^{-x/2} \end{bmatrix} \operatorname{sech}(x).$$

Note that $\ker(L) = \operatorname{span}\{\Psi_1\}$ and $\ker(L^*) = \operatorname{span}\{\Theta\}$. The system (17) can be rewritten into the block-diagonal form

$$L\Phi = P[(\lambda I + M(S))(\Psi_1 + \Phi)] \tag{23}$$

and

$$\langle \Theta, (\lambda I + M(S))(\Psi + \Phi) \rangle_{L^2} = 0. \tag{24}$$

Since L_0 is a closed operator on $L^2(\mathbb{R}; \mathbb{C}^2)$ with $\text{Range}(L_0) = L^2(\mathbb{R}; \mathbb{C}^2)$ and $M(Q_1)L_0^{-1}$ is a compact operator on $L^2(\mathbb{R}; \mathbb{C}^2)$, we see that L is Fredholm and

$$\text{Range}(L) = \{\Phi \in L^2(\mathbb{R}; \mathbb{C}^2) : \langle \Phi, \Theta \rangle_{L^2} = 0\}.$$

Thus, we can define L^{-1} as a bounded operator

$$L^{-1} : L^2(\mathbb{R}; \mathbb{C}^2) \cap {}^\perp \ker(L^*) \rightarrow H^1(\mathbb{R}; \mathbb{C}^2) \cap {}^\perp \ker(L).$$

If $S \in L^2(\mathbb{R})$ and $\lambda \in \mathbb{C}$ are sufficiently small, then there exists a unique solution $\Phi \in H^1(\mathbb{R}^2; \mathbb{C}^2)$ of (23) such that

$$\|\Phi\|_{H^1 \times H^1} \leq C(\|S\|_{L^2} + |\lambda|), \tag{25}$$

where C is a constant that does not depend on S and λ . On the other hand, Equation (24) can be written in the form

$$\begin{aligned} & \lambda \left(4 + \int_{\mathbb{R}} \text{sech}(x)[-e^{x/2}\Phi_1(x) + e^{-x/2}\Phi_2(x)]dx \right) \\ & = 2\langle Q_1, \Re(S) \rangle_{L^2} - 2i\langle \partial_x Q_1, \Im(S) \rangle_{L^2} - \int_{\mathbb{R}} \text{sech}(x)[e^{x/2}S(x)\Phi_2(x) + e^{-x/2}\overline{S(x)}\Phi_1(x)]dx \end{aligned}$$

In view of the bound (25), the latter equation gives

$$\exists C > 0 : \left| \lambda - \frac{1}{2}\langle Q_1, \Re(S) \rangle_{L^2} + \frac{i}{2}\langle \partial_x Q_1, \Im(S) \rangle_{L^2} \right| \leq C\|S\|_{L^2}^2, \tag{26}$$

which concludes the proof of Lemma 3.1 since $\lambda = \eta - \frac{1}{2}$ and $S = Q - Q_1$. ■

Remark 3.1. If the eigenvalue η is forced to stay at $\frac{1}{2}$, constraints on $S(x)$ need to be enforced, which are given at the leading order by

$$\langle Q_1, \Re(S) \rangle_{L^2} = 0, \quad \langle \partial_x Q_1, \Im(S) \rangle_{L^2} = 0. \tag{27}$$

□

Constraints (27) are nothing but the symplectic orthogonality conditions to the eigenvectors of the linearized time-evolution problem that correspond to the zero eigenvalue induced by the gauge and translational symmetries of the NLS equation. The

symplectic orthogonality conditions were used in [7, 24] to derive modulation equations for varying parameters of the solitary wave and to prove its asymptotic stability in the time evolution of the generalized NLS equation (6).

Let us generalize the symplectic orthogonal conditions (27) and decompose Q into a sum of all four secular modes and the residual part. This decomposition is standard and follows from the implicit function theorem arguments (see, e.g. [7, 24]).

Lemma 3.2. There exist positive constants C and ε and real constants $\alpha, \beta, \theta,$ and γ such that if $\|Q - Q_1\|_{L^2} \leq \varepsilon$, then Q can be represented by

$$e^{-i(vx+\theta)} Q(\cdot + \gamma) = Q_k + i\alpha x Q_k + \beta \partial_k Q_k + S, \tag{28}$$

with

$$\langle Q_k, \Re(S) \rangle_{L^2} = \langle \partial_x Q_k, \Im(S) \rangle_{L^2} = \langle x Q_k, \Re(S) \rangle_{L^2} = \langle \partial_k Q_k, \Im(S) \rangle_{L^2} = 0 \tag{29}$$

and

$$|\alpha| + |\beta| + |\theta| + |\gamma| + \|S\|_{L^2} \leq C \|Q - Q_1\|_{L^2}, \tag{30}$$

where k and v are real constants given in Lemma 3.1. □

In order to estimate the L^2 -norm of q defined by the Bäcklund transformation (16), we need to investigate solutions to the system (17).

Lemma 3.3. There exist positive constants C and ε such that if $\|Q - Q_1\|_{L^2} \leq \varepsilon$, then an H^1 -solution of the system (17) with $\eta = (k + iv)/2$ determined in Lemma 3.1 satisfies

$$\Psi(x + \gamma) = \operatorname{sech}(kx) e^{\frac{i}{2}(vx+\theta)\sigma_3} \begin{bmatrix} e^{-kx/2}(-1 + r_{11}(x)) + e^{kx/2}r_{12}(x) \\ e^{-kx/2}r_{21}(x) + e^{kx/2}(1 + r_{22}(x)) \end{bmatrix}, \tag{31}$$

$$\|r_{11}\|_{L^\infty} + \|r_{12}\|_{L^2 \cap L^\infty} + \|r_{21}\|_{L^2 \cap L^\infty} + \|r_{22}\|_{L^\infty} \leq C \|Q - Q_1\|_{L^2}, \tag{32}$$

where γ and θ are constants determined in Lemma 3.2. Moreover if $Q \in H^n(\mathbb{R})$ ($n \in \mathbb{N}$) in addition, then

$$\|\partial_x^m r_{11}\|_{L^\infty} + \|\partial_x^m r_{12}\|_{L^2 \cap L^\infty} + \|\partial_x^m r_{21}\|_{L^2 \cap L^\infty} + \|\partial_x^m r_{22}\|_{L^\infty} \leq C' (\|Q - Q_1\|_{H^m} + \|Q - Q_1\|_{H^m}^m) \tag{33}$$

for $0 \leq m \leq n$, where C' is a positive constant depending only on n . □

Lemma 3.3 will be proved at the end of this section. Assuming Lemma 3.3, we will prove that the Bäcklund transformation maps initial data around a 1-soliton to those around the zero solution.

Lemma 3.4. There exist positive constants C and ε satisfying the following: Let $Q \in H^3(\mathbb{R})$ and $\|Q - Q_1\|_{L^2} \leq \varepsilon$ and let Ψ be an H^1 -solution of the system (17) with $\eta = (k + iv)/2$ determined in Lemma 3.1. Suppose

$$q := -Q - \frac{2k\Psi_1\overline{\Psi_2}}{|\Psi_1|^2 + |\Psi_2|^2}.$$

Then $q \in H^3(\mathbb{R})$ and $\|q\|_{L^2} \leq C\|Q - Q_1\|_{L^2}$. □

Proof. By (31) and (32), we have

$$-\frac{2k\Psi_1\overline{\Psi_2}}{|\Psi_1|^2 + |\Psi_2|^2} = 2ke^{i(v(x-\gamma)+\theta)} \frac{1 + \varepsilon_1(x) + e^{k(x-\gamma)}\varepsilon_2(x) + e^{-k(x-\gamma)}\varepsilon_3(x)}{e^{k(x-\gamma)}(1 + \varepsilon_4(x)) + \varepsilon_5(x) + e^{-k(x-\gamma)}(1 + \varepsilon_6(x))}, \tag{34}$$

where

$$\begin{aligned} \varepsilon_1 &= \bar{r}_{22} - r_{11} - r_{12}\bar{r}_{21} - r_{11}\bar{r}_{22}, \\ \varepsilon_2 &= -(1 + \bar{r}_{22})r_{12}, \\ \varepsilon_3 &= \bar{r}_{21}(1 - r_{11}), \\ \varepsilon_4 &= 2\Re(r_{22}) + |r_{22}|^2 + |r_{12}|^2, \\ \varepsilon_5 &= -2\Re(\bar{r}_{12}(1 - r_{11})) + 2\Re(\bar{r}_{21}(1 + r_{22})), \\ \varepsilon_6 &= -2\Re(r_{11}) + |r_{11}|^2 + |r_{21}|^2. \end{aligned}$$

Lemmas 3.1–3.3 imply that

$$|k - 1| + |v| + |\theta| + |\gamma| \lesssim \|Q - Q_1\|_{L^2}$$

and

$$\|\varepsilon_1\|_{L^\infty} + \|\varepsilon_2\|_{L^2 \cap L^\infty} + \|\varepsilon_3\|_{L^2 \cap L^\infty} + \|\varepsilon_4\|_{L^\infty} + \|\varepsilon_5\|_{L^2 \cap L^\infty} + \|\varepsilon_6\|_{L^\infty} \lesssim \|Q - Q_1\|_{L^2},$$

where notation $A \lesssim B$ is used to say that there is a positive constant C such that $A \leq CB$.

