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Inertia law for spectral stability of solitary waves in coupled nonlinear Schrödinger equations

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Spectral stability analysis for solitary waves is developed in the context of the Hamiltonian system of coupled nonlinear Schrödinger equations. The linear eigenvalue problem for a non-self-adjoint operator is studied with two self-adjoint matrix Schrödinger operators. Sharp bounds on the number and type of unstable eigenvalues in the spectral problem are found from the inertia law for quadratic forms, associated with the two self-adjoint operators. Symmetry-breaking stability analysis is also developed with the same method.

Keywords: solitary waves; spectral stability; eigenvalues; matrix Schrödinger operators; inertia law for quadratic forms

1. The problem

This paper addresses spectral stability of solitary waves in the system of N coupled nonlinear Schrödinger (NLS) equations,

$$\mathrm{i}\frac{\partial \psi_n}{\partial z} + d_n \frac{\partial^2 \psi_n}{\partial x^2} + f_n(|\psi_1|^2, \ldots, |\psi_N|^2)\psi_n = 0, \quad n = 1, \ldots, N, \tag{1.1}$$

where $\psi_n(z,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$, $f_n : \mathbb{R}^N \to \mathbb{R}$ and $d_n \in \mathbb{R}$. We assume that $d_n > 0$, $f_n(0, \ldots, 0) = 0$, $n = 1, \ldots, N$ and

$$\frac{\partial f_n}{\partial |\psi_m|^2} = \frac{\partial f_m}{\partial |\psi_n|^2}, \quad n, m = 1, \ldots, N. \tag{1.2}$$

The system (1.1) has the following properties.

(i) The linear spectrum of (1.1) with $f_n \equiv 0$ is uncoupled:

$$\psi_n(z,x) = \int_{-\infty}^{\infty} \alpha_n(k_n) \mathrm{e}^{\mathrm{i}(k_n x + \omega_n(k_n) z)} \, \mathrm{d}k_n, \tag{1.3}$$

where $\omega_n = -d_n k_n^2 \leqslant 0$.

(ii) Any solution of (1.1) is invariant with respect to N phase rotations:

$$\psi_n(z,x) \mapsto \mathrm{e}^{\mathrm{i}\theta_n} \psi_n(z,x), \quad \theta_n \in \mathbb{R}, \; n = 1, \ldots, N, \tag{1.4}$$

which are associated with N conserved charge functionals,

$$Q_n[\boldsymbol{\psi}] = \int_{\mathbb{R}} |\psi_n|^2 \, \mathrm{d}x, \quad \boldsymbol{\psi} \in L^2(\mathbb{R}). \tag{1.5}$$

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(iii) Any solution of (1.1) is invariant with respect to space translation:

$$\psi_n(z,x) \mapsto \psi_n(z,x-s), \quad s \in \mathbb{R}. \tag{1.6}$$

(iv) Any solution of (1.1) is invariant with respect to Galileo translation:

$$\psi_n(z,x) \mapsto \psi_n(z,x-2vz)\mathrm{e}^{\mathrm{i}d_n^{-1}(vx-v^2z)}, \quad v \in \mathbb{R}. \tag{1.7}$$

(v) Under the condition (1.2), the system (1.1) conserves the Hamiltonian:

$$H[\boldsymbol{\psi}] = \int_{\mathbb{R}} \left[\sum_{n=1}^{N} d_n \left|\frac{\partial \psi_n}{\partial x}\right|^2 - U(|\psi_1|^2,\dots,|\psi_N|^2)\right] \mathrm{d}x, \quad \boldsymbol{\psi} \in H^1(\mathbb{R}), \tag{1.8}$$

where $f_n = \partial U/\partial|\psi_n|^2$, and the momentum associated with the symmetry (1.6) is given by

$$P[\boldsymbol{\psi}] = \mathrm{i}\int_{\mathbb{R}} \left[\sum_{n=1}^{N} \left(\bar{\psi}_n \frac{\partial \psi_n}{\partial x} - \psi_n \frac{\partial \bar{\psi}_n}{\partial x}\right)\right] \mathrm{d}x. \tag{1.9}$$

Under the condition (1.2), the system (1.1) takes the Hamiltonian form in canonical variables $\boldsymbol{u} = (u_1,\dots,u_N)^{\mathrm{T}}$ and $\boldsymbol{w} = (w_1,\dots,w_N)^{\mathrm{T}}$:

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{w}\end{pmatrix} = \tfrac{1}{2}\mathcal{J}H'[\boldsymbol{u},\boldsymbol{w}], \quad \mathcal{J} = \begin{pmatrix}\mathcal{O}_N & \mathcal{I}_N \\ -\mathcal{I}_N & \mathcal{O}_N\end{pmatrix}, \tag{1.10}$$

where $(\boldsymbol{u},\boldsymbol{w})^{\mathrm{T}} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^{2N}$, $\mathcal{I}_N$ and $\mathcal{O}_N$ are identity and zero matrices in $\mathbb{R}^N$, $\mathcal{J}^{\mathrm{T}} = -\mathcal{J}$ and the Hamiltonian $H[\boldsymbol{u},\boldsymbol{w}]$ follows from $H[\boldsymbol{\psi}]$ with $\psi_n = u_n + \mathrm{i}w_n$ and $\bar{\psi}_n = u_n - \mathrm{i}w_n$, $n = 1,\dots,N$.

2. The formalism

Stationary solutions of the coupled NLS equations (1.1) are defined by the standard ansatz:

$$\psi_n(z,x) = \Phi_n(x)\mathrm{e}^{\mathrm{i}\beta_n z}, \tag{2.1}$$

where $\Phi_n : \mathbb{R} \to \mathbb{R}$. Components $\Phi_n(x)$ satisfy the system of equations

$$d_n \frac{\mathrm{d}^2\Phi_n}{\mathrm{d}x^2} - \beta_n\Phi_n + f_n(\Phi_1^2,\dots,\Phi_N^2)\Phi_n = 0, \quad \lim_{|x|\to\infty} \Phi_n(x) = 0. \tag{2.2}$$

Throughout the paper, we assume that the existence problem has a solution with the following properties.

Assumption 2.1. *There exists an exponentially decaying solution* $\boldsymbol{\Phi}(x) = (\Phi_1,\dots,\Phi_N)^{\mathrm{T}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} \in H^1(\mathbb{R})$ *in an open domain* $\boldsymbol{\beta} = (\beta_1,\dots,\beta_N)^{\mathrm{T}} \in \mathcal{B} \subset \mathbb{R}^N$. *The stationary solution is not degenerate, such that* $\Phi_n(x) = 0$ *only in a finite number of points* $x \in \mathbb{R}$, $n = 1,\dots,N$. *The functions* $H[\boldsymbol{\Phi}]$ *and* $Q_n[\boldsymbol{\Phi}]$, $n = 1,\dots,N$ *are* C^1 *on* $\boldsymbol{\beta} \in \mathcal{B}$.

Exponentially decaying solutions of (2.2) may exist only if $\beta_n > 0$ (assuming $d_n > 0$), $n = 1, \dots, N$, when components $\Phi_n(x)$ decay asymptotically as

$$\lim_{x \to \pm\infty} |\Phi_n(x) \mathrm{e}^{a_n |x|} - c_n^{\pm}| = 0, \quad a_n = \sqrt{\frac{\beta_n}{d_n}} (> 0), \tag{2.3}$$

where $c_n^{\pm}$ are some non-zero constants. The constraint $\beta_n > 0$ is related to the constraint $\omega_n \leqslant 0$ in the linear spectrum (1.3). The spectrum of exponentially decaying stationary solutions (2.1) is isolated from the linear spectrum (1.3), when $\beta_n > 0$. Otherwise, as for other systems of coupled NLS equations (Pelinovsky & Yang 2002), the exponentially decaying solutions become embedded into the linear spectrum (embedded solitons). Such solutions are semi-stable due to nonlinearity-induced radiative decay, even if they are linearly stable (Pelinovsky & Yang 2002). We note that the algebraically decaying solutions may also exist in the system (2.2) for $\beta_n = 0$ and they are embedded into the edge of the linear spectrum at $\omega_n = 0$. We do not consider algebraically decaying solutions in this paper.

Definition 2.2. Families of stationary solutions $\boldsymbol{\Phi}(x)$ are classified by the nodal index $\boldsymbol{i} = (i_1, \dots, i_N)^{\mathrm{T}}$, where i_n is the number of zeros of $\Phi_n(x)$ for $x \in \mathbb{R}$. The stationary solution $\boldsymbol{\Phi}(x)$ with $\boldsymbol{i} = \boldsymbol{0}$ is called the ground state.

Lemma 2.3. *Stationary solutions $\boldsymbol{\Phi}(x)$ are critical points of the Lyapunov functional*

$$\Lambda[\boldsymbol{\psi}] = H[\boldsymbol{\psi}] + \sum_{n=1}^{N} \beta_n Q_n[\boldsymbol{\psi}], \tag{2.4}$$

where $Q_n[\boldsymbol{\psi}]$ and $H[\boldsymbol{\psi}]$ are given by (1.5) and (1.8).

Proof. The first variation of $\Lambda[\boldsymbol{\psi}]$ vanishes if $\boldsymbol{\psi} = \boldsymbol{\Phi}(x)$ satisfies the system (2.2). ■

Let $\Lambda_{\mathrm{s}}(\boldsymbol{\beta})$ be the energy surface of the stationary solutions,

$$\Lambda_{\mathrm{s}}(\boldsymbol{\beta}) = H_{\mathrm{s}}(\boldsymbol{\beta}) + \sum_{n=1}^{N} \beta_n Q_{n\mathrm{s}}(\boldsymbol{\beta}), \tag{2.5}$$

where $H_{\mathrm{s}}(\boldsymbol{\beta}) = H[\boldsymbol{\Phi}]$ and $Q_{n\mathrm{s}}(\boldsymbol{\beta}) = Q_n[\boldsymbol{\Phi}]$. The Hessian matrix $\mathcal{U}$ of the energy surface $\Lambda_{\mathrm{s}}(\boldsymbol{\beta})$ is a symmetric matrix with the elements

$$\mathcal{U}_{n,m} = \frac{\partial^2 \Lambda_{\mathrm{s}}}{\partial \beta_n \partial \beta_m}. \tag{2.6}$$

Lemma 2.4. *Let assumption 2.1 be satisfied. Matrix elements of the Hessian matrix $\mathcal{U}$ are continuous functions of $\boldsymbol{\beta}$ in $\mathcal{B}$, computed as*

$$\mathcal{U}_{n,m} = \frac{\partial Q_{n\mathrm{s}}}{\partial \beta_m} = 2 \left\langle \Phi_n \boldsymbol{e}_n, \frac{\partial \boldsymbol{\Phi}}{\partial \beta_m} \right\rangle. \tag{2.7}$$

Matrix $\mathcal{U}$ has N real bounded eigenvalues in the domain $\boldsymbol{\beta} \in \mathcal{B}$.

Proof. By lemma 2.3, we have

$$\frac{\partial \Lambda_{\rm s}}{\partial \beta_n} = Q_{ns} + \left(\frac{\partial H_{\rm s}}{\partial \beta_n} + \sum_{m=1}^{N} \beta_m \frac{\partial Q_{ms}}{\partial \beta_n} \right) = Q_{ns}. \tag{2.8}$$

By assumption 2.1, the second derivatives of $\Lambda_{\rm s}(\boldsymbol{\beta})$ are continuous in $\boldsymbol{\beta} \in \mathcal{B}$. Therefore, $\mathcal{U}$ is a matrix with bounded eigenvalues in $\boldsymbol{\beta} \in \mathcal{B}$. Since $\mathcal{U}$ is symmetric, all eigenvalues of $\mathcal{U}$ are real. ■

Let $n(\mathcal{U})$, $z(\mathcal{U})$ and $p(\mathcal{U})$ be the numbers of negative, zero and positive eigenvalues of $\mathcal{U}$, respectively, such that $n(\mathcal{U}) + z(\mathcal{U}) + p(\mathcal{U}) = N$.

Linearization at the stationary solutions (2.1) is defined by the expansion

$$\psi_n(z, x) = [\Phi_n(x) + U_n(z, x) + \mathrm{i} W_n(z, x)] \mathrm{e}^{\mathrm{i}\beta_n z}, \tag{2.9}$$

where $(U_n, W_n)^{\rm T} \in \mathbb{R}^2$ are perturbations functions. Neglecting nonlinear terms, we find that the perturbation vectors $\boldsymbol{U} = (U_1, \ldots, U_N)^{\rm T}$ and $\boldsymbol{W} = (W_1, \ldots, W_N)^{\rm T}$ satisfy the linearized system in Hamiltonian form,

$$\frac{\mathrm{d}}{\mathrm{d}z} \begin{pmatrix} \boldsymbol{U} \\ \boldsymbol{W} \end{pmatrix} = \tfrac{1}{2} \mathcal{J} h'[\boldsymbol{U}, \boldsymbol{W}], \tag{2.10}$$

where the linearized Hamiltonian $h[\boldsymbol{U}, \boldsymbol{W}]$ is the second variation of the Lyapunov functional (2.4),

$$h = \langle \boldsymbol{U}, \mathcal{L}_1 \boldsymbol{U} \rangle + \langle \boldsymbol{W}, \mathcal{L}_0 \boldsymbol{W} \rangle, \tag{2.11}$$

and $\mathcal{L}_0$ and $\mathcal{L}_1$ are matrix Schrödinger operators with the elements

$$(\mathcal{L}_0)_{n,m} = \left(-d_n \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta_n - f_n(\Phi_1^2, \ldots, \Phi_N^2) \right) \delta_{n,m}, \tag{2.12}$$

$$(\mathcal{L}_1)_{n,m} = \left(-d_n \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta_n - f_n(\Phi_1^2, \ldots, \Phi_N^2) \right) \delta_{n,m} - 2 \frac{\partial f_n}{\partial \Phi_m^2} \Phi_n \Phi_m. \tag{2.13}$$

The diagonal operator $\mathcal{L}_0$ is a composition of N scalar Schrödinger operators. The matrix operator $\mathcal{L}_1$ is symmetric in the Hamiltonian case (1.2). Both quadratic forms in (2.11) are real valued. The linearized problem (2.10) reduces to a linear eigenvalue problem after separation of variables: $\boldsymbol{U} = \boldsymbol{u}(x)\mathrm{e}^{\lambda z}$, $\boldsymbol{W} = \boldsymbol{w}(x)\mathrm{e}^{\lambda z}$. Eigenvalues λ are defined by the spectrum of the non-self-adjoint operator $\mathcal{A}$:

$$\mathcal{A} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{w} \end{pmatrix} = \lambda \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{w} \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \mathcal{O}_N & \mathcal{L}_0 \\ -\mathcal{L}_1 & \mathcal{O}_N \end{pmatrix}. \tag{2.14}$$

The operator $\mathcal{A}$ is defined on $L^2(\mathbb{R})$, equipped with the inner product

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \int_{\mathbb{R}} \left(\sum_{n=1}^{N} \bar{f}_n(x) g_n(x) \right) \mathrm{d}x. \tag{2.15}$$

We use standard definitions of eigenvalues of $\mathcal{A}$ from (Hislop & Sigal 1996, definition 1.4).

Definition 2.5. The value λ is an eigenvalue of $\mathcal{A}$ if $\ker(\mathcal{A}-\lambda) \neq \{0\}$ in $L^2(\mathbb{R})$, such that there exists a non-zero vector function $(\boldsymbol{u}, \boldsymbol{w})^{\mathrm{T}} \in \ker(\mathcal{A}-\lambda)$ called an eigenvector of $\mathcal{A}$. The dimension of $\ker(\mathcal{A}-\lambda)$ is called the geometric multiplicity of λ.

Definition 2.6. The discrete spectrum of $\mathcal{A}$, $\sigma_{\mathrm{dis}}(\mathcal{A})$, is the set of all eigenvalues of $\mathcal{A}$ with finite algebraic multiplicity which are isolated from the continuous spectrum of $\mathcal{A}$, $\sigma_{\mathrm{con}}(\mathcal{A})$. The embedded spectrum of $\mathcal{A}$, $\sigma_{\mathrm{emb}}(\mathcal{A})$, is the set of all eigenvalues with finite algebraic multiplicity which belong to the continuous spectrum of $\mathcal{A}$, including the boundary points. The essential spectrum of $\mathcal{A}$ is $\sigma_{\mathrm{ess}}(\mathcal{A}) = \sigma_{\mathrm{con}}(\mathcal{A}) \cup \sigma_{\mathrm{emb}}(\mathcal{A})$ and the point spectrum of $\mathcal{A}$ is $\sigma_p(\mathcal{A}) = \sigma_{\mathrm{dis}}(\mathcal{A}) \cup \sigma_{\mathrm{emb}}(\mathcal{A})$. The total spectrum of $\mathcal{A}$ is $\sigma(\mathcal{A}) = \sigma_{\mathrm{dis}}(\mathcal{A}) \cup \sigma_{\mathrm{ess}}(\mathcal{A}) = \sigma_{\mathrm{p}}(\mathcal{A}) \cup \sigma_{\mathrm{con}}(\mathcal{A})$.

