Stability of dark solitons in a bubble Bose-Einstein condensate
Supplemental Material

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This supplemental material provides analytical and numerical details to corroborate the main results presented in our related letter. We study the stability of dark solitons in a Bose-Einstein condensate trapped on the two-dimensional surface of a spherical bubble.

DARK SOLITONS FOR SMALL ε

Let us consider the profile $f(\theta) : [0,\pi] \to \mathbb{R}$, bounded at the end points $[\theta = 0,\pi]$, defined from the nonlinear differential equation,

$$-f''(\theta) - \cot\theta f'(\theta) + \varepsilon f^3(\theta) = \mu f(\theta), \tag{S1}$$

where we use primes for derivatives, with $\varepsilon > 0$ being a small parameter for the defocusing nonlinearity. By imposing the normalization constraint,

$$\int_0^\pi d\theta \sin\theta |f(\theta)|^2 = 1, \tag{S2}$$

then there exists a countable set of solutions $\{\mu_\ell\}_{\ell\in\mathbb{N}_0}$, uniquely parameterized by $\varepsilon > 0$, which bifurcate from the linear modes $\Theta_\ell(\theta) = P_\ell(\cos\theta)$ of the Laplace equation

$$-\left(\frac{d^2}{d\theta^2} - \cot\theta \frac{d}{d\theta}\right)\Theta_\ell = \ell(\ell+1)\Theta_\ell, \tag{S3}$$

where $P_\ell(\cos\theta)$ is the ℓ-degree Legendre polynomial. Therefore, for small ε, a solution of (S1) can be written as

$$f(\theta) = \frac{P_\ell(\cos\theta)}{\left[\int_0^\pi d\theta \sin\theta |P_\ell(\cos\theta)|^2\right]^{1/2}} + \mathcal{O}(\varepsilon), \tag{S4}$$

with μ having the following dependence on ε:

$$\mu(\varepsilon) = \ell(\ell+1) + \varepsilon\frac{\int_0^\pi d\theta \sin\theta |P_\ell(\cos\theta)|^4}{\left[\int_0^\pi d\theta \sin\theta |P_\ell(\cos\theta)|^2\right]^2} + \mathcal{O}(\varepsilon^2). \tag{S5}$$

For $\ell = 0$, we have the trivial constant solution

$$f(\theta) = \frac{1}{\sqrt{2}}, \quad \mu(\varepsilon) = \frac{\varepsilon}{2}. \tag{S6}$$

For $\ell = 1$, we have a dark-soliton solution, derived from (S4) and (S5), for a small expansion in ε, as

$$f(\theta) = \sqrt{\frac{3}{2}}\cos\theta + \varepsilon f_1(\theta) + \mathcal{O}(\varepsilon^2), \tag{S7}$$

$$\mu(\varepsilon) = 2 + \frac{9}{10}\varepsilon + \mathcal{O}(\varepsilon^2), \tag{S8}$$

where $f_1(\theta)$ is a solution of

$$-\left[\frac{d^2}{d\theta^2} + \cot\theta\frac{d}{d\theta} + 2\right] f_1(\theta) = \frac{3}{10}\sqrt{\frac{3}{2}}\cos\theta\left(3 - 5\cos^2\theta\right). \tag{S9}$$

Under the normalization (S2), there exist a unique solution $f_1(\theta)$ of (S9). Due to the orthogonality condition, $\int_0^\pi d\theta \sin\theta\cos\theta f_1(\theta) = 0$, which follows from (S2), it is proportional to $P_3(\cos\theta)$. Hence, Laplace equation (S3) with $\ell = 3$ implies that

$$f_1(\theta) = \frac{3}{100}\sqrt{\frac{3}{2}}\cos\theta\left(3 - 5\cos^2\theta\right). \tag{S10}$$

By substituting in (S7), the solution for small ε is

$$f(\theta) = \sqrt{\frac{3}{2}}\cos\theta\left[1 + \varepsilon\frac{3}{100}\left(3 - 5\cos^2\theta\right) + \mathcal{O}(\varepsilon^2)\right], \tag{S11}$$

which obeys (S2) up to $\mathcal{O}(\varepsilon^2)$.