Combining the above bounds with the expansion,

$$\begin{aligned} & \frac{1 + \varepsilon_1(x) + e^{k(x-\gamma)}\varepsilon_2(x) + e^{-k(x-\gamma)}\varepsilon_3(x)}{e^{k(x-\gamma)}(1 + \varepsilon_4(x)) + \varepsilon_5(x) + e^{-k(x-\gamma)}(1 + \varepsilon_6(x))} \\ &= \frac{1 + \varepsilon_1(x)}{e^{k(x-\gamma)}(1 + \varepsilon_4(x)) + \varepsilon_5(x) + e^{-k(x-\gamma)}(1 + \varepsilon_6(x))} + \mathcal{O}(|\varepsilon_2(x)| + |\varepsilon_3(x)|) \\ &= \frac{1}{2} \operatorname{sech}(k(x - \gamma))(1 + \mathcal{O}(|\varepsilon_1(x)| + |\varepsilon_4(x)| + |\varepsilon_5(x)| + |\varepsilon_6(x)|)) + \mathcal{O}(|\varepsilon_2(x)| + |\varepsilon_3(x)|), \end{aligned}$$

we get

$$\exists C > 0 : \left\| \frac{2k\Psi_1\overline{\Psi_2}}{|\Psi_1|^2 + |\Psi_2|^2} + Q_1 \right\|_{L^2} \leq C \|Q - Q_1\|_{L^2}. \tag{35}$$

Thus by (16) and (35),

$$\|q\|_{L^2} \leq \|Q - Q_1\|_{L^2} + \left\| \frac{2k\Psi_1\overline{\Psi_2}}{|\Psi_1|^2 + |\Psi_2|^2} + Q_1 \right\|_{L^2} \leq (C + 1) \|Q - Q_1\|_{L^2}.$$

If $Q \in H^3(\mathbb{R})$ in addition, then it follows from (16), (33), and (34) that $q \in H^3(\mathbb{R})$. ■

Corollary 3.5. Under conditions of Lemma 3.4, let

$$\psi_1 = \frac{\overline{\Psi_2}}{|\Psi_1|^2 + |\Psi_2|^2}, \quad \psi_2 = \frac{\Psi_1}{|\Psi_1|^2 + |\Psi_2|^2}.$$

Then (ψ_1, ψ_2) are C^2 -functions satisfying (7). □

Proof. Lemma 3.3 implies that ψ_1 and ψ_2 are C^2 -functions. By a direct substitution, we see that (ψ_1, ψ_2) is a solution of the system (7). ■

Remark 3.2. Using the change of variables

$$\begin{aligned} \Psi'_1(\gamma) &= e^{-\frac{i}{2}(v\gamma+\theta)} \Psi_1(x + \gamma), \\ \Psi'_2(\gamma) &= e^{\frac{i}{2}(v\gamma+\theta)} \Psi_2(x + \gamma), \\ Q'(\gamma) &= k^{-1} e^{-i(v\gamma+\theta)} Q(x + \gamma), \end{aligned}$$

where $y = kx$, we can translate the system (17) with $\eta = (k + iv)/2$ into

$$\partial_y \begin{bmatrix} \psi'_1 \\ \psi'_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & Q' \\ -\overline{Q'} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \psi'_1 \\ \psi'_2 \end{bmatrix}.$$

Therefore, we will assume $k=1$ and $v = \gamma = \theta = 0$ in (28) and (29) for the sake of simplicity. \square

Next, we will give an estimate of solutions to the linear inhomogeneous equation

$$L\mathbf{u} = \mathbf{f}. \quad (36)$$

To prove Lemma 3.3, we introduce Banach spaces $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$ such that for $\mathbf{u} = {}^t(u_1, u_2) \in X$ and $\mathbf{f} = {}^t(f_1, f_2) \in Y$, we have

$$\|\mathbf{u}\|_X = \|u_1\|_{X_1} + \|u_2\|_{X_2}, \quad \|\mathbf{f}\|_Y = \|f_1\|_{Y_1} + \|f_2\|_{Y_2},$$

equipped with the norms

$$\|u_1\|_{X_1} := \inf_{u_1 = v_1 + w_1} (\|e^{x/2} \cosh(x)v_1\|_{L^\infty} + \|e^{-x/2} \cosh(x)w_1\|_{L^2 \cap L^\infty}),$$

$$\|u_2\|_{X_2} := \inf_{u_2 = v_2 + w_2} (\|e^{-x/2} \cosh(x)v_2\|_{L^\infty} + \|e^{x/2} \cosh(x)w_2\|_{L^2 \cap L^\infty})$$

and

$$\|f_1\|_{Y_1} := \inf_{f_1 = g_1 + h_1} (\|e^{-x/2} \cosh(x)g_1\|_{L^2} + \|e^{x/2} \cosh(x)h_1\|_{L^1 \cap L^2}),$$

$$\|f_2\|_{Y_2} := \inf_{f_2 = g_2 + h_2} (\|e^{x/2} \cosh(x)g_2\|_{L^2} + \|e^{-x/2} \cosh(x)h_2\|_{L^1 \cap L^2}).$$

Lemma 3.6. Let $\mathbf{f} = {}^t(f_1, f_2) \in Y \cap {}^\perp \ker(L^*)$ and let \mathbf{u} be a solution of the system (36) such that $\mathbf{u} \perp \ker(L)$. Then, there is an \mathbf{f} -independent constant $C > 0$ such that $\|\mathbf{u}\|_X \leq C \|\mathbf{f}\|_Y$. \square

Remark 3.3. For an arbitrary $\mathbf{f} \in L^2(\mathbb{R}; \mathbb{C}^2) \cap {}^\perp \ker(L^*)$, an H^1 -solution \mathbf{u} of the system (36) does not necessarily decay as fast as its fundamental solution. However, since the potential matrix $M(S)$ in (20) is off-diagonal, solutions have a better decay property, according to the norm in X . \square

To prove Lemma 3.6, we will use an explicit formula of $L^{-1}\mathbf{f}$.

Lemma 3.7. For any $\mathbf{f} = {}^t(f_1, f_2) \in L^2(\mathbb{R}; \mathbb{C}^2) \cap \perp \ker(L^*)$, there exists a unique solution $\mathbf{u} \in H^1(\mathbb{R}; \mathbb{C}^2) \cap \perp \ker(L)$ of the system (36) that can be written as

$$\begin{aligned} \mathbf{u}(x) = & \zeta(\mathbf{f})\Psi_1(x) + \frac{1}{4}\Psi_1(x) \int_x^\infty e^{y/2}(e^{-2y} - 2y) \operatorname{sech}(y) f_1(y) dy \\ & - \frac{1}{4}\Psi_1(x) \int_{-\infty}^x e^{-y/2}(e^{2y} + 2 + 2y) \operatorname{sech}(y) f_2(y) dy + \frac{1}{4}\Psi_2(x) \int_x^{\pm\infty} \mathbf{f}(y) \cdot \Theta(y) dy, \end{aligned} \quad (37)$$

where $\zeta(\mathbf{f})$ is a continuous linear functional on L^2 . □

Remark 3.4. If $\langle \mathbf{f}, \Theta \rangle_{L^2} = 0$, then

$$\int_x^\infty \mathbf{f}(y) \cdot \Theta(y) dy = - \int_{-\infty}^x \mathbf{f}(y) \cdot \Theta(y) dy. \quad (38)$$

□

Proof of Lemma 3.7. Since $L : H^1(\mathbb{R}; \mathbb{C}^2) \rightarrow L^2(\mathbb{R}; \mathbb{C}^2)$ is a Fredholm operator, Equation (36) has a solution in $L^2(\mathbb{R}; \mathbb{C}^2)$ if \mathbf{f} is orthogonal to $\ker(L^*) = \operatorname{span}\{\Theta\}$.

Using a fundamental matrix $U(x) = [\Psi_1(x), \Psi_2(x)]$ of

$$\partial_x \Psi = \begin{bmatrix} \frac{1}{2} & Q_1 \\ -Q_1 & -\frac{1}{2} \end{bmatrix} \Psi,$$

we rewrite $L\mathbf{u} = \mathbf{f}$ as

$$\frac{d}{dx}(U(x)^{-1}\mathbf{u}) = U(x)^{-1}\sigma_3\mathbf{f} = \frac{-1}{4} \operatorname{sech}(x) \begin{bmatrix} e^{x/2}(e^{-2x} - 2x) & e^{-x/2}(e^{2x} + 2x + 2) \\ -e^{x/2} & e^{-x/2} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}.$$

Thus, we have

$$\mathbf{u}(x) = U(x)\mathbf{c} - \frac{1}{4}U(x)\mathbf{g}(x),$$

where \mathbf{c} is a constant vector, $\mathbf{g}(x) = {}^t(g_1(x), g_2(x))$ and

$$g_1(x) = \int_{x_1}^x e^{y/2}(e^{-2y} - 2y) \operatorname{sech}(y) f_1(y) dy + \int_{x_2}^x e^{-y/2}(e^{2y} + 2y + 2) \operatorname{sech}(y) f_2(y) dy,$$

$$g_2(x) = - \int_{x_3}^x e^{y/2} \operatorname{sech}(y) f_1(y) dy + \int_{x_4}^x e^{-y/2} \operatorname{sech}(y) f_2(y) dy.$$

Note that $x_1, x_2, x_3,$ and x_4 can be chosen freely. To let $\mathbf{u} \in L^2(\mathbb{R}; \mathbb{C}^2)$, we put $x_1 = \infty,$ $x_2 = -\infty, x_3 = x_4 = \pm\infty,$ and $\mathbf{c} = {}^t(\zeta, 0)$ and obtain (37).