Remark 2.7. The continuous spectrum $\sigma_{\mathrm{con}}(\mathcal{A})$ may contain resonances, corresponding to bounded non-decaying eigenvectors, and semi-eigenvalues, corresponding to eigenvectors, which are decaying at one infinity and bounded at the other infinity. Definitions of resonances and semi-eigenvalues will be given in terms of the scattering matrix for the problem (2.14) (see definition 7.1).

The non-self-adjoint linear eigenvalue problem (2.14) is formulated as a coupled system for two symmetric matrix Schrödinger operators $\mathcal{L}_0$ and $\mathcal{L}_1$. The spectrum of these operators is reviewed in the following statements.

Lemma 2.8. *Let $\mathcal{L}$ be a symmetric matrix Schrödinger operator, either $\mathcal{L}_0$ or $\mathcal{L}_1$. The continuous spectrum of $\mathcal{L}$ has N branches located at*

$$\sigma_{\mathrm{con}}(\mathcal{L}) = \bigcup_{1 \leqslant n \leqslant N} \{\lambda \in \mathbb{R} : \lambda \geqslant \beta_n\}. \tag{2.16}$$

The discrete and embedded spectrum of $\mathcal{L}$ has a finite number of eigenvalues located at

$$\sigma_{\mathrm{dis}}(\mathcal{L}) = \bigcup_m \{\lambda_m : \lambda_m \in \mathbb{R}, \lambda < \beta_{\min}\}, \tag{2.17}$$

$$\sigma_{\mathrm{emb}}(\mathcal{L}) = \bigcup_m \{\lambda_m : \lambda_m \in \mathbb{R}, \beta_{\min} \leqslant \lambda_m < \beta_{\max}\}, \tag{2.18}$$

where $\beta_{\min} = \min_{1 \leqslant n \leqslant N}(\beta_n)$ and $\beta_{\max} = \max_{1 \leqslant n \leqslant N}(\beta_n)$. The algebraic multiplicity of eigenvalues coincides with their geometric multiplicity and is at most N.

Proof. The matrix Schrödinger operator $\mathcal{L}$ has exponentially decaying potentials and becomes a diagonal differential operator in the limit $|x| \to \infty$. As a result, the continuous spectrum of $\mathcal{L}$ is defined by the Weyl criterion and the point spectrum of $\mathcal{L}$ is finite dimensional (Hislop & Sigal 1996, theorem 7.2). Furthermore, since $\mathcal{L}$ is self-adjoint, the algebraic multiplicity of eigenvalues always coincides with their geometric multiplicity (Hislop & Sigal 1996, theorem 6.7).

Exponentially decaying solutions of the spectral problem $\mathcal{L}\boldsymbol{u} = \lambda \boldsymbol{u}$ are superposed in the limit $|x| \to \infty$ over a basis of N vector functions $\boldsymbol{e}_n \mathrm{e}^{-b_n|x|}$, $n = 1, \ldots, N$, where

$$b_n = \sqrt{\frac{\beta_n - \lambda}{d_n}}.$$

For $\lambda < \beta_{\min}$, all vector functions are exponentially decaying and there exist no more than N linearly independent eigenvectors $\boldsymbol{u}(x)$ for some (isolated) values of λ. This contributes to eigenvalues of the discrete spectrum (2.17). For $\lambda \geqslant \beta_{\max}$, all vector functions are non-decaying and no embedded eigenvalues may exist. For $\beta_{\min} \leqslant \lambda \leqslant \beta_{\max}$, some components $u_n(x)$ are decaying, while the other components $u_n(x)$ are non-decaying. Let N_1 be the number of non-decaying components. Then, there exist N_1 branches of the continuous spectrum of $\mathcal{L}$ at this value of λ, and an embedded eigenvalue (if it exists) corresponds to at most $N-N_1$ linearly independent decaying eigenvectors. ∎

Lemma 2.9. *Let assumption 2.1 be satisfied. The kernel of $\mathcal{L}_0$ has a basis of N eigenvectors $\{\Phi_n(x)\boldsymbol{e}_n\}_{n=1}^N$. The kernel of $\mathcal{L}_1$ has at least one eigenvector $\boldsymbol{\Phi}'(x)$.*

Proof. The eigenvectors of the kernels of $\mathcal{L}_0$ and $\mathcal{L}_1$ are generated by the rotational and translational invariance (1.4) and (1.6), respectively. By assumption 2.1 and lemma 2.8, the set of non-empty eigenvectors $\{\Phi_n(x)\boldsymbol{e}_n\}_{n=1}^N$ forms a basis in the kernel of $\mathcal{L}_0$. ∎

Definition 2.10. Denote the number of negative and zero eigenvalues of $\sigma_{\mathrm{dis}}(\mathcal{L})$ as $n(\mathcal{L})$ and $z(\mathcal{L})$, respectively. The Morse index for stationary solutions is

$$n(h) = n(\mathcal{L}_1) + n(\mathcal{L}_0). \tag{2.19}$$

Lemma 2.11. *Let assumption 2.1 be satisfied. The negative index of $\mathcal{L}_0$ is*

$$n(\mathcal{L}_0) = \sum_{n=1}^{N} i_n, \tag{2.20}$$

where i_n is the number of zeros of $\Phi_n(x)$ for $x \in \mathbb{R}$. The negative index $n(\mathcal{L}_0)$ and the nodal index $\boldsymbol{i} = (i_1, \dots, i_N)^{\mathrm{T}}$ remain fixed in the open domain $\boldsymbol{\beta} \in \mathcal{B}$.

Proof. Since $\mathcal{L}_0$ is a diagonal composition of scalar Schrödinger operators, the Sturm oscillation theorem applies. Each operator $(\mathcal{L}_0)_{n,n}$ has a zero bound state $\Phi_n(x)$, such that $n((\mathcal{L}_0)_{n,n}) = i_n$. By assumption 2.1, the set of non-empty eigenvectors $\{\Phi_n(x)\boldsymbol{e}_n\}_{n=1}^N$ forms a basis in the kernel of $\mathcal{L}_0$ in $\boldsymbol{\beta} \in \mathcal{B}$. Therefore, the index $n(\mathcal{L}_0)$ remains fixed for any continuous deformations of $\boldsymbol{\Phi}(x)$ in $\boldsymbol{\beta} \in \mathcal{B}$. ∎

We finish this section with some general properties of the eigenvalue problem (2.14).

Lemma 2.12. *If λ is an eigenvalue of (2.14), so are $(-\lambda)$, $\bar{\lambda}$, and $(-\bar{\lambda})$.*

Proof. This standard result for linear Hamiltonian systems follows from the fact that, if $(\boldsymbol{u}, \boldsymbol{w})$ is the eigenvector of (2.14) with λ, then $(\boldsymbol{u}, -\boldsymbol{w})$, $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{w}})$, and $(\bar{\boldsymbol{u}}, -\bar{\boldsymbol{w}})$ are eigenvectors of (2.14) with $(-\lambda)$, $\bar{\lambda}$, and $(-\bar{\lambda})$, respectively. ∎

Definition 2.13. The stationary solution (2.1) is spectrally unstable if there exists at least one eigenvalue λ such that $\mathrm{Re}(\lambda) > 0$. It is weakly spectrally stable if all eigenvalues λ are zero or purely imaginary.

Spectral instability occurs when the eigenvalue problem (2.14) has a pair of real eigenvalues $(\lambda, -\lambda)$ or a quadruple of complex eigenvalues $(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda})$. Weak spectral stability does not yet guarantee strong spectral stability, since there may exist eigenvalues of higher algebraic multiplicity with $\mathrm{Re}(\lambda) = 0$, which lead to nonlinear instability of stationary solutions (Comech & Pelinovsky 2003). We shall study here the generic case, when no structurally unstable eigenvalues exist in the problem (2.14).

Assumption 2.14.

(i) $\sigma_{\mathrm{ess}}(\mathcal{A})$ *does not include semi-eigenvalues or embedded eigenvalues;*

(ii) $\sigma_{\mathrm{dis}}(\mathcal{A})$ *does not include non-zero eigenvalues of higher algebraic multiplicity;*

(iii) $z(\mathcal{L}_1) = 1$ *and* $z(\mathcal{U}) = 0$.

Bifurcations in the spectrum of $\mathcal{A}$ may occur when assumption 2.14 is violated. Bifurcations in the eigenvalue problem (2.14) will be studied elsewhere.

Lemma 2.15. *Define the constrained function space* $X_{\mathrm{c}}(\mathbb{R}) = X_{\mathrm{c}}^{(u)} \oplus X_{\mathrm{c}}^{(w)}$, *where*

$$X_{\mathrm{c}}^{(u)} = \{\boldsymbol{u} \in L^2(\mathbb{R}) : \langle \Phi_n \boldsymbol{e}_n, \boldsymbol{u} \rangle = 0,\ n = 1, \ldots, N\}, \tag{2.21}$$

$$X_{\mathrm{c}}^{(w)} = \{\boldsymbol{w} \in L^2(\mathbb{R}) : \langle \boldsymbol{\Phi}', \boldsymbol{w} \rangle = 0\}. \tag{2.22}$$

Eigenvectors $(\boldsymbol{u}, \boldsymbol{w})^{\mathrm{T}}$ *in the problem (2.14) for* $\lambda \neq 0$ *belong to the space* $X_{\mathrm{c}}(\mathbb{R})$.

Proof. The linear eigenvalue problem (2.14) is written as a coupled system:

$$\mathcal{L}_1 \boldsymbol{u} = -\lambda \boldsymbol{w}, \qquad \mathcal{L}_0 \boldsymbol{w} = \lambda \boldsymbol{u}. \tag{2.23}$$

The constraints in (2.21), (2.22) follow from the Fredholm alternative theorem applied to (2.23) for $\lambda \neq 0$, since the kernel of $\mathcal{L}_0$ has the set of eigenvectors $\{\Phi_n(x)\boldsymbol{e}_n\}_{n=1}^N$ and the kernel of $\mathcal{L}_1$ has the eigenvector $\boldsymbol{\Phi}'(x)$. ■

Lemma 2.16. *Let assumptions 2.1 and 2.14(iii) be satisfied. The geometric multiplicity of the null eigenvalue of* $\mathcal{A}$ *is exactly* $(N+1)$ *and the algebraic multiplicity of the null eigenvalue of* $\mathcal{A}$ *is exactly* $(2N+2)$.

Proof. The null space of $\mathcal{A}$ is spanned by at least $(N+1)$ eigenvectors:

$$\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{w} \end{pmatrix} = \left\{ \left\{ \begin{pmatrix} \mathbf{0}_N \\ \Phi_n(x)\boldsymbol{e}_n \end{pmatrix} \right\}_{n=1}^N, \begin{pmatrix} \boldsymbol{\Phi}'(x) \\ \mathbf{0}_N \end{pmatrix} \right\}. \tag{2.24}$$

The generalized null space of $\mathcal{A}$ includes at least $(N+1)$ generalized eigenvectors:

$$\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{w} \end{pmatrix} = \left\{ \left\{ \begin{pmatrix} \partial\boldsymbol{\Phi}/\partial\beta_n \\ \mathbf{0}_N \end{pmatrix} \right\}_{n=1}^N, \begin{pmatrix} \mathbf{0}_N \\ -\frac{1}{2}x\mathcal{D}^{-1}\boldsymbol{\Phi}(x) \end{pmatrix} \right\}, \tag{2.25}$$

where $\mathcal{D}$ is a diagonal matrix of $(d_1, \ldots, d_N)$. By lemma 2.9, the $(N+1)$ eigenvectors (2.24) form a basis for null space of $\mathcal{A}$, when $z(\mathcal{L}_1) = 1$. Fredholm's alternative theorem applied to the first N generalized eigenvectors in (2.25) fails when $z(\mathcal{U}) = 0$. The Fredholm theorem always fails for the last generalized eigenvector in (2.25). ■

Corollary 2.17. *When* $z(\mathcal{L}_1) = 1$ *and* $z(\mathcal{U}) = 0$, *the generalized eigenvectors (2.25) do not belong to the constrained space* $X_{\mathrm{c}}(\mathbb{R})$, *defined by (2.21), (2.22).*

3. Main results

Stability of solitary waves in NLS equations has been studied extensively in the recent past. The first stability–instability theorem for a scalar NLS equation (1.1) with $N=1$ was proven by Shatah & Strauss (1985) and Weinstein (1986). Only positive stationary solutions (ground states) were considered in one, two and three spatial dimensions. Ground states have the nodal index $i=0$ and the Morse index $n(h)=1$. A single negative eigenvalue of h does not necessary lead to spectral instability in the linearized problem (2.14) because of the constraints in (2.21), (2.22). If $p(\mathcal{U})=0$, the stationary solution $\Phi(x)$ is spectrally unstable and the linearized problem (2.14) has a single real positive eigenvalue λ. If $p(\mathcal{U})=1$, the solitary wave is weakly spectrally stable and all eigenvalues λ are purely imaginary (Shatah & Strauss 1985; Weinstein 1986).

A more formal and general analysis was developed by Grillakis *et al.* (1987, 1990) by using multi-dimensional Lie groups and spectral decompositions. The following theorems were proven for an abstract Hamiltonian system with symmetries, which includes the system of coupled NLS equations (1.10).

Theorem 3.1 (Grillakis *et al.* 1990). *Let $z(\mathcal{U})=0$. Then $p(\mathcal{U})\leqslant n(h)$. A stationary solution (2.1) is weakly spectrally stable if $n(h)=p(\mathcal{U})$ and it is spectrally unstable if $n(h)-p(\mathcal{U})$ is odd. The linearized problem (2.14) has at least one real positive eigenvalue λ if $n(h)-p(\mathcal{U})$ is odd.*

Theorem 3.2 (Grillakis *et al.* 1990). *The linearized problem (2.14) has at most $n(h)$ unstable eigenvalues λ such that* $\mathrm{Re}(\lambda)>0$.

Theorem 3.3 (Grillakis *et al.* 1990). *The linearized Hamiltonian h in constrained space $X_{\mathrm{c}}(\mathbb{R})$ has the negative index $n(h|_{X_{\mathrm{c}}})=n(h)-p(\mathcal{U})-z(\mathcal{U})$ and the null index $z(h|_{X_{\mathrm{c}}})=z(h)+z(\mathcal{U})$.*

Theorem 3.1 is the main stability–instability theorem in Grillakis *et al.* (1990). Theorem 3.2 is formulated in Grillakis *et al.* (1990, theorem 5.8) for a quadrant: $\mathrm{Re}(\lambda)<0$, $\mathrm{Im}(\lambda)\geqslant 0$. The method of the proof can, however, be applied to the left half-plane $\mathrm{Re}(\lambda)<0$, or equivalently, to the right half-plane $\mathrm{Re}(\lambda)>0$. Theorem 3.3 is formulated in Grillakis *et al.* (1990, theorem 3.1) as a more general statement, which is equivalent to theorem 3.3 under assumption 2.1 ($z_0=0$ in the notation of Grillakis *et al.* (1990)).

Theorem 3.1 generalizes stability–instability theory in finite-dimensional Hamiltonian systems with symmetries (Maddocks 1985). Since the positive ground state (2.1) with $N=1$ always has indices $n(\mathcal{L}_1)=1$ and $n(\mathcal{L}_0)=0$, its stability and instability are uniquely described by theorem 3.1. However, many examples showed insufficiency of theorem 3.1 for complete stability–instability analysis. For instance, a scalar NLS equation in two dimensions has radially symmetric excited states with the nodal index $i>0$ and the Morse index $n(h)\geqslant 1+2\mathrm{i}$ (Jones 1988*a*). When $p(\mathcal{U})=1$ and $n(h)-p(\mathcal{U})\geqslant 2\mathrm{i}$ is even, theorem 3.1 cannot be applied.

While a simple application of theorem 3.1 to the case of multi-component stationary solutions (2.1) with $N>1$ is given in Grillakis *et al.* (1990, theorem 9.1), we note that theorem 9.1 in Grillakis *et al.* (1990) derives a scalar stability criterion, computed from the minimal value $\beta_{\min}=\min_{1\leqslant n\leqslant N}(\beta_n)$. The scalar criterion generally fails for $N>1$, since the Morse index $n(h)$ of the stationary solution (2.1) is not necessarily one, unlike the assumption used in Grillakis *et al.* (1990).

More special instability theorems were found by Jones (1988*a*,*b*) and Grillakis (1988, 1990) for the scalar NLS equation (1.1) with $N = 1$. Jones (1988*a*,*b*) used topological and shooting methods of dynamical systems theory. When $n(\mathcal{L}_1) - p(\mathcal{U}) > n(\mathcal{L}_0)$, theorems in Jones (1988*a*,*b*) predict an unstable eigenvalue, no matter whether $n(h)$ is odd or even. The results apply to instability of radially symmetric solutions with nodal index $i > 0$ in two spatial dimensions (Jones 1988*a*), as well as to stability–instability of symmetric and antisymmetric solutions in the NLS equation with x-dependent nonlinear function $f = f(x; |\psi|^2)$ (Jones 1988*b*).