DARK SOLITONS FOR LARGE ε

We recall that the dark soliton profile $f(\theta)$ vanishes at $\theta = \frac{\pi}{2}$. As ε increases ($\varepsilon \gg 1$), its reduction becomes concentrated near $\theta = \frac{\pi}{2}$. The asymptotic solution is

$$-f''_\infty(\theta) + \varepsilon f^3_\infty(\theta) = \mu_\infty(\varepsilon) f_\infty(\theta). \tag{S12}$$

By connecting it with the constant solution (S6), and redefining it as $f_\infty(\theta) \equiv g_0(z)$, with $z \equiv \frac{\sqrt{\varepsilon}}{2}\left(\frac{\pi}{2} - \theta\right)$, we have $g_0(z) = \frac{1}{\sqrt{2}}\tanh(z)$ as an exact solution of (S12):

$$-\frac{\varepsilon}{4}g''_0(z) + \varepsilon g^3_0(z) = \frac{\varepsilon}{2}g_0(z) = \mu_\infty(\varepsilon) g_0(z), \tag{S13}$$

where $\frac{d}{d\theta} = -\frac{\sqrt{\varepsilon}}{2}\frac{d}{dz}$. Further, with $f(\theta) \equiv g(z)$, the original equation (S1) can be written as

$$-\frac{\varepsilon}{4}g''(z) + \frac{\sqrt{\varepsilon}}{2}\tan\left(\frac{2z}{\sqrt{\varepsilon}}\right)g'(z) + \varepsilon g^3(z) = \mu g(z). \tag{S14}$$

A formal expansion for $\varepsilon \to \infty$ generates the term $zg'(z)$ at lowest order $\mathcal{O}(1/\varepsilon)$. However, since we need to use the normalization condition (S2), the asymptotic expansions of $g(z)$ and $\mu(\varepsilon)$, for $\varepsilon \to \infty$, are modified by the $\mathcal{O}(1/\sqrt{\varepsilon})$ terms, such that

$$g(z) = g_0(z) + \frac{1}{\sqrt{\varepsilon}} g_1(z) + \mathcal{O}\left(\frac{1}{\varepsilon}\right), \tag{S15}$$

$$\mu(\varepsilon) = \frac{\varepsilon}{2}\left[1 + \frac{2\mu_1}{\sqrt{\varepsilon}} + \mathcal{O}\left(\frac{1}{\varepsilon}\right)\right]. \tag{S16}$$

Substituting in (S14), we obtain the equation for $g_1(z)$:

$$-g_1''(z) + [4 - 6\,\mathrm{sech}^2(z)]g_1(z) = 4\mu_1 g_0(z). \tag{S17}$$

Since $\mathcal{L}_+ := -\partial_z^2 + [4 - 6\,\mathrm{sech}^2(z)]$ has a kernel spanned by $\mathrm{sech}^2(z)$, and $g_0(z) = \frac{1}{\sqrt{2}}\tanh(z)$ is an odd function, a unique bounded solution exists for (S17), given by

$$g_1(z) = \frac{\mu_1}{\sqrt{2}}\left[\tanh(z) + z\,\mathrm{sech}^2(z)\right] = \mu_1 \frac{d}{dz}[z g_0(z)]. \tag{S18}$$