Next, we will show that $\zeta(\mathbf{f})$ is continuous on L^2 . Since $|\Psi_1(x)| \lesssim e^{-|x|/2}$ for all $x \in \mathbb{R},$

$$\begin{aligned} & \left\| \Psi_1(x) \int_x^\infty e^{y/2}(e^{-2y} - 2y) \operatorname{sech}(y) f_1(y) dy \right\|_{L^2} \\ & \lesssim \left\| \Psi_1(x) \int_x^\infty e^{-3y/2} \operatorname{sech}(y) f_1(y) dy \right\|_{L^2} + \left\| \Psi_1(x) \int_x^\infty y e^{y/2} \operatorname{sech}(y) f_1(y) dy \right\|_{L^2} \\ & \lesssim \left\| \int_x^\infty e^{(x-y)/2} |f_1(y)| dy \right\|_{L^2} + \left\| e^{-|x|/2} \int_x^\infty e^{-|y|/2} |y f_1(y)| dy \right\|_{L^2} \lesssim \|\mathbf{f}\|_{L^2}. \end{aligned}$$

Similarly, we have

$$\left\| \Psi_1(x) \int_{-\infty}^x e^{-y/2}(e^{2y} + 2 + 2y) \operatorname{sech}(y) f_2(y) dy \right\|_{L^2} \lesssim \|\mathbf{f}\|_{L^2}.$$

Using Remark 3.4 and the fact that $|\Psi_2(x)| \lesssim e^{|x|/2}$ and $|\Theta(x)| \lesssim e^{-|x|/2}$ for all $x \in \mathbb{R},$ we have

$$\begin{aligned} & \left\| \Psi_2(x) \int_x^{\pm\infty} \mathbf{f}(y) \cdot \Theta(y) dy \right\|_{L^2} \\ & \lesssim \left\| \int_x^\infty e^{(x-y)/2} |\mathbf{f}(y)| dy \right\|_{L^2(0,\infty)} + \left\| \int_{-\infty}^x e^{-(x-y)/2} |\mathbf{f}(y)| dy \right\|_{L^2(-\infty,0)} \lesssim \|\mathbf{f}\|_{L^2}. \end{aligned}$$

The constant $\zeta(\mathbf{f})$ in (37) is uniquely determined by the orthogonality condition $\mathbf{u} \perp \Psi_1.$ It follows from the bounds above that $\zeta(\mathbf{f})$ is continuous linear functional on $L^2.$ ■

Now we give a proof of Lemma 3.6.

Proof of Lemma 3.6. Since Y is continuously embedded into $L^2,$ the solution $\mathbf{u} = L^{-1}\mathbf{f}$ can be written as (37) and

$$\|\zeta(\mathbf{f})\Psi_1\|_X \lesssim \|\mathbf{f}\|_{L^2} \lesssim \|\mathbf{f}\|_Y.$$

Next we estimate the second term of (37). Noting that $\|a\psi_1\|_X \leq 2\|a\|_{L^\infty}$ for any $a \in L^\infty(\mathbb{R})$, we have

$$\begin{aligned} & \left\| \psi_1(x) \int_x^\infty e^{y/2}(e^{-2y} - 2y) \operatorname{sech}(y) f_1(y) dy \right\|_X \lesssim \left\| \int_x^\infty e^{y/2}(e^{-2y} - 2y) \operatorname{sech}(y) f_1(y) dy \right\|_{L^\infty} \\ & \leq \inf_{f_1=g_1+h_1} (\|g_1 e^{-y/2} \cosh(y)\|_{L^2} \| \operatorname{sech}^2(y)(e^{-y} - 2ye^y) \|_{L^2} \\ & \quad + \|h_1 e^{y/2} \cosh(y)\|_{L^1} \| \operatorname{sech}^2(y)(e^{-2y} - 2y) \|_{L^\infty}) \\ & \lesssim \inf_{f_1=g_1+h_1} (\|g_1 e^{-y/2} \cosh(y)\|_{L^2} + \|h_1 e^{y/2} \cosh(y)\|_{L^1}) \lesssim \|f_1\|_{Y_1}. \end{aligned}$$

Similarly, we have

$$\left\| \psi_1(x) \int_{-\infty}^x e^{-y/2}(e^{2y} + 2 + 2y) \operatorname{sech}(y) f_2(y) dy \right\|_X \lesssim \|f_2\|_{Y_2}.$$

Finally, we will estimate the fourth term of (37). Clearly,

$$\left\| \psi_2(x) \int_x^{\pm\infty} \mathbf{f}(y) \cdot \Theta(y) dy \right\|_X \leq \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4,$$

where

$$\begin{aligned} \Pi_1 &= \left\| e^x \int_x^\infty \mathbf{f}(y) \cdot \Theta(y) dy \right\|_{L^2 \cap L^\infty}, \\ \Pi_2 &= 2 \left\| (1+x) \int_{\pm\infty}^x \mathbf{f}(y) \cdot \Theta(y) dy \right\|_{L^\infty}, \\ \Pi_3 &= \left\| e^{-x} \int_{-\infty}^x \mathbf{f}(y) \cdot \Theta(y) dy \right\|_{L^2 \cap L^\infty}, \\ \Pi_4 &= 2 \left\| x \int_{\pm\infty}^x \mathbf{f}(y) \cdot \Theta(y) dy \right\|_{L^\infty}. \end{aligned}$$

Since $\|e^{y/2}\mathbf{f}\|_{L^2} \lesssim \|\mathbf{f}\|_Y$ and $|\Theta(y)| \lesssim e^{-|y|/2}$ for all $y \in \mathbb{R}$, we have

$$\begin{aligned} \Pi_1 &\leq \left\| \int_x^\infty e^{x-y} e^{|y|/2} (|f_1(y)| + |f_2(y)|) dy \right\|_{L^2 \cap L^\infty} \\ &\leq (\|e^{y/2} f_1\|_{L^2} + \|e^{y/2} f_2\|_{L^2}) \|e^{-x}\|_{L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)} \lesssim \|\mathbf{f}\|_Y. \end{aligned}$$

Similarly, we have $\text{II}_2 + \text{II}_3 + \text{II}_4 \lesssim \|\mathbf{f}\|_Y$. Therefore,

$$\left\| \Psi_2(x) \int_x^{\pm\infty} \mathbf{f}(y) \cdot \Theta(y) \, dy \right\|_X \lesssim \|\mathbf{f}\|_Y.$$

Thus, the proof of Lemma 3.6 is complete. ■

Now we are in a position to prove Lemma 3.3.

Proof of Lemma 3.3. Let Ψ be a solution of the system (17) in Lemma 3.1 such that

$$\Psi = \Psi_1 + \Phi, \quad \langle \Phi, \Psi_1 \rangle_{L^2} = 0.$$

Substituting (28) (with $k = 1$ and $v = \gamma = \theta = 0$) into the system (17), we obtain

$$L\Phi = \mathbf{R}_1 + \mathbf{R}_2 + R_3\Phi, \tag{39}$$

where

$$\mathbf{R}_1 = M(S)\Psi_1 = \begin{bmatrix} SQ_1 e^{x/2} \\ -\bar{S}Q_1 e^{-x/2} \end{bmatrix},$$

$$\mathbf{R}_2 = [-\alpha x Q_1 \sigma_2 + \beta(x\partial_x Q_1 + Q_1)\sigma_1]\Psi_1 = i\alpha x Q_1^2 \begin{bmatrix} e^{x/2} \\ e^{-x/2} \end{bmatrix} + \beta(x\partial_x Q_1 + Q_1)Q_1 \begin{bmatrix} e^{x/2} \\ -e^{-x/2} \end{bmatrix},$$

$$R_3 = M(S) - \alpha x Q_1 \sigma_2 + \beta(x\partial_x Q_1 + Q_1)\sigma_1.$$

Because $\Psi_1 \notin Y$ and $\|(I - P)\mathbf{f}\|_Y = \infty$ whatever \mathbf{f} is, we shall modify the projection operator compared to the proof of Lemma 3.1. Let $\tilde{P} : L^2(\mathbb{R}; \mathbb{C}^2) \rightarrow L^2(\mathbb{R}; \mathbb{C}^2) \cap \perp \ker(L^*)$ be a new projection defined by

$$\tilde{P}\mathbf{u} = \mathbf{u} - \frac{3}{4} \langle \mathbf{u}, \Theta \rangle_{L^2} \operatorname{sech}^2(x) \Psi_1.$$