Theorem 3.4 (Jones 1988*a*,*b*). *The linearized problem (2.14) with $N = 1$ has a real positive eigenvalue λ if $|n(\mathcal{L}_1) - n(\mathcal{L}_0)| > 1$.*

Grillakis (1988, 1990) used the theory of linear operators, orthogonal projections and quadratic forms and proved some general results for the linearized problem (2.14). In this context, the problem (2.14) is reformulated as a generalized eigenvalue problem for operators $\mathcal{L}_1$ and $\mathcal{L}_0^{-1}$. When $n(\mathcal{L}_0) = 0$, unstable eigenvalues λ may occur only as real positive eigenvalues (Grillakis 1988). When $n(h) - p(\mathcal{U}) > 1$ and $n(\mathcal{L}_0) \neq 0$, complex unstable eigenvalues λ may also occur in the linearized problem (2.14) (Grillakis 1990).

Theorem 3.5 (Grillakis 1988). *Let $n(\mathcal{L}_1|_{X_c^{(u)}})$ and $n(\mathcal{L}_0|_{X_c^{(u)}})$ be the negative indices of operators $\mathcal{L}_1$ and $\mathcal{L}_0$ in $X_c^{(u)}(\mathbb{R})$. The linearized problem (2.14) has at least $|n(\mathcal{L}_1|_{X_c^{(u)}}) - n(\mathcal{L}_0|_{X_c^{(u)}})|$ real positive eigenvalues λ. If $n(\mathcal{L}_0|_{X_c^{(u)}}) = 0$, the linearized problem has exactly $n(\mathcal{L}_1|_{X_c^{(u)}})$ real positive eigenvalues λ.*

Theorem 3.5 is formulated in Grillakis (1988, theorem 1.2). The theorem is more precise and general than theorem 3.4; the latter takes the worst case, when $p(\mathcal{U}) = 1$ for $N = 1$. It remains unclear how theorem 3.5, which exploits a special structure of the linearized problem (2.14), is related to general theorem 3.1 for an abstract Hamiltonian system. It also remains unclear how the bounds on the number of unstable eigenvalues can be extended in the case of complex eigenvalues in the linearized problem (2.14).

We shall revisit here the problem of spectral stability of stationary solutions (2.1) in the coupled NLS equations (1.1) with $N \geqslant 1$. We develop two new methods of analysis: (i) negative eigenvalues of a constrained spectral problem are counted from matrix analysis; (ii) the negative subspace of a linear differential matrix operator with positive continuous spectrum is proved to be invariant in two block-diagonal representations. The first method develops the matrix variant of the Vakhitov–Kolokolov method, previously studied in Pelinovsky & Kivshar (2000). The second method develops Sylvester's inertia theorem for quadratic forms associated with finite-dimensional (matrix) operators (Gelfand 1961), applied to finite-dimensional Hamiltonian systems in Maddocks (1988). The new methods of analysis are used to prove the following main results.

Theorem 3.6 (negative index of constrained operators). *Let assumption 2.1 be satisfied. Operator $\mathcal{L}_1$ in constrained space $X_c^{(u)}(\mathbb{R})$ has exactly*

$$n(\mathcal{L}_1|_{X_c^{(u)}}) = n(\mathcal{L}_1) - p(\mathcal{U}) - z(\mathcal{U})$$

negative eigenvalues and

$$z(\mathcal{L}_1|_{X_c^{(u)}}) = z(\mathcal{L}_1) + z(\mathcal{U})$$

zero eigenvalues. Operator $\mathcal{L}_0$ in the constrained space $X_{\rm c}^{(u)}(\mathbb{R})$ has exactly $n(\mathcal{L}_0|_{X_{\rm c}^{(u)}}) = n(\mathcal{L}_0)$ negative eigenvalues.

Corollary 3.7. *Let $z(\mathcal{U}) = 0$. Then $p(\mathcal{U}) \leqslant n(\mathcal{L}_1)$.*

Theorem 3.8 (closure relation for negative index). *Let assumptions 2.1 and 2.14 be satisfied. Let $N_{\rm real}$ be the number of real positive eigenvalues λ of the problem (2.14), $2N_{\rm comp}$ be the number of complex eigenvalues λ with $\mathrm{Re}(\lambda) > 0$, and $2N_{\rm imag}^-$ be the number of purely imaginary eigenvalues λ with $\langle \boldsymbol{u}, \mathcal{L}_1 \boldsymbol{u}\rangle = \langle \boldsymbol{w}, \mathcal{L}_0 \boldsymbol{w}\rangle < 0$. The dimension of the negative subspace of the linearized Hamiltonian h in $X_{\rm c}(\mathbb{R})$ is invariant as*

$$n(h|_{X_{\rm c}}) = n(\mathcal{L}_1) - p(\mathcal{U}) + n(\mathcal{L}_0) = N_{\rm real} + 2N_{\rm comp} + 2N_{\rm imag}^-. \tag{3.1}$$

Theorem 3.9 (bounds on unstable eigenvalues). *Let assumptions 2.1 and 2.14 be satisfied. The linearized problem (2.14) has $N_{\rm unst} = N_{\rm real} + 2N_{\rm comp}$ unstable eigenvalues λ with $\mathrm{Re}(\lambda) > 0$, such that*

$$\text{(i)}\quad |n(\mathcal{L}_1) - p(\mathcal{U}) - n(\mathcal{L}_0)| \leqslant N_{\rm unst} \leqslant (n(\mathcal{L}_1) - p(\mathcal{U}) + n(\mathcal{L}_0)), \tag{3.2}$$

$$\text{(ii)}\quad N_{\rm real} \geqslant |n(\mathcal{L}_1) - p(\mathcal{U}) - n(\mathcal{L}_0)|, \tag{3.3}$$

$$\text{(iii)}\quad N_{\rm comp} \leqslant \min(n(\mathcal{L}_0), n(\mathcal{L}_1) - p(\mathcal{U})). \tag{3.4}$$

Corollary 3.10. *Let $z(\mathcal{U}) = 0$. When $n(\mathcal{L}_0) = 0$, the linearized problem (2.14) has exactly $N_{\rm real} = n(\mathcal{L}_1) - p(\mathcal{U})$ real positive eigenvalues λ. If both $n(\mathcal{L}_0) = 0$ and $n(\mathcal{L}_1) = p(\mathcal{U})$, the stationary solution (2.1) is weakly spectrally stable.*

Theorem 3.6 decomposes general theorem 3.3 in the case when h is a sum of two quadratic forms for $\mathcal{L}_1$ and $\mathcal{L}_0$ as in (2.11). As a result, the upper bound on $p(\mathcal{U})$ of theorem 3.1 is improved as $p(\mathcal{U}) \leqslant n(\mathcal{L}_1) \leqslant n(h)$, as in corollary 3.7. Also the stability criterion of theorem 3.1 decomposes into two conditions: $n(\mathcal{L}_1) = p(\mathcal{U})$ and $n(\mathcal{L}_0) = 0$, as in corollary 3.10.

Theorem 3.8 gives a precise statement of the closure relation between indices $n(\mathcal{L}_0)$, $n(\mathcal{L}_1)$ and $p(\mathcal{U})$ on one side and $N_{\rm real}$, $N_{\rm comp}$ and $N_{\rm imag}^-$ on the other side. This theorem generalizes earlier results for

$$n(\mathcal{L}_1|_{X_{\rm c}^{(u)}}) = n(\mathcal{L}_0|_{X_{\rm c}^{(u)}})$$

formulated in Grillakis (1988, theorem 1.3) (when $N_{\rm comp} = N_{\rm imag}^- = 0$) and in Grillakis (1990, theorem 2.3) (when $N_{\rm real} = N_{\rm imag}^- = 0$).

Theorem 3.9 is a corollary of theorems 3.5, 3.6 and 3.8. The lower bound in (3.2) is identical to that in theorem 3.5 in view of theorem 3.6. The upper bound in (3.2) improves theorem 3.2. Theorem 3.9 also agrees with the instability criterion of theorem 3.1. Let $z(\mathcal{U}) = 0$ and $m = n(\mathcal{L}_1) - p(\mathcal{U}) + n(\mathcal{L}_0)$ be odd. Then $|n(\mathcal{L}_1) - p(\mathcal{U}) - n(\mathcal{L}_0)| = |m - 2n(\mathcal{L}_0)| > 0$ and $N_{\rm unst} > 0$. Therefore, theorem 3.9 also guarantees instability for odd m, as theorem 3.1. Corollary 3.10 of theorems 3.8 and 3.9 is equivalent to the second statement of theorem 3.5.

When $\mathcal{L}_1$ and $\mathcal{L}_0$ are finite-dimensional operators, theorems 3.6–3.9 reduce to stability–instability theorems for critical points in finite-dimensional Hamiltonian systems with symmetry constraints (Maddocks 1985, 1988). When $n(\mathcal{L}_0) = 0$, the quadratic form $\langle \boldsymbol{W}, \mathcal{L}_0 \boldsymbol{W}\rangle$ in (2.11) is equivalent to a positive-definite kinetic energy, while the quadratic form $\langle \boldsymbol{U}, \mathcal{L}_1 \boldsymbol{U}\rangle$ in (2.11) is equivalent to a sign indefinite potential energy. The Morse index of $\mathcal{L}_1$ under constraints (2.21) is $n(\mathcal{L}_1|_{X_c^{(u)}}) = n(\mathcal{L}_1) - p(\mathcal{U}) - z(\mathcal{U})$. When $n(\mathcal{L}_0) = 0$, the Morse index defines uniquely the unstable subspace of the linearized system according to corollary 3.10. When both $\mathcal{L}_0$ and $\mathcal{L}_1$ are not positive definite, complex instabilities may occur and they are defined by theorems 3.8 and 3.9.

In the end of this section, we show that the constraint (2.22) does not appear in theorems 3.6–3.9, due to the Galileo invariance (1.7). A general family of stationary solutions is defined as

$$\psi_n(z,x) = \Psi_n(x - 2vz - s)\mathrm{e}^{\mathrm{i}\omega_n z + \mathrm{i}\theta_n}. \tag{3.5}$$

The general stationary solutions (3.5) are critical points of the Lyapunov functional in the form:

$$\Lambda[\boldsymbol{\psi}] = H[\boldsymbol{\psi}] + \sum_{n=1}^{N} \omega_n Q_n[\boldsymbol{\psi}] + vP[\boldsymbol{\psi}], \tag{3.6}$$

where $Q_n[\boldsymbol{\psi}]$, $H[\boldsymbol{\psi}]$ and $P[\boldsymbol{\psi}]$ are given by (1.5), (1.8) and (1.9). The general Hessian matrix $\boldsymbol{U}_H$ has the structure

$$\mathcal{U}_H = \begin{bmatrix} \mathcal{U} & \dfrac{\partial \boldsymbol{Q}_\mathrm{s}}{\partial v} \\ \dfrac{\partial \boldsymbol{Q}_\mathrm{s}^\mathrm{T}}{\partial v} & \dfrac{\partial P_\mathrm{s}}{\partial v} \end{bmatrix}, \tag{3.7}$$

where $\boldsymbol{Q}_\mathrm{s} = (Q_{1s}, \ldots, Q_{Ns})^\mathrm{T}$, $Q_{ns}(\boldsymbol{\omega}, v) = Q_n[\boldsymbol{\Psi}]$ and $P_\mathrm{s}(\boldsymbol{\omega}, v) = P[\boldsymbol{\Psi}]$. It follows from the Galileo invariance (1.7) that a transformation,

$$\Psi_n(x) = \Phi_n(x)\mathrm{e}^{\mathrm{i}d_n^{-1}vx}, \qquad \omega_n = \beta_n + \frac{v^2}{d_n}, \tag{3.8}$$

expresses $Q_{ns}(\boldsymbol{\omega}, v)$, $P_s(\boldsymbol{\omega}, v)$ as functions of $\boldsymbol{\beta}$:

$$\frac{\partial Q_{ns}}{\partial v}(\boldsymbol{\omega}, v) = -\sum_{m=1}^{N} \frac{2v}{d_m} \frac{\partial Q_{ms}}{\partial \beta_n}(\boldsymbol{\beta}), \tag{3.9}$$

$$\frac{\partial P_\mathrm{s}}{\partial v}(\boldsymbol{\omega}, v) = -\sum_{n=1}^{N} \frac{2}{d_n} Q_{ns}(\boldsymbol{\beta}) + \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{4v^2}{d_n d_m} \frac{\partial Q_{ms}}{\partial \beta_n}(\boldsymbol{\beta}). \tag{3.10}$$

As a result, the quadratic form for $\boldsymbol{x} \in \mathbb{C}^{N+1}$ transforms to a quadratic form for $\boldsymbol{y} \in \mathbb{C}^N$ as

$$\langle \boldsymbol{x}, \mathcal{U}_H \boldsymbol{x} \rangle_{\mathbb{C}^{N+1}} = \langle \boldsymbol{y}, \mathcal{U} \boldsymbol{y} \rangle_{\mathbb{C}^N} - \left(\sum_{n=1}^{N} \frac{2Q_{ns}}{d_n} \right) |x_{N+1}|^2, \tag{3.11}$$

where

$$y_n = x_n - \frac{2v}{d_n} x_{N+1}, \quad n = 1, \ldots, N,$$

and $\langle \boldsymbol{a}, \boldsymbol{b} \rangle_{\mathbb{C}^N} = \sum_{n=1}^{N} \bar{a}_n b_n$. The additional eigenvalue for x_{N+1} is always negative, such that $p(\mathcal{U}_H) = p(\mathcal{U})$ and $z(\mathcal{U}_H) = z(\mathcal{U})$. Therefore, stability theorems are not affected by the constraint (2.22), due to the Galileo invariance (1.7).

4. Eigenvalues of constrained spectral problems for $\mathcal{L}_1$ and $\mathcal{L}_0$

Here we prove theorem 3.6 by counting eigenvalues of constrained spectral problems for $\mathcal{L}_1$ and $\mathcal{L}_0$ from matrix analysis. This method was first used in Pelinovsky & Kivshar (2000). Constrained spectral problems were also considered in Buslaev & Perelman (1993) and Grillakis (1988).

Given the spectrum of $\mathcal{L}_1$ and $\mathcal{L}_0$ in $L^2(\mathbb{R})$, we study the spectrum of operators $\mathcal{L}_1$ and $\mathcal{L}_0$ in $X_c^{(u)}(\mathbb{R})$, defined by (2.21). The constrained space $X_c^{(u)}(\mathbb{R})$ is an orthogonal complement of the kernel of $\mathcal{L}_0$ in $L^2(\mathbb{R})$. The spectrum of $\mathcal{L}_1$ and $\mathcal{L}_0$ is complete in $X_c^{(u)}(\mathbb{R})$, due to the abstract result in Hislop & Sigal (1996, proposition 2.7).

Proposition 4.1 (Hislop & Sigal 1996). *Let M be a closed subspace of a Hilbert space $\mathcal{H}$ and $M^{\perp}$ be the orthogonal complement of M in $\mathcal{H}$, such that $M^{\perp} = \{x \in \mathcal{H} : \langle x, m\rangle = 0\ \forall m \in M\}$. The subset $M^{\perp}$ is a closed subspace of $\mathcal{H}$ and is therefore a Hilbert space.*

Proposition 4.2. *Let negative eigenvalues of $\mathcal{L}_0$ in $X_c^{(u)}(\mathbb{R})$ be defined by the problem*

$$\mathcal{L}_0 \boldsymbol{u} = \lambda \boldsymbol{u}, \quad \boldsymbol{u} \in X_c^{(u)}(\mathbb{R}),\ \lambda < 0. \tag{4.1}$$

Then, $n(\mathcal{L}_0|_{X_c^{(u)}}) = n(\mathcal{L}_0)$.

Proof. Eigenvectors of $\mathcal{L}_0$ for negative eigenvalues λ are orthogonal to the eigenvectors $\{\Phi_n(x)\boldsymbol{e}_n\}_{n=1}^N$ of the kernel of $\mathcal{L}_0$. Therefore, they belong to $X_c^{(u)}(\mathbb{R})$. ■

Proposition 4.3. *Let $z(\mathcal{L}_1) = 1$. Let negative and zero eigenvalues of $\mathcal{L}_1$ in $X_c^{(u)}(\mathbb{R})$ be defined by the problem*

$$\mathcal{L}_1 \boldsymbol{u} = \lambda \boldsymbol{u} - \sum_{m=1}^{N} \nu_m \Phi_m(x) \boldsymbol{e}_m, \quad \boldsymbol{u} \in X_c^{(u)}(\mathbb{R}),\ \lambda \leqslant 0, \tag{4.2}$$

where $\nu_1, \ldots, \nu_N$ are Lagrange multipliers. Then, $n(\mathcal{L}_1|_{X_c^{(u)}}) = n(\mathcal{L}_1) - p(\mathcal{U}) - z(\mathcal{U})$ and $z(\mathcal{L}_1|_{X_c^{(u)}}) = 1 + z(\mathcal{U})$.