Next, we can fix μ_1 using the normalization constraint (S2), together with (S15) and (S16). From the expansion

$$\begin{aligned} g^2(z) &= g_0^2(z) + \frac{2}{\sqrt{\varepsilon}} g_0(z) g_1(z) + \mathcal{O}\left(\frac{1}{\varepsilon}\right) \\ &= g_0^2(z)\left[1 + \frac{2\mu_1}{\sqrt{\varepsilon}}\right] + \frac{\mu_1}{\sqrt{\varepsilon}} z \frac{d}{dz} g_0^2(z) + \mathcal{O}\left(\frac{1}{\varepsilon}\right), \end{aligned} \tag{S19}$$

and observing that $g_0(z) = f_\infty(\theta)$ is normalized to one only asymptotically, for $\sqrt{\varepsilon} \to \infty$, such that

$$\begin{aligned} &\int_0^\pi d\theta \sin\theta\, g_0^2(z)\Big|_{z=\frac{\sqrt{\varepsilon}}{2}\left(\frac{\pi}{2}-\theta\right)} \\ &= 1 - \frac{2}{\sqrt{\varepsilon}} + \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon^3}}\right), \end{aligned} \tag{S20}$$

the normalization constraint (S2) yields

$$\begin{aligned} 1 &= \int_0^\pi d\theta \sin\theta f^2(\theta) = \frac{2}{\sqrt{\varepsilon}} \int_{-\frac{\pi\sqrt{\varepsilon}}{4}}^{\frac{\pi\sqrt{\varepsilon}}{4}} dz \cos\left(\frac{2z}{\sqrt{\varepsilon}}\right) g^2(z) \\ &= \left(1 + \frac{2\mu_1}{\sqrt{\varepsilon}}\right)\left(1 - \frac{2}{\sqrt{\varepsilon}}\right) + \mathcal{O}\left(\frac{1}{\varepsilon}\right) \\ &+ \frac{2\mu_1}{\varepsilon} \int_{-\frac{\pi\sqrt{\varepsilon}}{4}}^{\frac{\pi\sqrt{\varepsilon}}{4}} dz \cos\left(\frac{2z}{\sqrt{\varepsilon}}\right) z \frac{d}{dz} g_0^2(z) \\ &= 1 + \frac{2(\mu_1 - 1)}{\sqrt{\varepsilon}} + \mathcal{O}\left(\frac{1}{\varepsilon}\right). \end{aligned} \tag{S21}$$

Hence, $\mu_1 = 1$ in (S15), to satisfy (S21) up to the $\mathcal{O}(1/\varepsilon)$ order, implying

$$\mu(\varepsilon) = \frac{\varepsilon}{2}\left[1 + \frac{2}{\sqrt{\varepsilon}} + \mathcal{O}\left(\frac{1}{\varepsilon}\right)\right], \quad \text{as } \varepsilon \to \infty. \tag{S22}$$

STABILITY ANALYSIS

For the stability analysis, the spectrum for each m-angular mode is considered separately, with the following coupled system for the operators $L_m^\pm$:

$$\begin{cases} \omega \hat{u}_m = L_m^- \hat{v}_m, & L_m^- = -\Delta_m + \varepsilon f^2(\theta) - \mu, \\ \omega \hat{v}_m = L_m^+ \hat{u}_m, & L_m^+ = -\Delta_m + 3\varepsilon f^2(\theta) - \mu. \end{cases} \tag{S23}$$

where

$$\Delta_m = \frac{d^2}{d\theta^2} + \cot\theta \frac{d}{d\theta} - \frac{m^2}{\sin^2\theta}.$$

We need to determine the number of negative and zero eigenvalues of the operators $L_m^\pm$, if μ and $f(\theta)$ are defined along the branch of dark soliton solutions (S7). If $L_m^\pm$ are strictly positive, then the respective eigenvalues $\omega_m^\pm$ in the spectral stability problem (S23) are real [1], implying that the dark solitons are spectrally stable with respect to the $m-$angular Fourier mode. The number of unstable eigenvalues ω with $\mathrm{Im}(\omega) \neq 0$ can be controlled by the number of negative eigenvalues of $L_m^\pm$ and the multiplicity of their zero eigenvalues (see Theorems 1.7, 1.8, and 3.10 in [1]).