Since $\Re \langle S, Q_1 \rangle_{L^2} = \Im \langle S, \partial_x Q_1 \rangle_{L^2} = 0$ by (29), we have

$$\langle M(S)\Psi_1, \Theta \rangle_{L^2} = -2\Re \langle S, Q_1 \rangle_{L^2} + 2i\Im \langle S, \partial_x Q_1 \rangle_{L^2} = 0. \tag{40}$$

By (40) and the fact that $\Theta \perp \text{Range}(L)$, we obtain

$$L\Phi = \tilde{P}L\Phi = \mathbf{R}_1 + \tilde{P}(\mathbf{R}_2 + R_3\Phi).$$

Thus, the system (39) is transformed into

$$(I - L^{-1}\tilde{P}R_3)\Phi = L^{-1}\mathbf{R}_1 + L^{-1}\tilde{P}\mathbf{R}_2. \tag{41}$$

Lemma 3.6 and the bound (30) imply

$$\begin{aligned} \|L^{-1}\mathbf{R}_1\|_X &\lesssim \|\mathbf{R}_1\|_Y \lesssim \|SQ_1 \cosh(x)\|_{L^2} \lesssim \|S\|_{L^2}, \\ \|L^{-1}\tilde{P}\mathbf{R}_2\|_X &\lesssim \|\tilde{P}\mathbf{R}_2\|_Y \lesssim \|\mathbf{R}_2\|_Y + |\langle \mathbf{R}_2, \Theta \rangle_{L^2}| \\ &\lesssim |\alpha| \|xQ_1^2 \cosh(x)\|_{L^2} + |\beta| \|(x\partial_x Q_1 + Q_1)Q_1 \cosh(x)\|_{L^2} + \|\mathbf{R}_2\|_{L^2} \\ &\lesssim \|Q - Q_1\|_{L^2}, \end{aligned}$$

and for $\mathbf{u} \in X$,

$$\begin{aligned} \|L^{-1}\tilde{P}R_3\mathbf{u}\|_X &\lesssim \|\tilde{P}R_3\mathbf{u}\|_Y \\ &\lesssim \|(\Re S)\sigma_1\mathbf{u}\|_Y + \|(\Im S)\sigma_2\mathbf{u}\|_Y + |\alpha| \|xQ_1\sigma_2\mathbf{u}\|_Y \\ &\quad + |\beta| \|(x\partial_x Q_1 + Q_1)\sigma_1\mathbf{u}\|_Y + \|R_3\mathbf{u}\|_{L^1} \\ &\lesssim (\|S\|_{L^2} + |\alpha| \|xQ_1\|_{L^2} + |\beta| \|x\partial_x Q_1 + Q_1\|_{L^2}) \|\mathbf{u}\|_X \\ &\lesssim \|Q - Q_1\|_{L^2} \|\mathbf{u}\|_X. \end{aligned}$$

If $\|Q - Q_1\|_{L^2}$ is sufficiently small, then $I - L^{-1}\tilde{P}R_3$ is invertible on X and

$$\|\Phi\|_X \leq \|(I - L^{-1}\tilde{P}R_3)^{-1}(L^{-1}\mathbf{R}_1 + L^{-1}\tilde{P}\mathbf{R}_2)\|_X \lesssim \|Q - Q_1\|_{L^2}.$$

Thus, we prove (32).

Next, we will prove (33). Differentiating (39) m times ($0 \leq m \leq n$), we have

$$L\partial_x^m \Phi = \partial_x^m(\mathbf{R}_1 + \mathbf{R}_2 + R_3\Phi) + [L, \partial_x^m]\Phi. \tag{42}$$

Let $\hat{P} : L^2(\mathbb{R}; \mathbb{C}^2) \rightarrow L^2(\mathbb{R}; \mathbb{C}^2) \cap \perp \ker(L)$ be another projection defined by

$$\hat{P}\mathbf{u} = \mathbf{u} - \frac{1}{\sqrt{2\pi}} \langle \mathbf{u}, \boldsymbol{\Psi}_1 \rangle_{L^2} \boldsymbol{\Psi}_1,$$

where we used $\|\boldsymbol{\Psi}_1\|_{L^2}^2 = 4 \int_0^\infty \operatorname{sech}(x) dx = 2\pi$. Since $L = L\hat{P} = \tilde{P}L\hat{P}$, Equation (42) can be rewritten as

$$(L - \tilde{P}R_3)\hat{P}\partial_x^m \boldsymbol{\Phi} = \tilde{P}\mathbf{R}_{4,m},$$

where $\mathbf{R}_{4,m} = \partial_x^m(\mathbf{R}_1 + \mathbf{R}_2) + \{[\partial_x^m, Q_1\sigma_1] + [\partial_x^m, R_3] + R_3[\partial_x^m, \hat{P}]\}\boldsymbol{\Phi}$. Note that $\hat{P}\boldsymbol{\Phi} = \boldsymbol{\Phi}$. Suppose that $\|\partial_x^l \boldsymbol{\Phi}\|_X \lesssim \|Q - Q_1\|_{H^l} + \|Q - Q_1\|_{H^l}^l$ for $0 \leq l < m \leq n$. Then by the induction hypothesis, we have

$$\|\mathbf{R}_{4,m}\|_Y \lesssim \|Q - Q_1\|_{H^m} + \|Q - Q_1\|_{H^m}^m.$$

Therefore, if $\|Q - Q_1\|_{L^2}$ is sufficiently small, then $I - L^{-1}\tilde{P}R_3$ is invertible on X and

$$\|\hat{P}\partial_x^m \boldsymbol{\Phi}\|_X \leq \|(I - L^{-1}\tilde{P}R_3)^{-1}L^{-1}\mathbf{R}_{4,m}\|_X \lesssim \|\mathbf{R}_{4,m}\|_Y \lesssim \|Q - Q_1\|_{H^m} + \|Q - Q_1\|_{H^m}^m$$

and

$$\|\partial_x^m \boldsymbol{\Phi}\|_X \leq \|\hat{P}\partial_x^m \boldsymbol{\Phi}\|_X + \|[\partial_x^m, \hat{P}]\boldsymbol{\Phi}\|_X \lesssim \|\hat{P}\partial_x^m \boldsymbol{\Phi}\|_X + \|\boldsymbol{\Phi}\|_{L^2} \lesssim \|Q - Q_1\|_{H^m} + \|Q - Q_1\|_{H^m}^m.$$

This completes the proof of Lemma 3.3. ■

4 From the Zero Solution to a 1-soliton

In this section, we will prove Theorem 1.1 by showing that a Bäcklund transformation (10) maps smooth solutions of the NLS equation in an L^2 -neighborhood of the zero solution to those in an L^2 -neighborhood of a 1-soliton.

First of all, we construct a fundamental system of solutions of the spectral problem (7) with $\eta = \frac{1}{2}$, which will be assumed throughout this section. If $q = 0$, then the fundamental system of solutions of (7) with $\eta = \frac{1}{2}$ is given by the two solutions

$$\boldsymbol{\psi}_1(x) = \begin{bmatrix} e^{x/2} \\ 0 \end{bmatrix}, \quad \boldsymbol{\psi}_2(x) = \begin{bmatrix} 0 \\ -e^{-x/2} \end{bmatrix}. \tag{43}$$

When q is small in L^2 , a fundamental system of (7) with $\eta = \frac{1}{2}$ can be found as a perturbation of the two linearly independent solutions (43).

Let us consider the following boundary value problems

$$\begin{aligned}
 \varphi_1' &= q\varphi_2, \\
 \varphi_2' &= -\bar{q}\varphi_1 - \varphi_2, \\
 \lim_{x \rightarrow \infty} \varphi_1(x) &= 1, \\
 \lim_{x \rightarrow -\infty} e^x \varphi_2(x) &= 0,
 \end{aligned} \tag{44}$$

and

$$\begin{aligned}
 \chi_1' &= \chi_1 + q\chi_2, \\
 \chi_2' &= -\bar{q}\chi_1, \\
 \lim_{x \rightarrow \infty} e^{-x} \chi_1(x) &= 0, \\
 \lim_{x \rightarrow -\infty} \chi_2(x) &= -1.
 \end{aligned} \tag{45}$$

If the boundary value problems (44) and (45) have a unique solution, then

$$\psi_1(x) = e^{x/2} \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \end{bmatrix}, \quad \psi_2(x) = e^{-x/2} \begin{bmatrix} \chi_1(x) \\ \chi_2(x) \end{bmatrix} \tag{46}$$

become linearly independent solutions of the system (7) with $\eta = \frac{1}{2}$. It follows from a standard ODE theory that every solution of the system (7) with $q \in C(\mathbb{R})$ can be written as a linear superposition of the two solutions (46).

Uniqueness of solutions of the boundary value problems (44) and (45) follows from the following lemma.