In order to prove proposition 4.3, we introduce some notation. We denote negative eigenvalues of $\mathcal{L}_1$ in $L^2(\mathbb{R})$ as λ_{-k} with orthonormal eigenvectors $\boldsymbol{u}_{-k}(x)$, accounting for their multiplicity. We order negative eigenvalues from the minimal eigenvalue $\lambda_{-n(\mathcal{L}_1)}$ to the maximal eigenvalue $\lambda_{-1} < 0$. We also write spectral decomposition in $L^2(\mathbb{R})$ as a sum of three terms, $\sum_{\lambda_{-k}<0}$, $\sum_{\lambda_{-k}=0}$ and $\sum_{\lambda_{-k}>0}$, where $\sum_{\lambda_{-k}<0}$ denotes $n(\mathcal{L}_1)$ terms from the negative discrete spectrum of $\mathcal{L}_1$, $\sum_{\lambda_{-k}=0}$ denotes the $z(\mathcal{L}_1) = 1$ term from the kernel of $\mathcal{L}_1$, and $\sum_{\lambda_k>0}$ denotes infinite-dimensional positive spectrum of $\mathcal{L}_1$, which includes isolated and embedded eigenvalues and N branches of the continuous spectrum.

When $\ker(\mathcal{L}_1 - \lambda) = \{0\}$ in $L^2(\mathbb{R})$, the constrained spectral problem (4.2) has a solution only if there exists a non-zero solution of the homogeneous linear system for $\nu_1, \ldots, \nu_N$:

$$\sum_{m=1}^{N} \langle \Phi_n \boldsymbol{e}_n, (\lambda - \mathcal{L}_1)^{-1} \Phi_m \boldsymbol{e}_m \rangle \nu_m = 0, \quad n = 1, \ldots, N. \tag{4.3}$$

By spectral calculus (Reed & Simon 1978), the linear system (4.3) is equivalent to zero eigenvalues of the matrix $\mathcal{A}(\lambda)$ with the elements

$$\mathcal{A}_{n,m}(\lambda) = \sum_{\lambda_{-k}<0} \frac{\langle \Phi_n \boldsymbol{e}_n, \boldsymbol{u}_{-k}\rangle \langle \boldsymbol{u}_{-k}, \Phi_m \boldsymbol{e}_m\rangle}{\lambda - \lambda_{-k}} + \sum_{\lambda_k>0} \frac{\langle \Phi_n \boldsymbol{e}_n, \boldsymbol{u}_k\rangle \langle \boldsymbol{u}_k, \Phi_m \boldsymbol{e}_m\rangle}{\lambda - \lambda_k}, \quad (4.4)$$

where we have used that

$$\langle \boldsymbol{\Phi}'(x), \Phi_m \boldsymbol{e}_m\rangle = 0.$$

When there exists a zero eigenvalue of $\mathcal{A}(\lambda)$, there exists a solution $\boldsymbol{u} \in X_{\mathrm{c}}^{(u)}(\mathbb{R})$, which is represented with the spectral decomposition in $L^2(\mathbb{R})$:

$$\boldsymbol{u}(x) = \sum_{m=1}^{N} \nu_m \left[\sum_{\lambda_{-k}<0} \frac{\langle \boldsymbol{u}_{-k}, \Phi_m \boldsymbol{e}_m\rangle}{\lambda - \lambda_{-k}} \boldsymbol{u}_{-k}(x) + \sum_{\lambda_k>0} \frac{\langle \boldsymbol{u}_k, \Phi_m \boldsymbol{e}_m\rangle}{\lambda - \lambda_k} \boldsymbol{u}_k(x) \right]. \quad (4.5)$$

The following results follow from analysis of eigenvalues of $\mathcal{A}(\lambda)$.

Lemma 4.4. *The matrix eigenvalue problem $\mathcal{A}(\lambda)\boldsymbol{\nu} = \alpha(\lambda)\boldsymbol{\nu}$, $\lambda \in \mathbb{R}$ has N real eigenvalues $\alpha_1(\lambda), \dots, \alpha_N(\lambda)$, which are meromorphic functions of λ for $\lambda \leqslant 0$.*

Proof. The matrix $\mathcal{A}(\lambda)$ has N real eigenvalues $\alpha(\lambda)$, since it is Hermitian for $\lambda \in \mathbb{R}$ and $\lambda \leqslant 0$. Coefficients of $\mathcal{A}(\lambda)$ have pole singularities at $\lambda = \lambda_{-k}$ for $\lambda \leqslant 0$, unless $\langle \Phi_n \boldsymbol{e}_n, \boldsymbol{u}_{-k}\rangle = 0$, $n = 1, \dots, N$. Since $\Phi_n \in L^2(\mathbb{R})$, $\boldsymbol{u} \in L^2(\mathbb{R})$ and $\langle \Phi_n \boldsymbol{e}_n, \boldsymbol{u}\rangle < \infty$, the series for $\mathcal{A}_{n,m}(\lambda)$ are bounded and converge for $\lambda \neq \lambda_{-k}$. In the limit $\lambda \to -\infty$, $\mathcal{A}_{n,m}(\lambda)$ converges to zero uniformly. As a result, all eigenvalues $\alpha_n(\lambda)$, $n = 1, \dots, N$ are meromorphic functions for $\lambda \leqslant 0$, which may have only pole singularities at $\lambda = \lambda_{-k}$. ■

Lemma 4.5. *Eigenvalues $\alpha_1(\lambda), \dots, \alpha_N(\lambda)$ are decreasing functions of λ in the domain $\mathcal{D} = \{\lambda \leqslant 0 : \lambda \neq \lambda_{-k},\ k = 1, \dots, n(\mathcal{L}_1)\}$. All eigenvalues $\alpha_n(\lambda)$, $n = 1, \dots, N$ are negative for $\lambda < \lambda_{-n(\mathcal{L}_1)}$.*

Proof. For Hermitian matrices, the set of eigenvalues $\{\alpha_n(\lambda)\}_{n=1}^{N}$ corresponds to the set of orthonormal eigenvectors $\{\boldsymbol{\nu}^{(n)}\}_{n=1}^{N}$ such that $\langle \boldsymbol{\nu}^{(n')}, \boldsymbol{\nu}^{(n)}\rangle_{\mathbb{C}^N} = \delta_{n',n}$, where $\langle \boldsymbol{f}, \boldsymbol{g}\rangle_{\mathbb{C}^N} = \sum_{n=1}^{N} \bar{f}_n g_n$. We construct quadratic forms associated to the eigenvalue-eigenvector pairs $(\alpha_n, \boldsymbol{\nu}^{(n)})$, $n = 1, \dots, N$:

$$\alpha_n(\lambda) = \langle \boldsymbol{\nu}^{(n)}, \mathcal{A}(\lambda)\boldsymbol{\nu}^{(n)}\rangle_{\mathbb{C}^N}, \alpha_n'(\lambda) = \langle \boldsymbol{\nu}^{(n)}, \mathcal{A}'(\lambda)\boldsymbol{\nu}^{(n)}\rangle_{\mathbb{C}^N}. \quad (4.6)$$

Computing the derivative of $\mathcal{A}'(\lambda)$, we rewrite the second equality in (4.6) as

$$\alpha_n'(\lambda) = -\left(\sum_{\lambda_{-k}<0} \frac{b_{-k}}{(\lambda - \lambda_{-k})^2} + \sum_{\lambda_k>0} \frac{b_k}{(\lambda - \lambda_k)^2} \right) = -\langle \boldsymbol{u}^{(n)}, \boldsymbol{u}^{(n)}\rangle < 0, \quad (4.7)$$

where $\boldsymbol{u}^{(n)}$ is given by (4.5) with $\boldsymbol{\nu} = \boldsymbol{\nu}^{(n)}$ and

$$b_{\pm k} = \left| \sum_{m=1}^{N} \langle \Phi_m \boldsymbol{e}_m, \boldsymbol{u}_{\pm k}\rangle \nu_m^{(n)} \right|^2 \geqslant 0. \quad (4.8)$$

As a result, all eigenvalues $\alpha_n(\lambda)$, $n = 1, \ldots, N$, are decreasing functions of λ in the domain $\mathcal{D}$. In order to prove that all eigenvalues $\alpha_n(\lambda)$, $n = 1, \ldots, N$, are negative for $\lambda < \lambda_{-n(\mathcal{L}_1)}$, we find from (4.3) and (4.4) that

$$\lim_{\lambda \to -\infty} (\lambda \mathcal{A}_{n,m}(\lambda)) = \langle \Phi_n \boldsymbol{e}_n, \Phi_m \boldsymbol{e}_m \rangle = Q_{ns} \delta_{n,m}, \tag{4.9}$$

where $Q_{ns}(\boldsymbol{\beta}) = Q_n(\Phi)$ is defined by (1.5). It follows from the first equality in (4.6) and (4.9) that

$$\lim_{\lambda \to -\infty} \lambda \alpha_n(\lambda) = Q_{ns}(\boldsymbol{\beta}) > 0, \quad n = 1, \ldots, N,$$

such that $\lim_{\lambda \to -\infty} \alpha_n(\lambda) = -0$, $n = 1, \ldots, N$. Since eigenvalues $\alpha(\lambda)$ are continuous and decreasing for $\lambda < \lambda_{-n(\mathcal{L}_1)}$, they remain negative for all values of $\lambda < \lambda_{-n(\mathcal{L}_1)}$. ■

Lemma 4.6. *Let λ_{-k} be a negative eigenvalue of $\mathcal{L}_1$ in $L^2(\mathbb{R})$ with multiplicity q_{-k}, such that at most $q_{-k}^{\parallel}$ linearly independent eigenvectors $\boldsymbol{u}_{-k}(x)$ belong to the constrained space $X_c^{(u)}(\mathbb{R})$ and $q_{-k}^{\perp} = q_{-k} - q_{-k}^{\parallel}$. There exist $(N - q_{-k}^{\perp})$ eigenvalues $\alpha_n(\lambda)$ that are continuous at $\lambda = \lambda_{-k}$ and $q_{-k}^{\perp}$ eigenvalues $\alpha_n(\lambda)$ that have infinity discontinuities, jumping from negative infinity for $\lambda = \lambda_{-k} - 0$ to positive infinity for $\lambda = \lambda_{-k} + 0$.*

Proof. In the limit $\lambda \to \lambda_{-k}$, we find that

$$\begin{aligned} \lim_{\lambda \to \lambda_{-k}} (\lambda - \lambda_{-k}) \mathcal{A}_{n,m}(\lambda) &= \sum_{r=1}^{q_{-k}} \langle \Phi_n \boldsymbol{e}_n, \boldsymbol{u}_{-k_r} \rangle \langle \boldsymbol{u}_{-k_r}, \Phi_m \boldsymbol{e}_m \rangle \\ &= \sum_{r=1}^{q_{-k}^{\perp}} \langle \Phi_n \boldsymbol{e}_n, \boldsymbol{u}_{-k_r} \rangle \langle \boldsymbol{u}_{-k_r}, \Phi_m \boldsymbol{e}_m \rangle. \end{aligned} \tag{4.10}$$

Denote $\mathcal{B}_{-k} = \lim_{\lambda \to \lambda_{-k}} (\lambda - \lambda_{-k}) \mathcal{A}(\lambda)$. The quadratic form $\langle \boldsymbol{\nu}, \mathcal{B}_{-k} \boldsymbol{\nu} \rangle_{\mathbb{C}^N}$ is diagonalized in normal variables,

$$x_r = \sum_{m=1}^{N} \langle \boldsymbol{u}_{-k_r}, \Phi_m \boldsymbol{e}_m \rangle \nu_m,$$

such that

$$\langle \boldsymbol{\nu}, \mathcal{B}_{-k} \boldsymbol{\nu} \rangle_{\mathbb{C}^N} = \sum_{r=1}^{q_{-k}^{\perp}} |x_r|^2.$$

Therefore, the matrix $\mathcal{B}_{-k}$ has exactly $q_{-k}^{\perp}$ positive eigenvalues and $(N - q_{-k}^{\perp})$ zero eigenvalues. Positive eigenvalues of $\mathcal{B}_{-k}$ correspond to $q_{-k}^{\perp}$ eigenvalues $\alpha_n(\lambda)$ jumping from negative infinity for $\lambda = \lambda_{-k} - 0$ to positive infinity for $\lambda = \lambda_{-k} + 0$. Zero eigenvalues of $\mathcal{B}_{-k}$ correspond to $(N - q_{-k}^{\perp})$ eigenvalues $\alpha_n(\lambda)$ that are continuous and have convergent Taylor series at $\lambda = \lambda_{-k}$. ■

Lemma 4.7. *At $\lambda = 0$, there exist $p(\mathcal{U})$ positive, $z(\mathcal{U})$ zero and $n(\mathcal{U})$ negative eigenvalues $\alpha_n(0)$, $n = 1, \ldots, N$.*

Proof. At $\lambda = 0$, the constrained eigenvalue problem (4.2) has an exact solution in $L^2(\mathbb{R})$:

$$\boldsymbol{u}_{\lambda=0}(x) = \sum_{m=1}^{N} \nu_m \frac{\partial \boldsymbol{\Phi}(x)}{\partial \beta_m} + c_0 \boldsymbol{\Phi}'(x), \tag{4.11}$$

where c_0 is not defined and

$$\mathcal{L}_1 \frac{\partial \boldsymbol{\Phi}}{\partial \beta_m} = -\Phi_m(x) \boldsymbol{e}_m. \tag{4.12}$$

Substituting (4.11) into (2.21), we find that $\mathcal{A}(0) = \frac{1}{2}\mathcal{U}$, where $\mathcal{U}$ is defined in (2.7). ■

Proof of proposition 4.3. We consider eigenvalues $\alpha_n(\lambda)$, $n = 1, \ldots, N$, as meromorphic functions of λ for $\lambda \leqslant 0$. Starting with small negative values at $\lambda \to -\infty$, eigenvalues $\alpha_n(\lambda)$, $n = 1, \ldots, N$, decrease as λ increases toward $n(\mathcal{L}_1)$ pole singularities at $\lambda = \lambda_{-k}$. At each pole singularity, $q_{-k}^{\perp}$ eigenvalues $\alpha(\lambda)$ jump and pop up to the positive half-plane. The total number of jumping eigenvalues for $\lambda < 0$ is

$$\sum_{\lambda_{-k}<0} q_{-k}^{\perp}.$$

Only jumping eigenvalues may cross the value $\alpha(\lambda) = 0$, which corresponds to a negative eigenvalue λ of the constrained problem (4.2) in space $X_{\mathrm{c}}^{(u)}(\mathbb{R})$. By lemma 4.7, the total number of zeros of $\alpha(\lambda)$ for $\lambda \leqslant 0$ is

$$\sum_{\lambda_{-k}<0} q_{-k}^{\perp} - p(\mathcal{U}).$$

At each $\lambda = \lambda_{-k}$, there are $q_{-k}^{\parallel}$ eigenfunctions $\boldsymbol{u}_{-k}(x)$ that lie in the constrained space $X_{\mathrm{c}}^{(u)}(\mathbb{R})$. Therefore, the total number of eigenvalues λ in $X_{\mathrm{c}}^{(u)}(\mathbb{R})$ for $\lambda \leqslant 0$ is

$$\sum_{\lambda_{-k}} q_{-k}^{\perp} - p(\mathcal{U}) + \sum_{\lambda_{-k}} q_{-k}^{\parallel} = \sum_{\lambda_{-k}} q_{-k} - p(\mathcal{U}) = n(\mathcal{L}_1) - p(\mathcal{U}).$$

The lemma is proved by subtracting the number $z(\mathcal{U})$ of zero eigenvalues for $\lambda = 0$. ■

Proposition 4.8. *Let assumption 2.1 apply. The kernel of $\mathcal{L}_1$ in $L^2(\mathbb{R})$ lies in $X_{\mathrm{c}}^{(u)}(\mathbb{R})$ for any $1 \leqslant z(\mathcal{L}_1) \leqslant N$.*

Proof. By lemma 2.4, the Hessian matrix $\mathcal{U}$ has only bounded eigenvalues. Suppose there exists an eigenvector $\boldsymbol{u}_0(x)$ of the kernel of $\mathcal{L}_1$ in $L^2(\mathbb{R})$ such that $\boldsymbol{u}_0 \notin X_{\mathrm{c}}^{(u)}(\mathbb{R})$. By lemma 4.6, there exists an eigenvalue $\alpha_n(\lambda)$ that diverges at $\lambda \to -0$. The eigenvalue contradicts lemma 4.7 since all eigenvalues of $\mathcal{U}$ are bounded. ■

Theorem 3.6 is proved with propositions 4.2, 4.3 and 4.8. Using theorem 3.6 and proposition 4.1, we formulate the following main result of this section.