About the eigenvalues of $L_m^\pm$, the following facts are applied for every integer $m \geq 0$:

- They are simple because there may be at most one bounded solution of the second-order differential equation $L_m^\pm \chi_m^\pm(\theta) = \omega_m^\pm \chi_m^\pm(\theta)$ at each endpoint of the interval $[0, \pi]$. The eigenfunctions $\chi_m^\pm(\theta)$ in the domain of $L_m^\pm$ provide the connections between the bounded solutions as $\theta \to 0$ and $\to \pi$.

- The eigenfunctions are either even or odd with respect to the midpoint $\theta = \frac{\pi}{2}$, since $f^2(\theta)$ is even about $\theta = \frac{\pi}{2}$. Hence, if the eigenvalues are simple, then the normalized eigenfunctions satisfy $\chi_m^\pm(\pi - \theta) = \pm\chi_m^\pm(\theta)$ (plus sign for even and minus sign for odd functions).

- The smallest eigenvalue of $L_m^\pm$ is associated with the even eigenfunction about $\theta = \frac{\pi}{2}$, with the second smallest associated with the odd eigenfunction about $\theta = \frac{\pi}{2}$.

The operators $L_m^\pm$ enjoy two comparative relations:

$$L_m^+ - L_m^- = 2\varepsilon f^2(\theta) \geq 0, \tag{S24}$$

$$L_{m+1}^\pm - L_m^\pm = \frac{2m+1}{\sin^2\theta} \geq 0. \tag{S25}$$

Since $L_m^+ > L_m^-$ for all $\theta \in [0, \pi]\backslash\{\frac{\pi}{2}\}$ and $L_{m+1}^\pm > L_m^\pm$ for all $\theta \in (0, \pi)$, the smallest eigenvalue of L_m^- is smaller than the smallest one of L_m^+; and the smallest eigenvalue of $L_m^\pm$ is smaller than the smallest one of $L_{m+1}^\pm$. The same holds for the second smallest eigenvalues of the same operators in the subspace of odd functions about $\theta = \frac{\pi}{2}$. The main results for the modes $m = 0$, $m = 1$, and $m \geq 2$ are presented below.

Mode $m = 0$: We prove for the spectral problem (S23) that there exists a single pair of eigenvalues ω of negative energy which are smaller than all other pairs of eigenvalues ω of positive energy for small values of ε. These pairs can coalesce hypothetically for large values of ε, triggering another instability bifurcation. However, our numerical data does not support evidence that this instability can occur for $m = 0$.

- There exists a simple zero eigenvalue of L_0^- with the eigenfunction given by the profile $f(\theta)$ of the dark soliton because $L_0^- f(\theta) = 0$ is equivalent to (S1) for every $\varepsilon > 0$.
- By Sturm's nodal theory, L_0^- has a simple negative eigenvalue since $f(\theta)$ has a single node on $[0, \pi]$. The rest of the spectrum of L_0^- is strictly positive for every $\varepsilon > 0$.
- Since the smallest eigenvalue of L_0^- in the space of odd functions about $\theta = \frac{\pi}{2}$ is located at 0, the comparison (S24) implies that L_0^+ has at most one negative eigenvalue and the second eigenvalue of L_0^+ is strictly positive for every $\varepsilon > 0$.
- Since the smallest eigenvalue of L_1^+ is 0 for every $\varepsilon > 0$ (see the case $m = 1$), the comparison (S25) for $m = 0$ implies that L_0^+ has exactly one negative eigenvalue for every $\varepsilon > 0$. See the left panel of Fig. S1 for illustration.
- For $\varepsilon = 0$, we have $L_m^\pm|_{\varepsilon=0} = -\Delta_m - 2$. By using the eigenvalues of the Laplace equation with $m = 0$, we obtain the asymptotic approximations of eigenvalues in the stability problem (S23). For small values of $\varepsilon > 0$, a special pair of simple eigenvalues exists,

$$\pm\omega_0 = \pm 2 + \mathcal{O}(\varepsilon), \tag{S26}$$

  which has negative energy, since we have for the eigenvector $(\hat{u}, \hat{v})$ of (S23) with $\pm\omega_0$:

$$\langle L_0^+ \hat{u}, \hat{u}\rangle = \langle L_0^- \hat{v}, \hat{v}\rangle < 0. \tag{S27}$$