Lemma 4.1. There exists a $\delta > 0$ such that if $\|q\|_{L^2} \leq \delta$, then the boundary value problems (44) and (45) have a solution in the class

$$(\varphi_1, \varphi_2) \in L^\infty \times (L^2 \cap L^\infty), \quad (\chi_1, \chi_2) \in (L^2 \cap L^\infty) \times L^\infty.$$

Moreover, there exists a $C > 0$ such that

$$\|\varphi_1 - 1\|_{L^\infty} + \|\varphi_2\|_{L^2 \cap L^\infty} \leq C \|q\|_{L^2},$$

$$\|\chi_1\|_{L^2 \cap L^\infty} + \|\chi_2 + 1\|_{L^\infty} \leq C \|q\|_{L^2}.$$

□

Proof. Let us translate the boundary value problem (44) into a system of integral equations

$$\begin{aligned}\varphi_1(x) &= 1 - \int_x^\infty q(y)\varphi_2(y) dy =: T_1(\varphi_1, \varphi_2)(x), \\ \varphi_2(x) &= - \int_{-\infty}^x e^{-(x-y)} \overline{q(y)} \varphi_1(y) dy =: T_2(\varphi_1, \varphi_2)(x).\end{aligned}\tag{47}$$

Let us introduce a Banach space $Z := L^\infty \times (L^\infty \cap L^2)$ equipped with the norm

$$\|(u_1, u_2)\|_Z = \|u_1\|_{L^\infty} + \|u_2\|_{L^\infty \cap L^2}.$$

In order to find a solution of the system (47), we will show that $T = (T_1, T_2) : Z \rightarrow Z$ is a contraction mapping.

Using the Schwarz inequality and Young's inequality, we have for (φ_1, φ_2) and $(\tilde{\varphi}_1, \tilde{\varphi}_2) \in Z$,

$$\|T_1(\varphi_1, \varphi_2) - T_1(\tilde{\varphi}_1, \tilde{\varphi}_2)\|_{L^\infty} = \sup_{x \in \mathbb{R}} \left| \int_x^\infty q(y)(\varphi_2(y) - \tilde{\varphi}_2(y)) dy \right| \leq \|q\|_{L^2} \|\varphi_2 - \tilde{\varphi}_2\|_{L^2},$$

and

$$\begin{aligned}\|T_2(\varphi_1, \varphi_2) - T_2(\tilde{\varphi}_1, \tilde{\varphi}_2)\|_{L^2 \cap L^\infty} &= \left\| \int_{-\infty}^x e^{-(x-y)} \overline{q(y)} (\varphi_1(y) - \tilde{\varphi}_1(y)) dy \right\|_{L^2 \cap L^\infty} \\ &\leq \|e^{-x}\|_{L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)} \|q\|_{L^2} \|\varphi_1 - \tilde{\varphi}_1\|_{L^\infty} \\ &\leq \|q\|_{L^2} \|\varphi_1 - \tilde{\varphi}_1\|_{L^\infty}.\end{aligned}$$

If $\|q\|_{L^2}$ is sufficiently small, then $T = (T_1, T_2)$ is a contraction mapping on Z . Therefore, $T = (T_1, T_2)$ has a unique fixed point $(\varphi_1, \varphi_2) \in Z$ and

$$\begin{aligned}\|\varphi_1 - 1\|_{L^\infty} + \|\varphi_2\|_{L^\infty \cap L^2} &= \|T(\varphi_1, \varphi_2) - T(0, 0)\|_Z \leq \|q\|_{L^2} \|(\varphi_1, \varphi_2)\|_Z \\ &\leq \|q\|_{L^2} (1 + \|\varphi_1 - 1\|_{L^\infty} + \|\varphi_2\|_{L^\infty \cap L^2}).\end{aligned}$$

Thus there exists $C > 0$ such that

$$\|\varphi_1 - 1\|_{L^\infty} + \|\varphi_2\|_{L^\infty} + \|\varphi_2\|_{L^2} \leq C \|q\|_{L^2}.$$

Finally, we confirm the boundary conditions in the system (44). By (47) and the fact that $q \in L^2$ and $\varphi_2 \in L^2$, we have $\lim_{x \rightarrow \infty} \varphi_1(x) = 1$. Since φ_2 is bounded and continuous, it is clear that $\lim_{x \rightarrow -\infty} e^x \varphi_2(x) = 0$.

In the same way, we can prove that the boundary value problem (45) has a unique solution $(\chi_1, \chi_2) \in \tilde{Z} := (L^\infty \cap L^2) \times L^\infty$ satisfying

$$\|\chi_1\|_{L^2 \cap L^\infty} + \|\chi_2 + 1\|_{L^\infty} \leq C \|q\|_{L^2}$$

and the boundary conditions $\lim_{x \rightarrow \infty} e^{-x} \chi_1(x) = 0$ and $\lim_{x \rightarrow -\infty} \chi_2(x) = -1$. ■

Next we will consider the time evolution of (ψ_1, ψ_2) . We will evolve (ψ_1, ψ_2) by the linear time evolution (8) for initial data $(\psi_1(0, x), \psi_2(0, x))$ satisfying the spectral problem (7) at $t = 0$ assuming that $q(t, x)$ is a solution of the NLS equation.

Suppose that $\varphi(t, x) = {}^t(\varphi_1(t, x), \varphi_2(t, x))$ satisfies the boundary value problem (44) at $t = 0$ with $q = q(0, x)$ and that $e^{x/2} \varphi(t, x)$ satisfies (8) for every $t \geq 0$ and $x \in \mathbb{R}$. Then the linear time evolution of $\varphi(t, x)$ can be written in the matrix form

$$\partial_t \varphi(t, x) = A(t, x) \varphi(t, x), \quad A(t, x) = \begin{bmatrix} a(t, x) & b(t, x) \\ c(t, x) & -a(t, x) \end{bmatrix}, \tag{48}$$

where

$$a = i \left(\frac{1}{2} + |q|^2 \right), \quad b = i(\partial_x q + q), \quad c = i(\partial_x \bar{q} - \bar{q}).$$

Similarly, let $\chi(t, x) = {}^t(\chi_1(t, x), \chi_2(t, x))$ be a solution of the boundary value problem (45) at $t = 0$ with $q = q(0, x)$ whose time evolution is written in the same matrix form (48) for $\chi(t, x)$. Solutions $\varphi(t, x)$ and $\chi(t, x)$ are characterized by the following lemma.

Lemma 4.2. Suppose that $q \in C(\mathbb{R}; H^3(\mathbb{R}))$ is a solution of the NLS equation and that $\|q(0, \cdot)\|_{L^2}$ is sufficiently small. Let $\varphi = {}^t(\varphi_1, \varphi_2)$ and $\chi = {}^t(\chi_1, \chi_2)$ be solutions of the linear equation (48) starting with the initial data given by solutions of the boundary value problems (44) and (45), respectively, with $q = q(0, x)$. Then $\partial_x^i \varphi \in C(\mathbb{R}; Z)$ and $\partial_x^i \chi \in C(\mathbb{R}; \tilde{Z})$ for $0 \leq i \leq 3$ and for every $t \in \mathbb{R}$,

$$\begin{aligned} \partial_x \varphi_1(t, x) &= q(t, x) \varphi_2(t, x), \\ \partial_x \varphi_2(t, x) &= -\overline{q(t, x)} \varphi_1(t, x) - \varphi_2(t, x), \\ \lim_{x \rightarrow \infty} \varphi_1(t, x) &= e^{it/2}, \\ \lim_{x \rightarrow -\infty} e^x \varphi_2(t, x) &= 0, \end{aligned} \tag{49}$$

and

$$\begin{aligned} \partial_x \chi_1(t, x) &= \chi_1(t, x) + q(t, x)\chi_2(t, x), \\ \partial_x \chi_2(t, x) &= -\overline{q(t, x)}\chi_1(t, x), \\ \lim_{x \rightarrow \infty} e^{-x} \chi_1(t, x) &= 0, \\ \lim_{x \rightarrow -\infty} \chi_2(t, x) &= -e^{-it/2}. \end{aligned} \tag{50}$$

□

Proof. First, we will prove that the boundary value problem (49) holds for every $t \in \mathbb{R}$.

The coefficient matrix $A(t, x)$ of the system (48) is continuous in (t, x) and C^1 in x since $q(t, x) \in C(\mathbb{R}; H^3(\mathbb{R}))$. By a bootstrapping argument for the system (44), Lemma 4.1 implies that $\varphi_1(0, x)$ and $\varphi_2(0, x)$ are C^1 in x . Solving the Cauchy problem for the linear evolution equation (48), we find that $\varphi_1(t, x)$ and $\varphi_2(t, x)$ are in $C^1(\mathbb{R} \times \mathbb{R})$. By a bootstrapping argument for the systems (44) and (48), we conclude that $\partial_x \partial_t \varphi(t, x)$ and $\partial_t \partial_x \varphi(t, x)$ are in $C(\mathbb{R} \times \mathbb{R}; \mathbb{R}^2)$ and thus $\partial_x \partial_t \varphi(t, x) = \partial_t \partial_x \varphi(t, x)$.