Proposition 4.9. *Let $\mathcal{L}$ be a symmetric matrix Schrödinger operator, either $\mathcal{L}_0$ or $\mathcal{L}_1$. There exists $\mathcal{L}$-invariant decomposition in $X_{\rm c}^{(u)}(\mathbb{R})$, such that*

$$\forall \boldsymbol{u} \in X_{\rm c}^{(u)}(\mathbb{R}), \quad \boldsymbol{u}(x) = \sum_{m=1}^{n_{\rm c}} a_m \boldsymbol{u}_m(x) + \sum_{m=0}^{z_{\rm c}} b_m \boldsymbol{u}_m(x) + \boldsymbol{u}^+(x), \tag{4.13}$$

where $\boldsymbol{u}_m(x)$ are eigenvectors of $\mathcal{L}$ in $X_{\rm c}^{(u)}(\mathbb{R})$ for negative and zero eigenvalues,

$$n_{\rm c} = n(\mathcal{L}|_{X_{\rm c}^{(u)}}), \qquad z_{\rm c} = z(\mathcal{L}|_{X_{\rm c}^{(u)}}), \qquad \langle \boldsymbol{u}^+, \mathcal{L}\boldsymbol{u}^+ \rangle \geqslant c\langle \boldsymbol{u}^+, \boldsymbol{u}^+ \rangle, \quad c > 0.$$

The quadratic form for $\mathcal{L}$ is diagonalized as follows:

$$\langle \boldsymbol{u}, \mathcal{L}\boldsymbol{u} \rangle = \sum_{m=1}^{n(\mathcal{L}|_{X_{\rm c}^{(u)}})} \lambda_m |a_m|^2 + \langle \boldsymbol{u}^+, \mathcal{L}\boldsymbol{u}^+ \rangle. \tag{4.14}$$

5. Eigenvalues of the linearized problem for $\boldsymbol{\mathcal{A}}$

Here we study the spectrum of the non-self-adjoint linearized problem (2.14) with the simultaneous block-diagonalization of two self-adjoint operators $\mathcal{L}_1$ and $\mathcal{L}_0$. A similar generalized eigenvalue problem for $\mathcal{L}_1$ and $\mathcal{L}_0^{-1}$ was used in Grillakis (1988, 1990). We analyse this problem in order to establish the inertia law for spectral stability. The inertia law was not considered in Grillakis (1988, 1990).

Lemma 5.1. *There exists a mapping $\gamma = -\lambda^2$ of the non-zero spectrum of $\mathcal{A}$ in $X_{\rm c}(\mathbb{R})$ to the non-zero spectrum of the problem*

$$\mathcal{L}_1 \boldsymbol{u} = \gamma \mathcal{L}_0^{-1} \boldsymbol{u}, \quad \boldsymbol{u} \in X_{\rm c}^{(u)}(\mathbb{R}). \tag{5.1}$$

Proof. Eigenvectors $\{\Phi_n(x)\boldsymbol{e}_n\}_{n=1}^N$ form a basis in the kernel of $\mathcal{L}_0$. The operator $\mathcal{L}_0$ is invertible on the subspace $\boldsymbol{u} \in X_{\rm c}^{(u)}(\mathbb{R})$. It follows from (2.23) that $\boldsymbol{w} = \lambda \mathcal{L}_0^{-1}\boldsymbol{u}$ for any $\lambda \neq 0$, such that the problem (2.23) is equivalent to (5.1) for any $\gamma \neq 0$. Two eigenvectors $(\boldsymbol{u}, \boldsymbol{w})^{\rm T}$ and $(\boldsymbol{u}, -\boldsymbol{w})^{\rm T}$ of $\mathcal{A}$ in $X_{\rm c}(\mathbb{R})$ corresponds to a single eigenvector $\boldsymbol{u}$ of the problem (5.1) in $X_{\rm c}^{(u)}(\mathbb{R})$. ∎

Corollary 5.2. *Let $\gamma = \gamma_m$ be a non-zero eigenvalue of (5.1) with the eigenvector $\boldsymbol{u} = \boldsymbol{u}_m(x)$ in $X_{\rm c}^{(u)}(\mathbb{R})$, such that*

$$\langle \boldsymbol{u}_m, \mathcal{L}_1 \boldsymbol{u}_m \rangle = \gamma_m \langle \boldsymbol{u}_m, \mathcal{L}_0^{-1} \boldsymbol{u}_m \rangle. \tag{5.2}$$

Eigenvalue γ_m is real if either $\mathcal{L}_1$ or $\mathcal{L}_0$ is positive definite.

Problem (5.1) is a classical problem of simultaneous block-diagonalization of two self-adjoint operators $\mathcal{L}_1$ and $\mathcal{L}_0^{-1}$. Each operator can be orthogonally diagonalized due to proposition 4.9. However, the orthogonal diagonalization (4.14) is relevant for the problem (5.1) only if the operators $\mathcal{L}_1$ and $\mathcal{L}_0^{-1}$ commute, such that there exists a common basis for $\langle \boldsymbol{u}, \mathcal{L}_1 \boldsymbol{u} \rangle$ and $\langle \boldsymbol{u}, \mathcal{L}_0^{-1} \boldsymbol{u} \rangle$ (Gelfand 1961). Since operators $\mathcal{L}_1$ and $\mathcal{L}_0^{-1}$ do not commute, eigenvectors of $\mathcal{L}_1$ and $\mathcal{L}_0^{-1}$ in $X_{\rm c}^{(u)}(\mathbb{R})$ are not related to eigenvectors of problem (5.1). Moreover, complex eigenvalues and multiple eigenvalues with higher algebraic multiplicity may generally occur in problem (5.1).

Lemma 5.3. *The spectrum of (5.1) is real if $\boldsymbol{\Phi}(x)$ is the ground state with $\Phi_n(x) > 0\ \forall x \in \mathbb{R}$, $n = 1, \dots, N$. Moreover, the positive-definite operator $\mathcal{L}_0$ can be factorized as $\mathcal{L}_0 = \mathcal{S}^+\mathcal{S}^-$, where $\mathcal{S}^\pm$ are diagonal operators with the elements*

$$\mathcal{S}^{\pm}_{n,m} = \sqrt{d_n}\left(\frac{\Phi_n'(x)}{\Phi_n(x)} \pm \frac{\mathrm{d}}{\mathrm{d}x}\right)\delta_{n,m}. \tag{5.3}$$

Proof. The factorization formula (5.3) follows from explicit computations:

$$\mathcal{S}^+_{n,n}\mathcal{S}^-_{n,n} = d_n\left(\frac{\Phi_n''}{\Phi_n} - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) = (\mathcal{L}_0)_{n,n}.$$

Using the transformation $\boldsymbol{u} = \mathcal{S}^+\boldsymbol{v}$, we rewrite (5.1) in the form

$$\mathcal{S}^-\mathcal{L}_1\mathcal{S}^+\boldsymbol{v} = \gamma\boldsymbol{v}.$$

Since $\mathcal{S}^-\mathcal{L}_1\mathcal{S}^+$ is a self-adjoint operator, all eigenvalues γ are real. It is also clear from (5.3) that the kernel of $\mathcal{S}^+$ is empty, such that the transformation $\boldsymbol{u} = \mathcal{S}^+\boldsymbol{v}$ is invertible. ■

Lemma 5.4. *Let $\gamma = \gamma_k = \gamma_{\mathrm{R}k} + \mathrm{i}\gamma_{\mathrm{I}k}$ be a complex eigenvalue of (5.1) such that $\gamma_{\mathrm{R}k}, \gamma_{\mathrm{I}k} \neq 0$, with a complex-valued eigenvector $\boldsymbol{u}_k(x) = \boldsymbol{u}_{\mathrm{R}k}(x) + \mathrm{i}\boldsymbol{u}_{\mathrm{I}k}(x)$. A linear combination of two real-valued eigenvectors $\boldsymbol{u}(x) = a_k\boldsymbol{u}_{\mathrm{R}k}(x) + b_k\boldsymbol{u}_{\mathrm{I}k}(x)$ diagonalizes the quadratic forms $\langle \boldsymbol{u}, \mathcal{L}_1\boldsymbol{u}\rangle$ and $\langle \boldsymbol{u}, \mathcal{L}_0^{-1}\boldsymbol{u}\rangle$ with respect to Jordan blocks,*

$$\langle \boldsymbol{u}, \mathcal{L}_1\boldsymbol{u}\rangle = \boldsymbol{a}_k^{\mathrm{T}}\hat{\gamma}_k\hat{l}_k\boldsymbol{a}_k, \qquad \langle \boldsymbol{u}, \mathcal{L}_0^{-1}\boldsymbol{u}\rangle = \boldsymbol{a}_k^{\mathrm{T}}\hat{l}_k\boldsymbol{a}_k, \tag{5.4}$$

where $\boldsymbol{a}_k = (a_k, b_k)^{\mathrm{T}}$,

$$\hat{\gamma}_k = \begin{pmatrix} \gamma_{\mathrm{R}k} & -\gamma_{\mathrm{I}k} \\ \gamma_{\mathrm{I}k} & \gamma_{\mathrm{R}k} \end{pmatrix}, \qquad \hat{l}_k = \begin{pmatrix} l_{\mathrm{R}k} & l_{\mathrm{I}k} \\ l_{\mathrm{I}k} & -l_{\mathrm{R}k} \end{pmatrix} \tag{5.5}$$

and

$$l_{\mathrm{R}k} = \langle \boldsymbol{u}_{\mathrm{R}k}, \mathcal{L}_0^{-1}\boldsymbol{u}_{\mathrm{R}k}\rangle = -\langle \boldsymbol{u}_{\mathrm{I}k}, \mathcal{L}_0^{-1}\boldsymbol{u}_{\mathrm{I}k}\rangle, \tag{5.6}$$

$$l_{\mathrm{I}k} = \langle \boldsymbol{u}_{\mathrm{I}k}, \mathcal{L}_0^{-1}\boldsymbol{u}_{\mathrm{R}k}\rangle = \langle \boldsymbol{u}_{\mathrm{R}k}, \mathcal{L}_0^{-1}\boldsymbol{u}_{\mathrm{I}k}\rangle. \tag{5.7}$$

Proof. Since the quadratic forms $\langle \boldsymbol{u}_k, \mathcal{L}_1\boldsymbol{u}_k\rangle$ and $\langle \boldsymbol{u}_k, \mathcal{L}_0^{-1}\boldsymbol{u}_k\rangle$ in (5.2) are real valued, the eigenvalues γ_k can be complex only if

$$\langle \boldsymbol{u}_k, \mathcal{L}_1\boldsymbol{u}_k\rangle = \langle \boldsymbol{u}_k, \mathcal{L}_0^{-1}\boldsymbol{u}_k\rangle = 0. \tag{5.8}$$

The zero inner product (5.8) for $\mathcal{L}_0^{-1}$ results in relations (5.6) and (5.7). The Jordan blocks (5.4) and (5.5) follow from direct computations. ■

Lemma 5.5. *Let $\gamma = \gamma_m$ be a real eigenvalue of (5.1) with a single real-valued eigenvector $\boldsymbol{u}_m(x)$ in $X_{\mathrm{c}}^{(u)}(\mathbb{R})$. The eigenvalue $\gamma = \gamma_m$ is a multiple eigenvalue of higher algebraic multiplicity if and only if*

$$l_m = \langle \boldsymbol{u}_m, \mathcal{L}_0^{-1}\boldsymbol{u}_m\rangle = 0. \tag{5.9}$$

Proof. The eigenvalue $\gamma = \gamma_m$ is a degenerate eigenvalue of higher algebraic multiplicity if and only if there exists a solution of the derivative problem:

$$\mathcal{L}_1\boldsymbol{u}_m' = \gamma_m\mathcal{L}_0^{-1}\boldsymbol{u}_m' + \mathcal{L}_0^{-1}\boldsymbol{u}_m, \qquad \boldsymbol{u}_m' \in X_{\mathrm{c}}^{(u)}(\mathbb{R}). \tag{5.10}$$

The condition (5.9) follows by the Fredholm alternative theorem. ■

A complex eigenvalue $\gamma = \gamma_k = \gamma_{Rk} + i\gamma_{Ik}$, such that $\gamma_{Rk}, \gamma_{Ik} \neq 0$, with a single complex-valued eigenvector $\boldsymbol{u}_k(x) = \boldsymbol{u}_{Rk}(x) + i\boldsymbol{u}_{Ik}(x)$ is a multiple eigenvalue of higher algebraic multiplicity if and only if $l_{Rk} = l_{Ik} = 0$ in (5.6) and (5.7). According to assumption 2.14(ii), we consider the generic case when non-zero eigenvalues of higher algebraic multiplicity do not occur in the problem (5.1).

Lemma 5.6. *Let assumption 2.14(ii) be satisfied. Eigenvectors $\boldsymbol{u}_m(x)$ for real eigenvalues γ_m and $(\boldsymbol{u}_{Rk}(x), \boldsymbol{u}_{Ik}(x))$ for complex eigenvalues $\gamma_k = \gamma_{Rk} + i\gamma_{Ik}$ are orthogonal with respect to operator $\mathcal{L}_0^{-1}$,*

$$\langle \boldsymbol{u}_{m'}, \mathcal{L}_0^{-1} \boldsymbol{u}_m \rangle = l_m \delta_{m',m}, \tag{5.11}$$

and

$$\left.\begin{aligned} \langle \boldsymbol{u}_{Rk'}, \mathcal{L}_0^{-1} \boldsymbol{u}_{Rk} \rangle = -\langle \boldsymbol{u}_{Ik'}, \mathcal{L}_0^{-1} \boldsymbol{u}_{Ik} \rangle = l_{Rk} \delta_{k',k}, \\ \langle \boldsymbol{u}_{Ik'}, \mathcal{L}_0^{-1} \boldsymbol{u}_{Rk} \rangle = \langle \boldsymbol{u}_{Rk'}, \mathcal{L}_0^{-1} \boldsymbol{u}_{Ik} \rangle = l_{Ik} \delta_{k',k}, \end{aligned}\right\} \tag{5.12}$$

where $l_m \neq 0$ and $|l_{Rk}|^2 + |l_{Ik}|^2 \neq 0$.

Proof. Orthogonality relations (5.11) and (5.12) for eigenvectors of the problem (5.1) follow from the identity

$$(\gamma_{m'} - \gamma_m) \langle \boldsymbol{u}_{m'}, \mathcal{L}_0^{-1} \boldsymbol{u}_m \rangle = 0. \tag{5.13}$$

By assumption 2.14(ii) and lemma 5.5, coefficients l_m and $|l_{Rk}|^2 + |l_{Ik}|^2$ are non-zero, since γ_m and $\gamma_k = \gamma_{Rk} + i\gamma_{Ik}$ are not eigenvalues of higher algebraic multiplicity. ■

Corollary 5.7. *The set of eigenvectors $\boldsymbol{u}_m(x)$ and $(\boldsymbol{u}_{Rk}(x), \boldsymbol{u}_{Ik}(x))$ is also orthogonal with respect to operator $\mathcal{L}_1$.*

We shall also consider the quadratic forms $\langle \boldsymbol{u}, \mathcal{L}_1 \boldsymbol{u} \rangle$ and $\langle \boldsymbol{u}, \mathcal{L}_0^{-1} \boldsymbol{u} \rangle$ for eigenvectors of the continuous spectrum of the problem (5.1). Let us introduce the $\mathcal{A}$-invariant decomposition of $X_c^{(u)}(\mathbb{R})$ into the discrete part for $\sigma_p(\mathcal{A})$ and the continuous part $Y_c^{(u)}(\mathcal{A})$ for $\sigma_{con}(\mathcal{A})$:

$$X_c^{(u)}(\mathbb{R}) = \sum_{\lambda \in \sigma_p(\mathcal{A})} \mathcal{N}_g(\mathcal{A} - \lambda) \oplus Y_c^{(u)}(\mathcal{A}), \qquad Y_c^{(u)}(\mathcal{A}) = \left[\sum_{\lambda \in \sigma_p(\mathcal{A})} \mathcal{N}_g(\mathcal{A}^* - \lambda) \right]^{\perp}, \tag{5.14}$$

where $\mathcal{A}^*$ is the adjoint operator and $\sigma_p(\mathcal{A}^*) = \sigma_p(\mathcal{A})$. According to assumption 2.14(i), we consider the general case, when $\sigma_{ess}(\mathcal{A})$ does not include semi-eigenvalues nor embedded eigenvalues.

Proposition 5.8. *Let assumption 2.14(i) be satisfied. The quadratic forms $\langle \boldsymbol{u}, \mathcal{L}_1 \boldsymbol{u} \rangle$ and $\langle \boldsymbol{u}, \mathcal{L}_0^{-1} \boldsymbol{u} \rangle$ are strictly positive in $Y_c^{(u)}(\mathcal{A})$, such that*

$$\forall \boldsymbol{u}^+ \in Y_c^{(u)}(\mathcal{A}): \quad \langle \boldsymbol{u}^+, \mathcal{L}_1 \boldsymbol{u}^+ \rangle \geqslant c_1 \langle \boldsymbol{u}^+, \boldsymbol{u}^+ \rangle, \qquad \langle \boldsymbol{u}^+, \mathcal{L}_0^{-1} \boldsymbol{u}^+ \rangle \geqslant c_0 \langle \boldsymbol{u}^+, \boldsymbol{u}^+ \rangle, \tag{5.15}$$

where $c_1 > 0$, $c_0 > 0$.

Proof of proposition 5.8 is given in § 7 with the use of wave functions of the problem (2.14). Wave functions of $\mathcal{A}$ with $N = 1$ were introduced in Buslaev & Perelman (1993), where orthogonality and completeness relations between the wave functions were derived by spectral analysis.

Remark 5.9. By assumption 2.14(iii), lemma 2.16 and corollary 2.17, the zero eigenvalue is simple in $\boldsymbol{u} \in X_{\mathrm{c}}^{(u)}(\mathbb{R})$.

Combining lemmas 5.4 and 5.6 and proposition 5.8, we formulate the following main result of this section.