  In addition, there exists the double-zero eigenvalue due to the rotational invariance and a countable sequence of pairs of simple eigenvalues

$$\pm\omega_\ell = \pm[\ell(\ell+1) - 2] + \mathcal{O}(\varepsilon), \quad \ell \geq 2, \tag{S28}$$

  with the positive energies since we have for the eigenvector $(\hat{u}, \hat{v})$ of (S23) with $\pm\omega_\ell$:

$$\langle L_0^+ \hat{u}, \hat{u}\rangle = \langle L_0^- \hat{v}, \hat{v}\rangle > 0. \tag{S29}$$

- Since $\omega_\ell > \omega_0$ for every $\ell \geq 2$, there is some $\varepsilon_0 > 0$ (which might be infinite) such that the stability problem (S23) for $m = 0$ and $\varepsilon \in (0, \varepsilon_0)$ admits only real eigenvalues ω.
- The only way to get complex unstable eigenvalues ω of the stability problem (S23) for $\varepsilon \geqq \varepsilon_0$ is if eigenvalues $\pm\omega_0$ of negative energy coalesce with eigenvalues $\{\pm\omega_\ell\}_{\ell=2}^\infty$ of positive energy for some $\varepsilon = \varepsilon_0$. Numerical evidence shows that this does not occur and $\varepsilon_0 = \infty$.

Mode $m = 1$: We prove for the spectral problem (S23) that only real eigenvalues ω exist for every $\varepsilon > 0$.

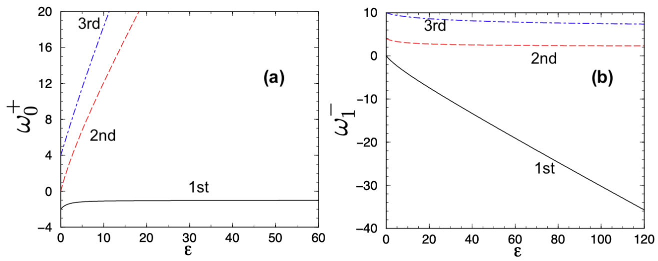

FIG. S1. The three smallest eigenvalues of L_0^+ and L_1^-, respectively, ω_0^+ and ω_1^-, are given in (a) and (b), as functions of ε. The eigenvalues were calculated numerically by discretizing the operators with finite differences up to 800 points.

- A simple zero eigenvalue of L_1^+ exists, with the eigenfunction $f'(\theta)$, as $L_1^+ f'(\theta) = 0$ is equivalent to differentiating (S1) with respect to θ for every $\varepsilon > 0$.
- From Sturm's nodal theory, L_1^+ does not have negative eigenvalues. $f'(\theta) < 0$ is sign-definite for $\theta \in (0, \pi)$ due to monotonicity of the dark soliton profile (S7). So, L_1^+ admits a simple zero eigenvalue, with the rest of its spectrum being strictly positive for all $\varepsilon > 0$.
- The comparison (S24) implies that L_1^- has at least one negative eigenvalue for every $\varepsilon > 0$.
- Since the smallest eigenvalue of L_0^- in the space of odd functions about $\theta = \frac{\pi}{2}$ is located at 0 for every $\varepsilon > 0$ (see the case $m = 0$), the comparison (S25) for $m = 0$ implies that L_1^- has exactly one negative eigenvalue and the second eigenvalue of L_1^- is strictly positive for every $\varepsilon > 0$. See the right panel of Fig. S1 for illustration.
- For $\varepsilon = 0$, we have $L_1^\pm|_{\varepsilon=0} = -\Delta_1 - 2$. By using the eigenvalues of the Laplace equation for $m = 1$, we obtain the asymptotic approximations of the eigenvalues in the stability problem (S23) for $m = 1$ and small values of $\varepsilon > 0$. There exists a double-zero eigenvalue due to the derivative mode $f'(\theta)$ and a countable sequence of pairs of simple eigenvalues $\{\omega_\ell\}_{\ell=2}^\infty$, defined by exactly the same formula (S28).
- Since all eigenvalues have positive energy for every $\ell \geq 2$ and no change of eigenvalues of $L_1^\pm$ occurs in ε, the stability problem (S23) for $m = 1$ admits only real eigenvalues ω for every $\varepsilon > 0$.