Let

$$B(t, x) = \begin{bmatrix} 0 & q(t, x) \\ -\overline{q(t, x)} & -1 \end{bmatrix}, \quad \mathbf{F}(t, x) = \partial_x \varphi(t, x) - B(t, x)\varphi(t, x).$$

Since q is a solution of the NLS equation, the matrices A and B satisfy the Zakharov–Shabat compatibility condition

$$\partial_x A - \partial_t B + [A, B] = 0. \tag{51}$$

As a result, we obtain

$$\begin{aligned} \partial_t \mathbf{F} &= \partial_t \partial_x \varphi - (\partial_t B)\varphi - B \partial_t \varphi = \partial_x(A\varphi) - (\partial_t B)\varphi - B A \varphi \\ &= (\partial_x A + [A, B] - \partial_t B)\varphi + A\mathbf{F} = A\mathbf{F}. \end{aligned}$$

Applying Gronwall’s equality, we see that for any $T > 0$, there exists a constant $C(T)$ such that

$$|\mathbf{F}(t)| \leq C(T)|\mathbf{F}(0)|, \quad t \in [-T, T].$$

Since $\mathbf{F}(0) = \mathbf{0}$ by the assumption, it follows that $\mathbf{F}(t) = \mathbf{0}$ for every $t \in \mathbb{R}$. Thus, we prove the differential part of the system (49).

Next we will prove $\varphi_1(t, \cdot) \in L^\infty(\mathbb{R})$ and $\varphi_2(t, \cdot) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for every $t \in \mathbb{R}$. By the linear evolution (48), we have

$$\begin{aligned} |\partial_t(|\varphi_1(t, x)|^2 + |\varphi_2(t, x)|^2)| &= 4|\Im q(t, x)\overline{\varphi_1(t, x)\varphi_2(t, x)}| \\ &\leq 2\|q(t, \cdot)\|_{L^\infty}(|\varphi_1(t, x)|^2 + |\varphi_2(t, x)|^2). \end{aligned}$$

Applying Gronwall’s inequality again, we have

$$|\varphi_1(t, x)|^2 + |\varphi_2(t, x)|^2 \leq e^{\alpha|t|}(|\varphi_1(0, x)|^2 + |\varphi_2(0, x)|^2), \quad t \in \mathbb{R}, \tag{52}$$

where $\alpha = 2 \sup_{(t,x) \in \mathbb{R} \times \mathbb{R}} |q(t, x)|$. Since $\varphi(0, \cdot) \in L^\infty(\mathbb{R}; \mathbb{C}^2)$, bound (52) shows that $\varphi(t, \cdot) \in L^\infty(\mathbb{R}; \mathbb{C}^2)$ for any $t \in \mathbb{R}$.

Using the linear system (48) again, we have

$$\begin{aligned} \partial_t|\varphi_2(t, x)|^2 &\leq 2|\partial_x q(t, x) - q(t, x)\varphi_1(t, x)| |\varphi_2(t, x)| \\ &\leq |\varphi_2(t, x)|^2 + |\varphi_1(t, x)|^2 |\partial_x q(t, x) - q(t, x)|^2. \end{aligned}$$

By Gronwall’s inequality, for any $T > 0$ there exists a $C(T) > 0$ such that

$$|\varphi_2(t, x)|^2 \leq |\varphi_2(s, x)|^2 + C(T) \int_s^t |\partial_x q(\tau, x) - q(\tau, x)|^2 d\tau, \quad 0 \leq s \leq t \leq T, \quad x \in \mathbb{R}.$$

Therefore, we have

$$\|\varphi_2(t, \cdot)\|_{L^2}^2 \leq \|\varphi_2(s, \cdot)\|_{L^2}^2 + C(T) \int_s^t \|\partial_x q(\tau, \cdot) - q(\tau, \cdot)\|_{L^2}^2 d\tau. \tag{53}$$

Since $\varphi_2(0, \cdot) \in L^2(\mathbb{R})$, bound (53) shows that $\varphi_2(t, \cdot) \in L^2(\mathbb{R})$ for every $t \in \mathbb{R}$ and $\|\varphi_2(t)\|_{L^2}$ is continuous in t . Since $A(t, \cdot) \in C(\mathbb{R}; H^2(\mathbb{R}))$ and $\|\varphi_1(t)\|_{L^\infty}$ and $\|\varphi_2(t)\|_{L^2 \cap L^\infty}$ are bounded locally in time, the linear system (48) implies that $\varphi_1(t, \cdot)$ and $\varphi_2(t, \cdot)$ are continuous in $L^\infty(\mathbb{R})$ and thus $\varphi_2(t, \cdot)$ is continuous in $L^2(\mathbb{R})$. Using the fact that $\varphi \in C(\mathbb{R}; Z)$ and a bootstrapping argument for the system (44), we have $\partial_x^i \varphi \in C(\mathbb{R}; Z)$ for $1 \leq i \leq 3$.

It remains to prove the boundary conditions of the system (49). Since $\varphi_2(t, x)$ is bounded and continuous in x for every fixed $t \in \mathbb{R}$, we have $\lim_{x \rightarrow -\infty} e^x \varphi_2(t, x) = 0$. By a variation of constants formula, we have

$$\varphi(t, x) = e^{i\sigma_3 t/2} \varphi(0, x) + \int_0^t e^{i\sigma_3(t-s)/2} A_1(s, x) \varphi(s, x) ds, \tag{54}$$

where $A_1(t, x) = A(t, x) - i\sigma_3/2$. By the assumption that $q \in C(\mathbb{R}; H^3(\mathbb{R}))$, we have

$$\sup_{x \in \mathbb{R}} \sup_{0 \leq s \leq t} |A_1(s, x)| < \infty \quad \text{and} \quad \lim_{x \rightarrow \pm\infty} A_1(s, x) = 0.$$

Applying Lebesgue’s dominated convergence theorem to the integral equation (54), we get

$$\lim_{x \rightarrow \infty} |\varphi(t, x) - e^{i\sigma_3 t/2} \varphi(0, x)| = 0.$$

Combining the above with the boundary condition $\lim_{x \rightarrow \infty} \varphi_1(0, x) = 1$, we obtain

$$\lim_{x \rightarrow \infty} \varphi_1(t, x) = e^{it/2}.$$

Properties of χ and the boundary value problem (50) can be proved in the same way as properties of φ and the boundary value problem (49). ■

Now, we have time global estimates of solutions to the linear evolution equation (48).

Lemma 4.3. Let $q \in C(\mathbb{R}; H^3(\mathbb{R}))$ be a solution of the NLS equation. Suppose that $\varphi(t, x)$ and $\chi(t, x)$ are solutions of the linear evolution equation (48) such that $\varphi(0, x) \in Z$ and $\chi(0, x) \in \tilde{Z}$, respectively. There exist positive constants ε and C such that if $\|q(0, \cdot)\|_{L^2} \leq \varepsilon$, then for every $t \in \mathbb{R}$,

$$\|\varphi_1(t, \cdot) - e^{it/2}\|_{L^\infty} + \|\varphi_2(t, \cdot)\|_{L^2 \cap L^\infty} \leq C \|q(0, \cdot)\|_{L^2}, \tag{55}$$

$$\|\chi_1(t, \cdot)\|_{L^2 \cap L^\infty} + \|\chi_2(t, \cdot) + e^{-it/2}\|_{L^\infty} \leq C \|q(0, \cdot)\|_{L^2}. \tag{56}$$

□

Proof. Since $\varphi(t, \cdot) \in Z$ and $\chi(t, \cdot) \in \tilde{Z}$ for each $t \in \mathbb{R}$ and satisfy the boundary value problem (49) and (50), Lemma 4.3 can be proved in exactly the same way as Lemma 4.1. ■

Our next result shows that the Bäcklund transformation (10) with $\eta = \frac{1}{2}$ generates a new solution Q in a L^2 -neighborhood of the 1-soliton $e^{i(t+\theta)} Q_1(x - \gamma)$, where $Q_1(x) = \text{sech}(x)$.

Lemma 4.4. Let ε be a sufficiently small positive number. Let $q(t, x) \in C(\mathbb{R}; H^3(\mathbb{R}))$ be a solution of the NLS equation such that $\|q(0, \cdot)\|_{L^2} \leq \varepsilon$ and let

$$\begin{aligned} \psi_1(t, x) &= c_1 e^{x/2} \varphi_1(t, x) + c_2 e^{-x/2} \chi_1(t, x), \\ \psi_2(t, x) &= c_1 e^{x/2} \varphi_2(t, x) + c_2 e^{-x/2} \chi_2(t, x), \end{aligned} \tag{57}$$

where $c_1 = a e^{(\gamma+i\theta)/2}$, $c_2 = a e^{-(\gamma+i\theta)/2}$ and $a \neq 0$, $\gamma \in \mathbb{R}$, $\theta \in \mathbb{R}$ are constants. Let

$$Q(t, x) = -q(t, x) - \frac{2\psi_1(t, x)\overline{\psi_2(t, x)}}{|\psi_1(t, x)|^2 + |\psi_2(t, x)|^2}, \tag{58}$$

Then $Q \in C(\mathbb{R}; H^3(\mathbb{R}))$ and $Q(t, x)$ is a solution of the NLS equation. Moreover, there is an ε -dependent constant $C > 0$ such that

$$\sup_{t \in \mathbb{R}} \|Q(t, \cdot) - e^{i(t+\theta)} Q_1(\cdot - \gamma)\|_{L^2} \leq C \|q(0, \cdot)\|_{L^2}. \tag{59}$$

□

Proof. Since ψ in (57) solve the Lax system (7) and (8), the Bäcklund transformation (58) implies that if $q(t, x)$ is a solution of the NLS equation, so is $Q(t, x)$. Let us still give a rigorous proof of this fact for the sake of self-containedness. Let

$$\Psi_1(t, x) := \frac{\overline{\psi_2(t, x)}}{|\psi_1(t, x)|^2 + |\psi_2(t, x)|^2}, \quad \Psi_2(t, x) := \frac{\overline{\psi_1(t, x)}}{|\psi_1(t, x)|^2 + |\psi_2(t, x)|^2}.$$

Thanks to (55) and (56), $\psi \neq 0$ for any $(t, x) \in \mathbb{R}^2$, hence Q and Ψ are well defined for every $t \in \mathbb{R}$. Since $\partial_x^i \varphi \in C(\mathbb{R}; Z)$ and $\partial_x^i \chi \in C(\mathbb{R}; \tilde{Z})$ for any $0 \leq i \leq 3$ and

$$q \in C(\mathbb{R}; H^3(\mathbb{R})) \cap C^1(\mathbb{R}; H^1(\mathbb{R})),$$

it follows from the linear evolution equation (48) that Ψ is of the class C^1 and $\partial_x \partial_t \Psi$ and $\partial_t \partial_x \Psi$ are continuous. Moreover, $Q(t, \cdot) \in C(\mathbb{R}; H^3(\mathbb{R}))$.