Proposition 5.10. *Let assumption 2.14 be satisfied. There exists $\mathcal{A}$-invariant decomposition in $X_{\mathrm{c}}^{(u)}(\mathbb{R})$ such that*

$$\forall \boldsymbol{u} \in X_{\mathrm{c}}^{(u)}(\mathbb{R}): \quad \boldsymbol{u}(x) = \sum_k [a_k \boldsymbol{u}_{\mathrm{R}k}(x) + b_k \boldsymbol{u}_{\mathrm{I}k}(x)] + \sum_m c_m \boldsymbol{u}_m(x) + \boldsymbol{u}^+(x), \quad (5.16)$$

where $\boldsymbol{u}_m(x)$ are eigenvectors for real and zero eigenvalues γ_m, $(\boldsymbol{u}_{\mathrm{R}k}(x), \boldsymbol{u}_{\mathrm{I}k}(x))$ are eigenvectors for complex eigenvalues

$$\gamma_k = \gamma_{\mathrm{R}k} + \mathrm{i}\gamma_{\mathrm{I}k},$$

and $\boldsymbol{u}^+ \in Y_{\mathrm{c}}^{(u)}(\mathcal{A})$, such that (5.15) holds. The quadratic forms for $\mathcal{L}_1$ and $\mathcal{L}_0^{-1}$ are simultaneously block-diagonalized as follows:

$$\langle \boldsymbol{u}, \mathcal{L}_1 \boldsymbol{u} \rangle = \sum_k \boldsymbol{a}_k^{\mathrm{T}} \hat{\gamma}_k \hat{l}_k \boldsymbol{a}_k + \sum_m \gamma_m l_m |c_m|^2 + \langle \boldsymbol{u}^+, \mathcal{L}_1 \boldsymbol{u}^+ \rangle, \quad (5.17)$$

$$\langle \boldsymbol{u}, \mathcal{L}_0^{-1} \boldsymbol{u} \rangle = \sum_k \boldsymbol{a}_k^{\mathrm{T}} \hat{l}_k \boldsymbol{a}_k + \sum_m l_m |c_m|^2 + \langle \boldsymbol{u}^+, \mathcal{L}_0^{-1} \boldsymbol{u}^+ \rangle, \quad (5.18)$$

where $\boldsymbol{a}_k = (a_k, b_k)^{\mathrm{T}}$ and the Jordan blocks $\hat{\gamma}_k$ and $\hat{l}_k$ are defined by (5.5).

6. Proof of theorems 3.8 and 3.9

Eigenvalues γ of the diagonalization problem (5.1) correspond to three different types of eigenvalues λ of the linear stability problem (2.14). When $\gamma = \gamma_m > 0$, the linear problem (2.14) has two purely imaginary eigenvalues λ, which are weakly spectrally stable. When $\gamma = \gamma_m < 0$, the linear problem (2.14) has two real eigenvalues λ, which include an unstable positive eigenvalue. When $\gamma = \gamma_k = \gamma_{\mathrm{R}k} + \mathrm{i}\gamma_{\mathrm{I}k}$ is complex, the linear problem (2.14) has four complex eigenvalues λ, which include two unstable eigenvalues with $\mathrm{Re}(\lambda) > 0$. We trace the unstable eigenvalues λ of the stability problem (2.14) from negative and complex eigenvalues γ of the diagonalization problem (5.1), according to the following proposition.

Proposition 6.1. *Let $\mathcal{L}$ be a symmetric matrix Schrödinger operator, either $\mathcal{L}_1$ or $\mathcal{L}_0^{-1}$. The negative index $n(\mathcal{L}|_{X_{\mathrm{c}}^{(u)}})$ of the quadratic form $\langle \boldsymbol{u}, \mathcal{L}\boldsymbol{u} \rangle$ in Hilbert space $X_{\mathrm{c}}^{(u)}(\mathbb{R})$ remains invariant in the diagonal representation (4.14) and the block-diagonal representation (5.17), (5.18).*

Proof. By proposition 4.9, operator $\mathcal{L}$ has the basis

$$S_u = E_u^- \wedge E_0 \wedge E_u^+,$$

where E_u^- is the negative subspace spanned by eigenvectors $\{\boldsymbol{u}_m\}_{m=1}^{M_u}$ such that $\lambda_m = \langle \boldsymbol{u}_m, \mathcal{L}\boldsymbol{u}_m \rangle < 0$, E_0 is the kernel of $\mathcal{L}$ in $X_{\mathrm{c}}^{(u)}(\mathbb{R})$, and E_u^+ is the positive subspace of $\mathcal{L}$ in $X_{\mathrm{c}}^{(u)}(\mathbb{R})$. The negative index of $\langle \boldsymbol{u}, \mathcal{L}\boldsymbol{u} \rangle$ is $n_u(\mathcal{L}|_{X_{\mathrm{c}}^{(u)}}) = M_u$. By proposition 5.10, operator $\mathcal{L}$ has also another basis

$$S_v = E_v^{\mathrm{c}} \wedge E_v^- \wedge E_0 \wedge E_v^+,$$

where E_v^- is the real negative subspace spanned by eigenvectors $\{\boldsymbol{v}_m\}_{m=1}^{M_v}$, such that $l_m = \langle \boldsymbol{v}_m, \mathcal{L}\boldsymbol{v}_m\rangle < 0$, E_v^{c} is the complex subspace spanned by eigenvectors $\{\boldsymbol{v}_{\mathrm{R}k}, \boldsymbol{v}_{\mathrm{I}k}\}_{k=1}^{K_v}$ such that $l_{\mathrm{R}k} = \langle \boldsymbol{v}_{\mathrm{R}k}, \mathcal{L}\boldsymbol{v}_{\mathrm{R}k}\rangle$, $l_{\mathrm{I}k} = \langle \boldsymbol{v}_{\mathrm{I}k}, \mathcal{L}\boldsymbol{v}_{\mathrm{R}k}\rangle$, and E_v^+ is the positive subspace of $\mathcal{L}$ in $X_{\mathrm{c}}^{(u)}(\mathbb{R})$. The Jordan block $\hat{l}_k$ in (5.5) has one positive and one negative eigenvalue

$$\pm l_k = \pm\sqrt{l_{\mathrm{R}k}^2 + l_{\mathrm{I}k}^2}.$$

The eigenvectors $\boldsymbol{v}_{\mathrm{R}k}(x)$ and $\boldsymbol{v}_{\mathrm{I}k}(x)$ can be orthogonalized with respect to operator $\mathcal{L}$ in the linear combination

$$\boldsymbol{v}_k^{\pm} = l_{\mathrm{I}k}\boldsymbol{v}_{\mathrm{R}k}(x) + (\pm l_k - l_{\mathrm{R}k})\boldsymbol{v}_{\mathrm{I}k}(x).$$

The negative index of $\langle \boldsymbol{u}, \mathcal{L}\boldsymbol{u}\rangle$ is $n_v(\mathcal{L}|_{X_{\mathrm{c}}^{(u)}}) = M_v + K_v$. We will prove that

$$n_u(\mathcal{L}|_{X_{\mathrm{c}}^{(u)}}) = n_v(\mathcal{L}|_{X_{\mathrm{c}}^{(u)}}).$$

We assume that $(M_v + K_v) > M_u$ and show that this is false. The case $(M_v + K_v) < M_u$ can be treated similarly. Consider a function $\boldsymbol{g}_u(x)$ given by

$$\boldsymbol{g}_u(x) = \sum_{k=1}^{K_v} a_k \boldsymbol{v}_k^-(x) + \sum_{m=1}^{M_v} c_m \boldsymbol{v}_m(x) + \boldsymbol{u}_0 + \boldsymbol{u}^+(x). \tag{6.1}$$

The eigenfunctions $\boldsymbol{v}_k^-(x)$ and $\boldsymbol{v}_m(x)$ can be decomposed over the basis of S_u:

$$\boldsymbol{v}_k^-(x) = \sum_{j=1}^{M_u} \alpha_{kj}\boldsymbol{u}_j(x) + \boldsymbol{u}_{0k}(x) + \boldsymbol{u}_k^+(x), \tag{6.2}$$

$$\boldsymbol{v}_m(x) = \sum_{j=1}^{M_u} \gamma_{mj}\boldsymbol{u}_j(x) + \boldsymbol{u}_{0m}(x) + \boldsymbol{u}_m^+(x). \tag{6.3}$$

Therefore, the function $\boldsymbol{g}_u(x)$ is decomposed as

$$\begin{aligned}\boldsymbol{g}_u(x) = {} & \sum_{j=1}^{M_u}\left(\sum_{k=1}^{K_v} \alpha_{kj}a_k + \sum_{m=1}^{M_v} \gamma_{mj}c_m\right)\boldsymbol{u}_j(x) \\ & + \left(\boldsymbol{u}_0(x) + \sum_{k=1}^{K_v} a_k\boldsymbol{u}_{0k}(x) + \sum_{m=1}^{M_v} c_m\boldsymbol{u}_{0m}(x)\right) \\ & + \left(\boldsymbol{u}^+(x) + \sum_{k=1}^{K_v} a_k\boldsymbol{u}_k^+(x) + \sum_{m=1}^{M_v} c_m\boldsymbol{u}_m^+(x)\right).\end{aligned} \tag{6.4}$$

Consider a particular case $\boldsymbol{g}_u(x) = \boldsymbol{0}$. Since the set S_u is complete, then

$$\left.\begin{aligned}\sum_{k=1}^{K_v} \alpha_{kj}a_k + \sum_{m=1}^{M_v} \gamma_{mj}c_m = 0, \quad j = 1, \ldots, M_u, \\ \boldsymbol{u}_0(x) + \sum_{k=1}^{K_v} a_k\boldsymbol{u}_{0k}(x) + \sum_{m=1}^{M_v} c_m\boldsymbol{u}_{0m}(x) = 0, \\ \boldsymbol{u}_+(x) + \sum_{k=1}^{K_v} a_k\boldsymbol{u}_k^+(x) + \sum_{m=1}^{M_v} c_m\boldsymbol{u}_m^+(x) = 0.\end{aligned}\right\} \tag{6.5}$$

The linear homogeneous system (6.5) is under-determined, such that at least $(M_v + K_v - M_u)$ unknowns are arbitrary. Therefore, there exists a non-zero solution of (6.5) such that a non-zero vector $\boldsymbol{s}_u(x)$ is defined by (6.1) with $\boldsymbol{g}_u(x) = \boldsymbol{0}$:

$$\boldsymbol{s}_u(x) = \sum_{k=1}^{K_v} a_k \boldsymbol{v}_k^-(x) + \sum_{m=1}^{M_v} c_m \boldsymbol{v}_m(x) = -\boldsymbol{u}_0(x) - \boldsymbol{u}^+(x).$$

Therefore, the quadratic form $\langle \boldsymbol{s}_u, \mathcal{L}\boldsymbol{s}_u\rangle$ can be bounded in two contradictory ways:

$$\langle \boldsymbol{s}_u, \mathcal{L}\boldsymbol{s}_u\rangle = -\sum_{k=1}^{K_v} 2(l_{\mathrm{R}k}^2 + l_{\mathrm{I}k}^2)\left(\sqrt{l_{\mathrm{R}k}^2 + l_{\mathrm{I}k}^2} + l_{\mathrm{R}k}\right)|a_k|^2 + \sum_{m=1}^{M_v} l_m |c_m|^2 < 0,$$

$$\langle \boldsymbol{s}_u, \mathcal{L}\boldsymbol{s}_u\rangle = \langle \boldsymbol{u}^+, \mathcal{L}\boldsymbol{u}^+\rangle > 0.$$

The contradiction is resolved if and only if $(M_v + K_v) = M_u$, when

$$n_u(\mathcal{L}|_{X_{\mathrm{c}}^{(u)}}) = n_v(\mathcal{L}|_{X_{\mathrm{c}}^{(u)}}).$$

■

Corollary 6.2. *Let N_{comp} be the number of complex eigenvalues in the problem (5.1). Let $n(\mathcal{L}_0|_{X_{\mathrm{c}}^{(u)}})$ and $n(\mathcal{L}_1|_{X_{\mathrm{c}}^{(u)}})$ be the numbers of negative eigenvalues in the problems (4.1) and (4.2). Then*

$$N_{\mathrm{comp}} \leqslant \min(n(\mathcal{L}_0|_{X_{\mathrm{c}}^{(u)}}), n(\mathcal{L}_1|_{X_{\mathrm{c}}^{(u)}}))$$

and there exist $(n(\mathcal{L}_1|_{X_{\mathrm{c}}^{(u)}}) - N_{\mathrm{comp}})$ eigenvectors $\boldsymbol{u}_m(x)$ in problem(5.1) such that $\langle \boldsymbol{u}_m, \mathcal{L}_1\boldsymbol{u}_m\rangle < 0$ and $(n(\mathcal{L}_0|_{X_{\mathrm{c}}^{(u)}}) - N_{\mathrm{comp}})$ eigenvectors $\boldsymbol{u}_m(x)$ such that $\langle \boldsymbol{u}_m, \mathcal{L}_0^{-1}\boldsymbol{u}_m\rangle < 0$.

Proposition 6.1 generalizes Sylvester's inertia theorem for finite-dimensional operators (Gelfand 1961). Using this result, we prove theorems 3.8 and 3.9, which define sharp bounds on the number of negative and complex eigenvalues of the problem (5.1) from the numbers of negative eigenvalues of $\mathcal{L}_1$ and $\mathcal{L}_0$ in $X_{\mathrm{c}}^{(u)}(\mathbb{R})$.

Proof of theorem 3.8. It follows from corollary 6.2 that

$$n(\mathcal{L}_1|_{X_{\mathrm{c}}^{(u)}}) = N_{\mathrm{comp}} + \#_{<0}(\gamma_m l_m), \tag{6.6}$$

$$n(\mathcal{L}_0|_{X_{\mathrm{c}}^{(u)}}) = N_{\mathrm{comp}} + \#_{<0}(l_m), \tag{6.7}$$

where $\#_{<0}(l_m)$ is the number of negative values of l_m. Taking the sum of (6.6) and (6.7), we find that

$$n(\mathcal{L}_1|_{X_{\mathrm{c}}^{(u)}}) + n(\mathcal{L}_0|_{X_{\mathrm{c}}^{(u)}}) = 2N_{\mathrm{comp}} + \#_{<0}(\gamma_m l_m) + \#_{<0}(l_m) = 2N_{\mathrm{comp}} + 2N_{\mathrm{imag}}^- + N_{\mathrm{real}}.$$

By theorem 3.6, the latter identity gives the closure relation (3.1). ■

Proof of theorem 3.9. Taking the difference of (6.6) and (6.7), we find that

$$|n(\mathcal{L}_1|_{X_{\mathrm{c}}^{(u)}}) - n(\mathcal{L}_0|_{X_{\mathrm{c}}^{(u)}})| = |\#_{<0}(\gamma_m l_m) - \#_{<0}(l_m)| \leqslant N_{\mathrm{real}} \leqslant N_{\mathrm{unst}},$$

which are the lower bounds (3.2) and (3.3). The upper bound in (3.2) is a corollary of theorem 3.8. The bound (3.4) is given by corollary 6.2. ■

Remark 6.3. The constrained problems (4.1) and (4.2) have a common set of eigenfunctions if and only if operators $\mathcal{L}_1$ and $\mathcal{L}_0$ commute. Assume that this is true. Let $\lambda = \xi_m$ be an eigenvalue of (4.1) and $\lambda = \eta_m$ be an eigenvalue of (4.2), with the same eigenvector $\boldsymbol{u}_m(x)$. There exists an eigenvalue $\gamma = \gamma_m$ of the problem (5.1), such that

$$\gamma_m = \eta_m \xi_m = \eta_m \frac{\langle \boldsymbol{u}_m, \mathcal{L}_0 \boldsymbol{u}_m \rangle}{\langle \boldsymbol{u}_m, \boldsymbol{u}_m \rangle} = \eta_m \frac{\langle \boldsymbol{u}_m, \boldsymbol{u}_m \rangle}{\langle \boldsymbol{u}_m, \mathcal{L}_0^{-1} \boldsymbol{u}_m \rangle}. \tag{6.8}$$

This formula is used in Pelinovsky & Kivshar (2000) to approximate $\lambda^2 = -\gamma_m$ from the given solution of the constrained problem (4.2) in the case when $\mathcal{L}_0$ is positive definite in $X_{\mathrm{c}}^{(u)}(\mathbb{R})$. Since the operators $\mathcal{L}_1$ and $\mathcal{L}_0$ do not commute for any $\boldsymbol{\Phi} \neq \mathbf{0}$, the approximation formula (6.8) does not give an exact relation between γ_m and η_m.

7. Proof of proposition 5.8

We introduce wave functions of the spectral problem (2.14), similarly to analysis in Buslaev & Perelman (1993) for $N = 1$. Since operators $\mathcal{L}_0$ and $\mathcal{L}_1$ in (2.12) and (2.13) are diagonal differential operators with exponentially decaying matrix potentials, there exist $2N$ branches of the continuous spectrum, located symmetrically at

$$\sigma_{\mathrm{con}}(\mathcal{A}) = \bigcup_{1 \leqslant n \leqslant N} \{\lambda \in \mathrm{i}\mathbb{R} : |\operatorname{Im}(\lambda)| \geqslant \beta_n\}. \tag{7.1}$$

Using a transformation, $\lambda \mapsto \mathrm{i}\Omega$, $\boldsymbol{u} \mapsto \boldsymbol{u}$, $\boldsymbol{w} \mapsto \mathrm{i}\boldsymbol{w}$, we rewrite the problem (2.14) as

$$\mathcal{L}_1 \boldsymbol{u} = \Omega \boldsymbol{w}, \qquad \mathcal{L}_0 \boldsymbol{w} = \Omega \boldsymbol{u}. \tag{7.2}$$

We use the order

$$\beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_N \tag{7.3}$$

and denote the number of end points β_n to the left of a given value Ω by K_Ω, $K_\Omega \leqslant N$. Let $\Omega_E = \{\beta_1, \ldots, \beta_N\}$. We introduce a set of continuous parameters $k_n \in \mathbb{R}$, $n = 1, \ldots, N$, where k_n parametrizes the nth positive branch of the continuous spectrum:

$$\Omega = \beta_n + d_n k_n^2, \quad n = 1, \ldots, N, \tag{7.4}$$

such that

$$k_n = k_n(\Omega) = \left(\frac{\Omega - \beta_n}{d_n} \right)^{1/2}. \tag{7.5}$$

Since the point $\Omega = \beta_n$ is a branch point in a two-sheet Riemann surface, we fix the argument of Ω by considering the sheet with $k_n > 0$ on $\Omega > \beta_n$. The set of wave functions $\boldsymbol{u}_n^{\pm}(\Omega) \equiv \boldsymbol{u}_n^{\pm}(x; \boldsymbol{k}(\Omega))$ is defined for $\Omega \neq \Omega_E$, according to the asymptotic values at infinity:

$$\boldsymbol{u}_n^{\pm}(\Omega) \to \boldsymbol{e}_n \mathrm{e}^{\pm \mathrm{i} k_n x} \quad \text{as } x \to \pm\infty, \ k_n > 0, \tag{7.6}$$

where $\boldsymbol{e}_n$ is the nth unit vector in $\mathbb{C}^N$. The set of scattering coefficients is defined from asymptotic values of $\boldsymbol{u}_n^{\pm}(k)$ at the other infinities:

$$\left. \begin{aligned} \boldsymbol{u}_n^{-}(\Omega) &\to \sum_{l=1}^{K_\Omega} \boldsymbol{e}_l [a_{n,l}(\Omega) \mathrm{e}^{-\mathrm{i} k_l x} + b_{n,l}(\Omega) \mathrm{e}^{\mathrm{i} k_l x}] \quad \text{as } x \to +\infty, \\ \boldsymbol{u}_n^{+}(\Omega) &\to \sum_{l=1}^{K_\Omega} \boldsymbol{e}_l [A_{n,l}(\Omega) \mathrm{e}^{\mathrm{i} k_l x} + B_{n,l}(\Omega) \mathrm{e}^{-\mathrm{i} k_l x}] \quad \text{as } x \to -\infty, \end{aligned} \right\} \tag{7.7}$$

where $k_l > 0$, $l = 1, \ldots, K_\Omega$, $\Omega \neq \Omega_E$. It follows from the system (7.2) that the components $\boldsymbol{w}_n^{\pm}(\Omega)$ have the same asymptotic representation (7.6), (7.7) for $\Omega > \beta_n$.

Definition 7.1. When the eigenvector of (7.2) with $\Omega \geqslant \beta_1$ is exponentially decaying as $|x| \to \infty$, Ω is called an embedded eigenvalue of $\sigma_{\text{emb}}(\mathcal{A})$. When the eigenvector of (7.2) with $\Omega \geqslant \beta_1$ is exponentially decaying at one infinity and bounded at the other infinity, Ω is called a semi-eigenvalue of $\sigma_{\text{con}}(\mathcal{A})$. When the set of wave functions

$$\{\boldsymbol{u}_n^-(\Omega)\}_{n=1}^{K_\Omega}, \quad K_\Omega \leqslant N$$

is linearly dependent on the set of wave functions

$$\{\boldsymbol{u}_n^+(\Omega)\}_{n=1}^{K_\Omega},$$

Ω is called a resonance of $\sigma_{\text{con}}(\mathcal{A})$.

According to assumption 2.14(i), we assume that no semi-eigenvalues of $\sigma_{\text{con}}(\mathcal{A})$ nor embedded eigenvalues of $\sigma_{\text{emb}}(\mathcal{A})$ exist for $\Omega \geqslant \beta_1$.

The existence of wave functions was shown in Buslaev & Perelman (1993) for $N = 1$, where all fundamental solutions of the linear system (7.2) were considered with Volterra integral equations, including exponentially decreasing and increasing solutions. Exponentially decreasing terms are neglected in the asymptotic representations (7.6), (7.7). The existence of the wave functions $\boldsymbol{u}_{\pm}^{(m)}(\Omega)$ follows from the following lemma.

Lemma 7.2. *Let assumption 2.14(i) be satisfied. The wave functions $\boldsymbol{u}_n^{\pm}(\Omega)$, $n = 1, \ldots, N$ exist and have unique asymptotic representations (7.6), (7.7), where coefficients $a_{n,l}(\Omega)$, $b_{n,l}(\Omega)$ are all bounded for any $\Omega > \beta_n$, $\Omega \neq \Omega_E$.*

Proof. For $\Omega > \beta_n$, besides a set of $2K_\Omega$ oscillatory functions, $K_\Omega \leqslant N$, there exist sets of $2N - K_\Omega$ exponentially decaying and $2N - K_\Omega$ exponentially growing solutions at each infinity $x \to \pm\infty$. When $\Omega \neq \Omega_E$, all functions are uniquely defined by standard theorems on solutions of linear differential equations with exponentially decaying coefficients (Coddington & Levinson 1955). In order to define $\boldsymbol{u}_n^{\pm}(\Omega)$, we construct a linear combination of $2N - K_\Omega$ exponentially decaying functions at $x \to \pm\infty$ with the oscillatory function $\boldsymbol{e}_n \mathrm{e}^{\pm \mathrm{i} k_n x}$ and uniquely define the coefficients of the linear combination from $2N - K_\Omega$ conditions that exponentially growing functions are removed in the limit $x \to \mp\infty$. If semi-eigenvalues and embedded eigenvalues do not exist for $\Omega > \beta_n$, the non-homogeneous linear system for the coefficients of the linear combination always has a unique solution. Therefore, the wave functions $\boldsymbol{u}_n^{\pm}(\Omega)$ are uniquely specified by the asymptotic representations (7.6), (7.7) for any $\Omega > \beta_n$, $\Omega \neq \Omega_E$ and the coefficients $a_{n,l}(\Omega)$, $b_{n,l}(\Omega)$, $l = 1, \ldots, K_\Omega$ are all bounded. ■

We define a 'scalar' Wronskian between two solutions of the system (7.2) with $\Omega = \Omega_1$ and $\Omega = \Omega_2$ by

$$W[\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}] = \sum_{n=1}^{N} d_n (u_n^{(1)} u_n^{(2)\prime} - u_n^{(1)\prime} u_n^{(2)} + w_n^{(1)} w_n^{(2)\prime} - w_n^{(1)\prime} w_n^{(2)}), \tag{7.8}$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}x} W[\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}] = (\Omega_1 - \Omega_2) \sum_{n=1}^{N} (u_n^{(1)} w_n^{(2)} + w_n^{(1)} u_n^{(2)}). \tag{7.9}$$

If $\Omega_1 = \Omega_2$, then $W[\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}]$ is constant in $x \in \mathbb{R}$. Using asymptotic values (7.6), (7.7) for $W[\boldsymbol{u}_m^-(\Omega), \boldsymbol{u}_n^+(\Omega)]$ and $W[\bar{\boldsymbol{u}}_m^-(\Omega), \boldsymbol{u}_n^+(\Omega)]$, we derive the linear relations between the scattering coefficients:

$$A_{n,m}(\Omega) = \frac{k_n d_n}{k_m d_m} a_{m,n}(\Omega), \qquad B_{n,m}(\Omega) = -\frac{k_n d_n}{k_m d_m} \bar{b}_{m,n}(\Omega). \tag{7.10}$$

Using asymptotic values (7.6), (7.7) for $W[\bar{\boldsymbol{u}}_m^-(\Omega), \boldsymbol{u}_n^-(\Omega)]$ and $W[\bar{\boldsymbol{u}}_m^+(\Omega), \boldsymbol{u}_n^+(\Omega)]$, we derive the quadratic relations between the scattering coefficients:

$$k_n d_n \delta_{m,n} = \sum_{l=1}^{K_\Omega} k_l d_l [\bar{a}_{m,l}(\Omega) a_{n,l}(\Omega) - \bar{b}_{m,l}(\Omega) b_{n,l}(\Omega)], \tag{7.11}$$

$$\frac{1}{k_n d_n} \delta_{m,n} = \sum_{l=1}^{K_\Omega} \frac{1}{k_l d_l} [\bar{a}_{l,m}(\Omega) a_{l,n}(\Omega) - \bar{b}_{l,m}(\Omega) b_{l,n}(\Omega)]. \tag{7.12}$$

We use the scalar Wronskians (7.8) to study the behaviour of wave functions at the end points and to derive the orthogonality relations between the wave functions.

Lemma 7.3. *No resonances may occur for any $\Omega \geqslant \beta_1$, $\Omega \neq \Omega_E$.*

Proof. By definition 7.1, Ω is a resonance, if there exists a non-zero eigenvector $\boldsymbol{u}(x)$ such that

$$\boldsymbol{u}(x) = \sum_{n=1}^{K_\Omega} c_n^- \boldsymbol{u}_n^-(\Omega) = \sum_{n=1}^{K_\Omega} c_n^+ \boldsymbol{u}_n^+(\Omega). \tag{7.13}$$

Computing $W[\bar{\boldsymbol{u}}, \boldsymbol{u}]$ in the limits $x \to \pm\infty$, we have

$$\sum_{n=1}^{K_\Omega} d_n k_n (|c_n^-|^2 + |c_n^+|^2) = 0. \tag{7.14}$$

When $\Omega \neq \Omega_E$, all $k_n > 0$, such that all $c_n^\pm = 0$. Therefore, no eigenvector $\boldsymbol{u}(x)$ exists for $\Omega \neq \Omega_E$. ∎

We define the symplectic inner product as

$$J[\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}] = \frac{1}{2\pi \mathrm{i}} \langle \boldsymbol{u}^{(1)}, \mathcal{J} \boldsymbol{u}^{(2)} \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{n=1}^{N} (\bar{u}_n^{(1)} w_n^{(2)} + \bar{w}_n^{(1)} u_n^{(2)})\, \mathrm{d}x. \tag{7.15}$$

The Dirac function $\delta(k)$ has the properties

$$\delta(k) = \frac{1}{\pi} \lim_{L \to \infty} \frac{\mathrm{e}^{\mathrm{i}kL}}{\mathrm{i}k} \tag{7.16}$$

and $|\alpha|\delta(\alpha k) = \delta(k)$, $\alpha \neq 0$. Using standard computations of the symplectic inner products (7.15), we derive the orthogonality relations between wave functions $\boldsymbol{u}_n^\pm(\Omega)$, $n = 1, \ldots, N$.

Lemma 7.4. *Let assumption 2.14(i) be satisfied. The set of wave functions* $\{\boldsymbol{u}_n^{\pm}(\Omega)\}_{n=1}^N$ *satisfies the orthogonality relations for* $\Omega \neq \Omega_E$*:*

$$J[\boldsymbol{u}_m^-(\Omega), \boldsymbol{u}_n^-(\Omega')] = \alpha_{m,n}(\Omega)\delta(\Omega - \Omega'), \tag{7.17}$$

$$J[\boldsymbol{u}_m^+(\Omega), \boldsymbol{u}_n^+(\Omega')] = \beta_{m,n}(\Omega)\delta(\Omega - \Omega'), \tag{7.18}$$

$$J[\boldsymbol{u}_m^-(\Omega), \boldsymbol{u}_n^+(\Omega')] = 0, \tag{7.19}$$

where

$$\alpha_{m,n}(\Omega) = 4\sum_{l=1}^{K_\Omega} k_l d_l \bar{a}_{m,l}(\Omega) a_{n,l}(\Omega), \qquad \beta_{m,n}(\Omega) = 4\sum_{l=1}^{K_\Omega} \frac{1}{k_l d_l} \bar{a}_{l,m}(\Omega) a_{l,n}(\Omega). \tag{7.20}$$

Proof. We integrate the Wronskian relation (7.9) as

$$J[\boldsymbol{u}_m^-(\Omega), \boldsymbol{u}_n^-(\Omega')] = \frac{1}{2\pi} \lim_{x\to\infty} \frac{W[\boldsymbol{u}_m^-(\Omega), \boldsymbol{u}_n^-(\Omega')]}{\Omega - \Omega'} - \frac{1}{2\pi} \lim_{x\to-\infty} \frac{W[\boldsymbol{u}_m^-(\Omega), \boldsymbol{u}_n^-(\Omega')]}{\Omega - \Omega'}. \tag{7.21}$$

The second term in (7.21) is computed with the use of (7.4), (7.6) and (7.16) as $\delta_{m,n}\delta(k_n - k_n')$. The first term in (7.21) is computed with the use of (7.4), (7.7) and (7.16) as

$$\sum_{l=1}^{K_\Omega} (\bar{a}_{m,l} a_{n,l} + \bar{b}_{m,l} b_{n,l})\delta(k_l - k_l') + \sum_{l=1}^{K_\Omega} (\bar{a}_{m,l} b_{n,l} + \bar{b}_{m,l} a_{n,l})\delta(k_l + k_l'),$$

where we have suppressed the arguments of $a_{n,l}(\Omega)$ and $b_{n,l}(\Omega)$. Since $k_l > 0$ and $k_l' > 0$ in the representation (7.7), we understand that $\delta(k_l + k_l') = 0$ and

$$\delta(k_l - k_l') = 2k_l d_l \delta(\Omega - \Omega').$$

Using (7.11), we derive (7.17). The other relations (7.18) and (7.19) are derived similarly, with the use of (7.10) and (7.12). ∎

Lemma 7.5. *The coefficient matrices*

$$[\alpha_{m,n}(\Omega)]_{1\leqslant m,n\leqslant K_\Omega} \quad \text{and} \quad [\beta_{m,n}(\Omega)]_{1\leqslant m,n\leqslant K_\Omega}$$

are strictly positive for $\Omega \neq \Omega_E$.

Proof. We consider a quadratic form in $\mathbb{C}^{K_\Omega}$:

$$\sum_{m=1}^{K_\Omega}\sum_{n=1}^{K_\Omega} \bar{x}_m \alpha_{n,m}(\Omega) x_n = 4\sum_{l=1}^{K_\Omega} k_l d_l \left| \sum_{n=1}^{K_\Omega} a_{n,l}(\Omega) x_n \right|^2 \geqslant 0. \tag{7.22}$$

Since all $k_l > 0$, $l = 1, \ldots, K_\Omega$ for $\Omega \neq \Omega_E$, the equality would mean that the determinant of the matrix $[a_{m,n}(\Omega)]_{1\leqslant m,n\leqslant K_\Omega}$ is zero, which contradicts lemma 7.3. Therefore, the quadratic form in (7.22) is strictly positive for $\Omega \neq \Omega_E$. Similar computations hold for the matrix $[\beta_{m,n}(\Omega)]_{1\leqslant m,n\leqslant K_\Omega}$. ∎

We define the normalized set of wave functions $\boldsymbol{e}_n(\Omega)$, such that

$$\boldsymbol{e}_n(\Omega) \equiv \boldsymbol{e}_n^+(\Omega) = \frac{1}{\sqrt{k_n d_n \beta_{n,n}}} \boldsymbol{u}_n^+(\Omega), \quad k_n > 0,$$

$$\boldsymbol{e}_n(\Omega) \equiv \boldsymbol{e}_n^-(\Omega) = \sqrt{\frac{k_n d_n}{\alpha_{n,n}}} \boldsymbol{u}_n^-(\Omega), \qquad k_n < 0.$$

It follows from (7.11), (7.12) and (7.20) at $m = n$ that $\alpha_{n,n}(\Omega) > 0$ and $\beta_{n,n}(\Omega) > 0$. By lemma 7.4, the wave functions $\{\boldsymbol{e}_n(\Omega)\}_{n=1}^N$ satisfy the orthogonality relations

$$J[\boldsymbol{e}_m(\Omega), \boldsymbol{e}_n(\Omega')] = \rho_{m,n}(\Omega)\delta(\Omega - \Omega'), \quad k_n \in \mathbb{R}, \tag{7.23}$$

where

$$\rho_{m,n}(\Omega) \equiv \begin{cases} \rho_{m,n}^+(\Omega) = \dfrac{\beta_{m,n}}{\sqrt{k_m k_n d_m d_n \beta_{m,m} \beta_{n,n}}}, & k_n > 0, \\[2ex] \rho_{m,n}^-(\Omega) = \sqrt{\dfrac{k_m k_n d_m d_n}{\alpha_{m,m}\alpha_{n,n}}}\,\alpha_{m,n}, & k_n < 0. \end{cases} \tag{7.24}$$

By lemma 7.5, the matrix $[\rho_{m,n}(\Omega)]_{1\leqslant m,n\leqslant K_\Omega}$ is positive for any $\Omega \in \sigma_{\mathrm{con}}(\mathcal{A})$.