By a straightforward but lengthy computation, we show that

$$\partial_x \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & Q \\ -\bar{Q} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \tag{60}$$

$$\partial_t \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = i \begin{bmatrix} \frac{1}{2} + |Q|^2 & \partial_x Q + Q \\ \partial_x \bar{Q} - \bar{Q} & -\frac{1}{2} - |Q|^2 \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}. \tag{61}$$

It is clear that $\Psi(x, t) \neq 0$ for every $(t, x) \in \mathbb{R} \times \mathbb{R}$. Combining (60) and (61) and the compatibility condition $\partial_t \partial_x \Psi = \partial_x \partial_t \Psi$, we obtain $iQ_t + Q_{xx} + 2|Q|^2 Q = 0$.

Now we will show the bound (59). Let

$$\begin{aligned} R(t, x) &:= -Q(t, x) - q(t, x) \\ &= \frac{2(c_1 \varphi_1(t, x) + c_2 e^{-x} \chi_1(t, x)) \overline{(c_1 e^x \varphi_2(t, x) + c_2 \chi_2(t, x))}}{|c_1 \varphi_1(t, x) + c_2 e^{-x} \chi_1(t, x)|^2 e^x + |c_1 e^x \varphi_2(t, x) + c_2 \chi_2(t, x)|^2 e^{-x}} = \frac{2R_1}{R_2}, \end{aligned} \tag{62}$$

where

$$\begin{aligned} R_1 &:= e^{x+\gamma} \varphi_1(t, x) \overline{\varphi_2(t, x)} + e^{-x-\gamma} \chi_1(t, x) \overline{\chi_2(t, x)} + e^{i\theta} \varphi_1(t, x) \overline{\chi_2(t, x)} + e^{-i\theta} \chi_1(t, x) \overline{\varphi_2(t, x)}, \\ R_2 &:= e^{x+\gamma} (|\varphi_1(t, x)|^2 + |\varphi_2(t, x)|^2) + e^{-x-\gamma} (|\chi_2(t, x)|^2 + |\chi_1(t, x)|^2) \\ &\quad + 2\Re[e^{i\theta} (\varphi_1(t, x) \overline{\chi_1(t, x)} + \varphi_2(t, x) \overline{\chi_2(t, x)})]. \end{aligned}$$

For $x \geq -\gamma$,

$$R = \frac{2 e^{-x-\gamma+i\theta} \varphi_1(t, x) \overline{\chi_2(t, x)}}{|\varphi_1(t, x)|^2 + e^{-2(x+\gamma)} |\chi_2(t, x)|^2} + \mathcal{O}(|\varphi_2(t, x)| + e^{-x-\gamma} |\chi_1(t, x)|) \tag{63}$$

since $|\varphi_1|, |\chi_2| \sim 1$ and $\varphi_2, \chi_1 \sim 0$ by Lemma 4.3. Similarly, for $x \leq -\gamma$,

$$R = \frac{2 e^{x+\gamma+i\theta} \varphi_1(t, x) \overline{\chi_2(t, x)}}{|\chi_2(t, x)|^2 + e^{2(x+\gamma)} |\varphi_1(t, x)|^2} + \mathcal{O}(|\chi_1(t, x)| + e^{x+\gamma} |\varphi_2(t, x)|). \tag{64}$$

Combining (63) and (64), we get

$$|R(t, x) + e^{i(t+\theta)} \operatorname{sech}(x + \gamma)| \leq C e^{-|x+\gamma|} (\|\varphi_1 - e^{it/2}\|_{L^\infty} + \|\chi_2 + e^{-it/2}\|_{L^\infty}) + C (|\varphi_2(t, x)| + |\chi_1(t, x)|),$$

where C is a constant depending only on $\|q(0, \cdot)\|_{L^2}$. Thus, by Lemma 4.3, there is $C > 0$ such that

$$\sup_{t \in \mathbb{R}} \|R(t, \cdot) + e^{i(t+\theta)} \operatorname{sech}(\cdot + \gamma)\|_{L^2} \leq C \|q(0, \cdot)\|_{L^2}.$$

Combining the above with the L^2 -conservation law $\|q(t, \cdot)\|_{L^2} = \|q(0, \cdot)\|_{L^2}$ of NLS, we conclude that

$$\|Q(t, \cdot) - e^{i(t+\theta)} \operatorname{sech}(\cdot + \gamma)\|_{L^2} \leq \|R(t, \cdot) + e^{i(t+\theta)} \operatorname{sech}(\cdot + \gamma)\|_{L^2} + \|q(t, \cdot)\|_{L^2} \lesssim \|q(0, \cdot)\|_{L^2}.$$

This completes the proof of the bound (59). ■

Remark 4.1. To prove Lemmas 4.2 and 4.3, we require H^3 -regularity of a solution q to the NLS equation. The high regularity assumption on $q(t, x)$ is only used to prove qualitative properties on a solution (ψ_1, ψ_2) of the Lax system (7) and (8) and has not been used quantitatively to prove the bounds (55) and (56). This is the reason why we can prove Theorem 1.1 for any initial data satisfying $\|u(0, \cdot) - Q_1\|_{L^2} \leq \varepsilon$ by using an approximation argument. □

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Thanks to the scaling invariance of NLS, we may choose $k = 1$, that is, $Q_k = Q_1$.

Step 1: First, we will show (5) assuming that $u(0, \cdot) \in H^3(\mathbb{R})$. Lemmas 3.1 and 3.4 imply that if $Q = u(0, \cdot) \in H^3(\mathbb{R})$ and $\|u(0, \cdot) - Q_1\|_{L^2}$ are sufficiently small, then there exists a solution Ψ of the system (17) with $\eta = (k + iv)/2$ satisfying

$$\exists C > 0 : |k - 1| + |v| \leq C \|u(0, \cdot) - Q_1\|_{L^2}.$$

Letting

$$q_0(x) = -u(0, x) - \frac{2k\Psi_1(x)\overline{\Psi_2(x)}}{|\Psi_1(x)|^2 + |\Psi_2(x)|^2}$$

and

$$\psi_{1,0}(x) = \frac{\overline{\Psi_2(x)}}{|\Psi_1(x)|^2 + |\Psi_2(x)|^2}, \quad \psi_{2,0}(x) = \frac{\Psi_1(x)}{|\Psi_1(x)|^2 + |\Psi_2(x)|^2},$$

we see that $(\psi_{1,0}, \psi_{2,0})$ is a solution of the system (7) with $q = q_0$. We may assume $k = 1$ and $v = 0$ without loss of generality thanks to the change of variables in Remark 3.2 and

the invariance of the NLS equation under the transformation

$$\lambda \tilde{q}(\lambda^2(t + t_0), \lambda(x + x_0)) = e^{i(vx/2 - v^2t/4)} q(t, x - vt),$$

where $\lambda > 0$ and $t_0, x_0, v \in \mathbb{R}$ are constants.

By the linear superposition principle, we can find complex constants c_1 and c_2 satisfying

$$\psi_0 = {}^t(\psi_{1,0}, \psi_{2,0}) = c_1 e^{x/2} \varphi(0, x) + c_2 e^{-x/2} \chi(0, x).$$

Let $q(t, x)$ be a solution of the NLS equation with $q(0, x) = q_0(x)$ and let

$$\psi(t, x) = {}^t(\psi_1(t, x), \psi_2(t, x)) = c_1 e^{x/2} \varphi(t, x) + c_2 e^{-x/2} \chi(t, x).$$

Lemma 4.4 implies that $\psi(t, x)$ is a solution of the Lax system (7) and (8) and that $Q(t, x)$ defined by (58) satisfies the stability result (59). Since $Q(t, x)$ is a solution of the NLS equation in the class $C(\mathbb{R}; H^3(\mathbb{R}))$ and

$$\begin{aligned} Q(0, x) &= -q(0, x) - \frac{2\psi_1(0, x)\overline{\psi_2(0, x)}}{|\psi_1(0, x)|^2 + |\psi_2(0, x)|^2} \\ &= -q_0(x) - \frac{2\Psi_1(x)\overline{\Psi_2(x)}}{|\Psi_1(x)|^2 + |\Psi_2(x)|^2} = u(0, x) \end{aligned}$$

by the definition, we have $Q(t, x) = u(t, x)$.