We define the projection operator

$$\mathcal{S} : X_{\mathrm{c}}^{(u)}(\mathbb{R}) \mapsto Y_{\mathrm{c}}^{(u)}(\mathcal{A}),$$

according to the standard formula (Buslaev & Perelman 1993)

$$\forall \boldsymbol{u} \in X_{\mathrm{c}}^{(u)}(\mathbb{R}),\ \exists \boldsymbol{u}^+ \in Y_{\mathrm{c}}^{(u)}(\mathcal{A}) : \quad \boldsymbol{u}^+ = \mathcal{S}\boldsymbol{u} = \sum_{n=1}^{N} \mathrm{p.v.} \int_{-\infty}^{\infty} \hat{u}_n(\Omega)\boldsymbol{e}_n(\Omega)\,\mathrm{d}k_n, \tag{7.25}$$

where 'p.v.' stands for the principal-value integral to exclude possible singularity at $k_n = 0$ in the case when $\Omega = \beta_n$ is a resonance of $\sigma_{\mathrm{con}}(\mathcal{A})$. Coefficients $\hat{u}_n(\Omega)$ in the projection formula (7.25) are uniquely defined by the orthogonality relations (7.23), since the matrix $[\rho_{m,n}(\Omega)]_{1\leqslant m,n\leqslant K_\Omega}$ is positive definite for any $\Omega \in \sigma_{\mathrm{con}}(\mathcal{A})$. Using (7.4), we can rewrite (7.25) in the form

$$\boldsymbol{u}^+(x) = \sum_{n=1}^{N} \mathrm{p.v.} \int_{\beta_n}^{\infty} \frac{\mathrm{d}\Omega}{2k_n d_n} (\hat{u}_n^+(\Omega)\boldsymbol{e}_n^+(\Omega) + \hat{u}_n^-(\Omega)\boldsymbol{e}_n^-(\Omega)), \quad k_n > 0. \tag{7.26}$$

With this construction, we finally prove proposition 5.8.

Proof of proposition 5.8. Using (7.2) and (7.15), we find that

$$\rho_{m,n}(\Omega)\delta(\Omega - \Omega') = J[\boldsymbol{e}_m(\Omega), \boldsymbol{e}_n(\Omega')] = \frac{\langle \boldsymbol{e}_m(\Omega), \mathcal{L}_1 \boldsymbol{e}_n(\Omega')\rangle}{2\pi\Omega'} + \frac{\langle \mathcal{L}_1 \boldsymbol{e}_m(\Omega), \boldsymbol{e}_n(\Omega')\rangle}{2\pi\Omega}. \tag{7.27}$$

Integrating by parts and using quadratic relations (7.11), (7.12) for asymptotic representations (7.6), (7.7), we confirm that

$$\langle \mathcal{L}_1 \boldsymbol{e}_m(\Omega), \boldsymbol{e}_n(\Omega')\rangle = \langle \boldsymbol{e}_m(\Omega), \mathcal{L}_1 \boldsymbol{e}_n(\Omega')\rangle. \tag{7.28}$$

As a result, we have the simultaneous orthogonality relations

$$\left.\begin{aligned}\langle \boldsymbol{e}_m(\Omega), \mathcal{L}_1 \boldsymbol{e}_n(\Omega')\rangle &= \pi\Omega\rho_{m,n}(\Omega)\delta(\Omega-\Omega'),\\ \langle \boldsymbol{e}_m(\Omega), \mathcal{L}_0^{-1} \boldsymbol{e}_n(\Omega')\rangle &= \frac{\pi}{\Omega}\rho_{m,n}(\Omega)\delta(\Omega-\Omega').\end{aligned}\right\} \tag{7.29}$$

A simple calculation of the quadratic form $\langle \boldsymbol{u}^+, \mathcal{L}_1 \boldsymbol{u}^+\rangle$ for $\boldsymbol{u}^+ \in Y_c^{(u)}(\mathcal{A})$ with the use of the spectral representation (7.26) and the orthogonality relations (7.29) leads to the formula

$$\langle \boldsymbol{u}^+, \mathcal{L}_1 \boldsymbol{u}^+\rangle = \text{p.v.} \int_{\beta_1}^{\infty} \left(\sum_{m=1}^{K_\Omega} \sum_{n=1}^{K_\Omega} \frac{\rho^+_{m,n}(\Omega)\hat{\bar{u}}^+_m(\Omega)\hat{u}^+_n(\Omega) + \rho^-_{m,n}(\Omega)\hat{\bar{u}}^-_m(\Omega)\hat{u}^-_n(\Omega)}{4k_m k_n d_m d_n}\right)\pi\Omega\,\mathrm{d}\Omega. \tag{7.30}$$

By lemma 7.5, we have $\langle \boldsymbol{u}^+, \mathcal{L}_1 \boldsymbol{u}^+\rangle > 0$, and therefore, $\langle \boldsymbol{u}^+, \mathcal{L}_0^{-1} \boldsymbol{u}^+\rangle > 0$. ∎

8. Symmetry-breaking stability analysis

Stability analysis based on simultaneous block-diagonalization of two linear operators can be applied to other Hamiltonian dynamical systems. We show here that similar analysis is applied also to symmetry-breaking instabilities of stationary solutions of coupled NLS equations.

Symmetry-breaking instabilities may occur when the stationary solutions in (z, x) are perturbed in another spatial dimension, say in y (Kivshar *et al.* 2000). The system of coupled NLS equations (1.1) in three spatial dimensions (z, x, y) takes the form

$$\mathrm{i}\frac{\partial \psi_n}{\partial z} + d_n\left(\frac{\partial^2 \psi_n}{\partial x^2} + \frac{\partial^2 \psi_n}{\partial y^2}\right) + f_n(|\psi_1|^2, \ldots, |\psi_N|^2)\psi_n = 0, \quad n = 1, \ldots, N. \tag{8.1}$$

We assume the same conditions on f_n and d_n as apply below (1.1). Linearization of the stationary solutions (2.1) is defined by the expansion

$$\psi_n(z, x, y) = [\Phi_n(x) + U_n(z, x, y) + \mathrm{i}W_n(z, x, y)]\mathrm{e}^{\mathrm{i}\beta_n z}, \tag{8.2}$$

where $(U_n, W_n)^{\mathrm{T}} \in \mathbb{R}^2$ are perturbation functions. Separating the variables (z, x, y) as

$$\boldsymbol{U} = \boldsymbol{u}(x)\mathrm{e}^{\lambda z + \mathrm{i}py}, \qquad \boldsymbol{W} = \boldsymbol{w}(x)\mathrm{e}^{\lambda z + \mathrm{i}py},$$

we arrive at the linear eigenvalue problem,

$$(\mathcal{L}_1 + p^2\mathcal{D})\boldsymbol{u} = -\lambda \boldsymbol{w}, \qquad (\mathcal{L}_0 + p^2\mathcal{D})\boldsymbol{w} = \lambda \boldsymbol{u}, \tag{8.3}$$

where $(\boldsymbol{u}, \boldsymbol{w})^{\mathrm{T}} \in \mathbb{R}^{2N}$, $p \in \mathbb{R}$, and $\mathcal{D}$ is a diagonal matrix of $(d_1, \ldots, d_N)$. Eigenvalues λ and eigenvectors $(\boldsymbol{u}, \boldsymbol{w})^{\mathrm{T}}$ of the linearized problem (8.3) depend on parameter p.

Lemma 8.1. *There exist exactly $n(\mathcal{L}_{1,0})$ negative eigenvalues λ of the problem*

$$\mathcal{L}_{1,0}\boldsymbol{u} = \lambda\mathcal{D}\boldsymbol{u}, \qquad \boldsymbol{u} \in L^2(\mathbb{R}),\ \lambda < 0. \tag{8.4}$$

Proof. Since $\mathcal{D}$ is positive definite, all eigenvalues λ in (8.4) are real. By proposition 6.1, the negative index of quadratic forms $\langle \boldsymbol{u}, \mathcal{L}_{1,0}\boldsymbol{u}\rangle$ in Hilbert space $L^2(\mathbb{R})$ is invariant in two diagonal representations, one with respect to $\langle \boldsymbol{u}_n, \boldsymbol{u}_n\rangle$ and the other one with respect to $\langle \boldsymbol{u}_n, \mathcal{D}\boldsymbol{u}_n\rangle > 0$. ∎

We define negative eigenvalues λ of the problem (8.4) for $\mathcal{L}_1$ as $\lambda = -\{\Lambda_{1,n}^2\}_{n=1}^{n(\mathcal{L}_1)}$ and for $\mathcal{L}_0$ as $\lambda = -\{\Lambda_{0,n}^2\}_{n=1}^{n(\mathcal{L}_0)}$. We split the domain $p^2 \in \mathbb{R}_+$ into sub-domains:

$$\mathcal{D}_{n_1,n_0} = \{p^2 \in \mathbb{R}_+ : \Lambda_{1,n_1}^2 < p^2 < \Lambda_{1,(n_1+1)}^2, \Lambda_{0,n_0}^2 < p^2 < \Lambda_{0,(n_0+1)}^2\}, \tag{8.5}$$

where $0 \leqslant n_1 \leqslant n(\mathcal{L}_1)$, $0 \leqslant n_0 \leqslant n(\mathcal{L}_0)$, such that

$$\Lambda_{1,0}^2 = \Lambda_{0,0}^2 \equiv 0 \quad \text{and} \quad \Lambda_{1,(n(\mathcal{L}_1)+1)}^2 = \Lambda_{0,(n(\mathcal{L}_0)+1)}^2 \equiv \infty.$$

Lemma 8.2. *In the domain $\mathcal{D}_{n_1,n_0}$, there are exactly $(n(\mathcal{L}_1)-n_1)$ negative eigenvalues of the problem*

$$(\mathcal{L}_1 + p^2\mathcal{D})\boldsymbol{u} = \lambda \boldsymbol{u}, \quad \boldsymbol{u} \in L^2(\mathbb{R}),\ \lambda < 0, \tag{8.6}$$

and exactly $(n(\mathcal{L}_0) - n_0)$ negative eigenvalues of the problem:

$$(\mathcal{L}_0 + p^2\mathcal{D})\boldsymbol{u} = \lambda \boldsymbol{u}, \quad \boldsymbol{u} \in L^2(\mathbb{R}),\ \lambda < 0. \tag{8.7}$$

Proof. The result follows from continuity of eigenvalues λ of the uncoupled problems (8.6) and (8.7) with respect to parameter p^2 in the domain $0 < p^2 < \infty$. Each negative eigenvalue $\lambda = \lambda(p^2)$ of operator $(\mathcal{L}_{1,0} + p^2\mathcal{D})$ is an increasing function of p^2 if $\mathcal{D}$ is positive definite, since

$$\lambda'(p^2) = \frac{\langle \boldsymbol{u}, \boldsymbol{u}\rangle}{\langle \boldsymbol{u}, \mathcal{D}\boldsymbol{u}\rangle} > 0. \tag{8.8}$$

When p^2 increases, eigenvalues $\lambda(p^2)$ pass through the zero value at the boundaries between domains $\mathcal{D}_{n_1,n_0}$ in (8.5), and the number of negative eigenvalues of (8.6), (8.7) reduces according to the multiplicity of eigenvalues $\lambda = -\Lambda_{1,n_1}^2$ and $\lambda = -\Lambda_{0,n_0}^2$ in (8.4). ■

Proposition 8.3. *Let assumption 2.14(i)–(ii) be satisfied for the problem (8.3) in the domain $\mathcal{D}_{n_1,n_0}$. The linearized problem (8.3) has $N_{\text{unst}} = N_{\text{real}} + 2N_{\text{comp}}$ unstable eigenvalues $\lambda = \lambda(p)$ with $\text{Re}(\lambda) > 0$, such that*

$$\text{(i)} \quad |n(\mathcal{L}_1) - n(\mathcal{L}_0) - n_1 + n_0| \leqslant N_{\text{unst}} \leqslant (n(\mathcal{L}_1) + n(\mathcal{L}_0) - n_1 - n_0), \tag{8.9}$$

$$\text{(ii)} \quad N_{\text{real}} \geqslant |n(\mathcal{L}_1) - n(\mathcal{L}_0) - n_1 + n_0|, \tag{8.10}$$

$$\text{(iii)} \quad N_{\text{comp}} \leqslant \min(n(\mathcal{L}_0) - n_0, n(\mathcal{L}_1) - n_1). \tag{8.11}$$

Proof. The linearization problem (8.3) can be rewritten in the form of a diagonalization problem,

$$(\mathcal{L}_1 + p^2\mathcal{D})\boldsymbol{u} = \gamma(\mathcal{L}_0 + p^2\mathcal{D})^{-1}\boldsymbol{u}, \quad \boldsymbol{u} \in L^2(\mathbb{R}), \tag{8.12}$$

where $\gamma = -\lambda^2$. If $p^2 > 0$ and $p^2 \neq \Lambda_{0,n}^2$, $n = 1, \ldots, n(\mathcal{L}_0)$, the operator $(\mathcal{L}_0 + p^2\mathcal{D})$ is invertible in $L^2(\mathbb{R})$. By lemma 8.2, we have $n(\mathcal{L}_1 + p^2\mathcal{D}) = n(\mathcal{L}_1) - n_1$ and $n(\mathcal{L}_0 + p^2\mathcal{D}) = n(\mathcal{L}_0) - n_0$ in the domain D_{n_1,n_0}. Proposition 8.3 is then equivalent to theorem 3.9. ■

Proposition 8.4. *Let assumption 2.14 be satisfied for the problem (2.14). Let N_{unst} be the number of unstable eigenvalues in the problem (2.14). There exists $p_*^2 > 0$ such that the linearized problem (8.3) has exactly $\hat{N}_{\text{unst}}$ unstable eigenvalues in the domain $0 < p^2 < p_*^2$, where $\hat{N}_{\text{unst}} = N_{\text{unst}} + p(\mathcal{U})$. The new $p(\mathcal{U})$ unstable eigenvalues λ are all real and positive.*

Proof. By lemma 2.16, the linearized problem (2.14) has $N+1$ double zero eigenvalues in $L^2(\mathbb{R})$. The symmetry-breaking perturbation with $p^2 > 0$ split these double eigenvalues into pairs of real or imaginary eigenvalues $\lambda(p)$ of the linearized problem (8.3). We show that $p(\mathcal{U})$ double eigenvalues split into pairs of real eigenvalues λ. Expanding solutions of (8.3) into power series of p, we have the following perturbation series expansions:

$$\left.\begin{aligned} \boldsymbol{u} &= p\lambda_1 \sum_{n=1}^{N} c_n \frac{\partial \boldsymbol{\Phi}}{\partial \beta_n} + O(p^3), \\ \boldsymbol{w} &= \sum_{n=1}^{N} c_n \Phi_n(x) \boldsymbol{e}_n + p^2 \boldsymbol{w}_2(x) + O(p^4), \end{aligned}\right\} \tag{8.13}$$

where $\lambda = p\lambda_1 + O(p^3)$. The function $\boldsymbol{w}_2(x)$ satisfies the non-homogeneous linear problem

$$\mathcal{L}_0 \boldsymbol{w}_2 = \lambda_1^2 \sum_{n=1}^{N} c_n \frac{\partial \boldsymbol{\Phi}}{\partial \beta_n} - \sum_{n=1}^{N} c_n d_n \Phi_n(x) \boldsymbol{e}_n. \tag{8.14}$$

Using the Fredholm alternative theorem, we find that $\boldsymbol{c} = (c_1, \dots, c_N)^{\mathrm{T}}$ satisfies the generalized eigenvalue problem,

$$\lambda_1^2 \mathcal{U} \boldsymbol{c} = 2\mathcal{D}\mathcal{Q}_{\mathrm{s}} \boldsymbol{c}, \tag{8.15}$$

where $\mathcal{Q}_{\mathrm{s}}$ is a diagonal matrix of $(Q_{1s}, \dots, Q_{ns})^{\mathrm{T}}$ and $\mathcal{U}$ is the Hessian matrix (2.7). Since $\mathcal{D}\mathcal{Q}_{\mathrm{s}}$ is positive-definite, Sylvester's inertia theorem suggests that the linear system (8.15) has exactly $p(\mathcal{U})$ positive eigenvalues and $n(\mathcal{U})$ negative eigenvalues λ_1^2. Therefore, positive eigenvalues of $\mathcal{U}$ are related to new unstable (real and positive) eigenvalues $\lambda = \lambda(p)$ in the linearization problem (8.3) for sufficiently small values of $p^2 > 0$, in addition to N_{unst} unstable eigenvalues $\lambda(p)$ existing in the limit $p^2 \to 0$ with $\mathrm{Re}(\lambda) > 0$. ∎

Remark 8.5. Proposition 8.4 agrees with proposition 8.3 for $n_1 = 0$ and $n_0 = 0$. We also notice that the $(N+1)$th double zero eigenvalue with the eigenvector $(\boldsymbol{\Phi}'(x), \mathbf{0}_N)^{\mathrm{T}}$ always splits into a pair of imaginary eigenvalues for $p^2 > 0$. This property is due to the Galileo invariance (1.7), since the translational symmetry (1.6) does not change the index $p(\mathcal{U}_H)$.

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