Step 2: Next, we prove (5) for any $u(0, \cdot)$ which is sufficiently close to Q_1 in $L^2(\mathbb{R})$. Let $\delta_1 = \|u(0, \cdot) - Q_1\|_{L^2}$. Let $u_{n,0} \in H^3(\mathbb{R})$ ($n \in \mathbb{N}$) be a sequence such that

$$\lim_{n \rightarrow \infty} \|u_{n,0} - u(0, \cdot)\|_{L^2} = 0,$$

and let $u_n(t, x)$ be a solution of the NLS equation with $u_n(0, x) = u_{0,n}(x)$. In view of the first step, we see that there exist a positive constant C and real numbers k_n, v_n, t_n, x_n ($n \in \mathbb{N}$) such that

$$\sup_{t \in \mathbb{R}} \|u_n(t + t_n, \cdot + x_n) - Q_{k_n, v_n}\|_{L^2} + |k_n - 1| + |v_n| + |t_n| + |x_n| \leq C \|u_{0,n} - Q_1\|_{L^2}. \quad (65)$$

By (65), there exist k_0, v_0, t_0 , and x_0 and subsequences of $\{k_n\}$, $\{v_n\}$, $\{t_n\}$, and $\{x_n\}$ such that

$$\lim_{j \rightarrow \infty} k_{n_j} = k_0, \quad \lim_{j \rightarrow \infty} v_{n_j} = v_0, \quad \lim_{j \rightarrow \infty} t_{n_j} = t_0, \quad \lim_{j \rightarrow \infty} x_{n_j} = x_0. \quad (66)$$

It follows from the main theorem in Tsutsumi [28] (see also [21, Theorem 5.2]) that the NLS equation is L^2 -well-posed in the class of solutions (4). Therefore, combining (65) and (66), we obtain (5). Thus, we complete the proof. ■

5 Discussions

We finish this article with three observations which give way for further work.

1. The Cauchy problem associated with the generalized nonlinear Schrödinger equation (6) is well studied in the context of dispersive decay of small-norm solutions. Since the decay rate of the $L^\infty - L^1$ norm for the semi-group

$$S(t) := e^{-it(-\partial_x^2 + V(x))}, \quad t > 0$$

is $\mathcal{O}(t^{-1/2})$, the nonlinear term $\|u(t, \cdot)\|_{L^\infty}^{2p}$ is absolute integrable if $p > 1$. The case $p = 1$ of the cubic NLS equation is critical with respect to this dispersive decay in the $L^\infty - L^1$ norm. The scattering theory for small solutions in the supercritical case $p > 1$ was studied long ago [4, 11, 16, 27]. The scattering theory was extended to the critical ($p = 1$) and subcritical ($p = \frac{1}{2}$) cases by Hayashi and Naumkin [14, 15] using more specialized properties of the fundamental solutions generated by the semi-group $S(t)$.

In particular, Hayashi and Naumkin proved that if $q_0 \in H^1(\mathbb{R}) \cap L_1^2(\mathbb{R})$ and $\|q_0\|_{H^1} + \|q_0\|_{L_1^2} \leq \varepsilon$ for sufficiently small $\varepsilon > 0$, then there exists a unique global solution $q(t, \cdot) \in C(\mathbb{R}; H^1(\mathbb{R}) \cap L_1^2(\mathbb{R}))$ of the NLS equation with $q(0) = q_0$ such that

$$\exists C > 0 : \|q(t, \cdot)\|_{H^1} \leq C\varepsilon, \quad \|q(t, \cdot)\|_{L^\infty} \leq C\varepsilon(1 + |t|)^{-1/2}, \quad t \in \mathbb{R}_+. \tag{67}$$

Space $L_1^2(\mathbb{R})$ is needed to control an initially small norm $\|q_0\|_{L^1}$. Recall from inverse scattering (see, e.g. [1]) that if $\|q_0\|_{L^1}$ is small, then the spectral problem (7) admits no isolated eigenvalue and produces no soliton in $q(t, \cdot)$ as $t \rightarrow \infty$. In other words, $q(t, \cdot)$ contains only the dispersive radiation part. Unfortunately, the norm $\|q(t, \cdot)\|_{L_1^2}$ (and the norm $\|q(t, \cdot)\|_{L^1}$) may grow as $t \rightarrow \infty$. Indeed, it is shown in [14] that there exists a small $\varepsilon > 0$ such that

$$\|(x + 2it\partial_x)q(t, \cdot)\|_{L^2} \lesssim (1 + |t|)^\varepsilon,$$

which implies that $\|q(t, \cdot)\|_{L_1^2} \geq C(1 + |t|)$ as $t \rightarrow \infty$ for some $C > 0$.

The possible growth of $\|q(t, \cdot)\|_{L^1}$ is an obstruction on the use of the Bäcklund transformation in our approach. If we can prove that the Bäcklund transformation provides an isomorphism between a ball $B_\delta(0) \ni q$ of small radius $\delta > 0$ centered at

0 in the energy space $H^1(\mathbb{R})$ and a ball $B_\varepsilon(Q_1) \ni Q$ of small radius $\varepsilon > 0$ centered at $Q_1(x) = \operatorname{sech}(x)$ in the same energy space $H^1(\mathbb{R})$ such that

$$\exists C > 0 : \|Q - Q_1\|_{L^\infty} \leq C \|q\|_{L^\infty},$$

then the asymptotic stability of 1-solitons holds in the following sense: There exist positive constants C and ε such that if $u(t, \cdot) \in C(\mathbb{R}_+, H^1(\mathbb{R}))$ is a solution of the NLS equation with $u(0) = u_0$ and $\|u_0 - Q_1\|_{H^1 \cap L^2_1} \leq \varepsilon$, then there exist constants $k \in \mathbb{R}$ and $v \in \mathbb{R}$ such that

$$|k - 1| \leq C\varepsilon, \quad |v| \leq C\varepsilon, \quad \inf_{(t_0, x_0) \in \mathbb{R}^2} \|u(t, \cdot) - Q_{k,v}(t - t_0, \cdot - x_0)\|_{H^1} \leq C \|u_0 - Q_1\|_{H^1 \cap L^2_1} \quad (68)$$

and

$$\lim_{t \rightarrow \infty} \|u(t, \cdot) - Q_{k,v}(t - t'_0, \cdot - x'_0)\|_{L^\infty} = 0, \quad (69)$$

where (t'_0, x'_0) are optimal values from the infimum in (68).

Unfortunately, unless $\|q\|_{L^1}$ is assumed to be small, we cannot prove the analog of Lemma 4.1 under the assumption of small $\|q\|_{L^\infty}$. The best bound we can derive is

$$\begin{aligned} \|\varphi_1 - 1\|_{L^\infty} + \|\varphi_2\|_{L^2} &\leq C \|q\|_{L^2}, & \|\varphi_2\|_{L^\infty} &\leq C \|q\|_{L^\infty}, \\ \|\chi_1\|_{L^2} + \|\chi_2 + 1\|_{L^\infty} &\leq C \|q\|_{L^2}, & \|\chi_1\|_{L^\infty} &\leq C \|q\|_{L^\infty}. \end{aligned}$$

This is good to control $\|Q - Q_1\|_{L^\infty((-\infty, -x_0) \cup (x_0, \infty))}$ in terms of $\|q\|_{L^\infty}$ for sufficiently large $x_0 > 0$ but it is not sufficient to control the L^∞ -norm over $(-x_0, x_0)$. More detailed analysis near the soliton core is needed and the asymptotic stability of 1-solitons in the cubic NLS equation is left as an open problem.

2. Another interesting development is a connection between the NLS equation and the integrable Landau–Lifshitz (LL) model

$$\mathbf{u}_t = \mathbf{u} \times \mathbf{u}_{xx}, \quad (\text{LL})$$

where $\mathbf{u}(t, x) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{S}^2$ such that $\mathbf{u} \cdot \mathbf{u} = 1$. A Bäcklund transformation which connects NLS and LL equations is called the Hasimoto transformation [13, 32]. The Hasimoto transformation can potentially be useful to deduce L^2 -orbital stability of 1-solitons of the NLS equation from H^1 -orbital stability of the domain wall solutions of the LL equation and H^1 -asymptotic stability of 1-solitons of the NLS equation from H^2 -asymptotic stability of domain wall solutions of the LL equation. More studies are

needed to see if our results can be deduced from the corresponding results on the LL equation using the Hasimoto transformation.

3. Our approach to employ the Bäcklund transformation for the proof of L^2 -orbital stability of solitary waves can be used to other nonlinear evolution equations integrable by the inverse scattering transform method. In particular, we expect it to work for systems where orbital stability of solitary waves in energy space cannot be deduced by standard methods [12]. Nonlinear Dirac equations in one dimension and Davey–Stewartson equations in two dimensions are possible examples for applications of our technique. These examples are left for further studies.

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