Modes $m \geq 2$: For these cases, it is demonstrated that all eigenvalues of $L_m^\pm$ are strictly positive for small $\varepsilon > 0$ and that the eigenvalues of L_m^+ remain strictly positive for every $\varepsilon > 0$. The smallest eigenvalue of L_m^- can cross 0 at $\varepsilon = \varepsilon_m > 0$ and become negative for $\varepsilon > \varepsilon_m$. If this happens, the smallest eigenvalue of L_{m+1}^- crosses 0 for larger values of ε compared to the smallest eigenvalue of L_m^-, that is, $\varepsilon_m < \varepsilon_{m+1}$. Since we show that $\varepsilon_m = 4m(m-1) + \mathcal{O}(1)$ as $m \to \infty$ in

(S32), this implies that the crossing at ε_m exists for every $m \geq 2$, triggering instability with exactly one unstable eigenvalue ω in the spectral stability problem (S23). These analytical predictions are well illustrated by the numerical results we are presenting in Fig. 2 of the main text, for angular modes m =2, 3, 4, and 5.

- All eigenvalues of $L_m^\pm$ are strictly positive for small $\varepsilon > 0$ since $L_m^\pm|_{\varepsilon=0} = -\Delta_m - 2$ and the eigenvalues of the Laplace equation for $m \geq 2$ yield $\ell(\ell+1) - 2 > 0$ for $\ell \geq m$. Hence, there exists $\varepsilon_0 > 0$ such that the stability problem (S23) for $m \geq 2$ and $\varepsilon \in (0, \varepsilon_0)$ admits only pairs of simple real eigenvalues $\{\omega_\ell\}_{\ell=m}^\infty$ of positive energy, defined by the same formula (S28).
- Since the smallest eigenvalue of L_1^+ is 0 for every $\varepsilon > 0$ (see the case $m = 1$), the comparison (S25) for $m \geq 1$ implies that the smallest eigenvalue of L_m^+ for $m \geq 2$ is strictly positive for every $\varepsilon > 0$.
- Since the smallest eigenvalue of L_1^- in the space of odd functions about $\theta = \frac{\pi}{2}$ is strictly positive for every $\varepsilon > 0$ (see the case $m = 1$), the comparison (S25) with $m \geq 1$ implies that the second eigenvalue of L_m^- for $m \geq 2$ is strictly positive for every $\varepsilon > 0$.
- The comparison (S24) implies that the smallest eigenvalue for L_m^- is always smaller than the smallest eigenvalue for L_m^+. Therefore, it can be both positive and negative. The complex eigenvalues ω in the spectral stability problem (S23) may arise if and only if the smallest positive eigenvalue of L_m^- crosses 0 at $\varepsilon = \varepsilon_m$. The comparison (S25) for $m \geq 2$ implies that the smallest eigenvalue of L_m^- always crosses 0 for smaller values of ε compared to the smallest eigenvalue of L_{m+1}^- so that $\varepsilon_m < \varepsilon_{m+1}$ for $m \geq 2$.

ASYMPTOTIC FORMULA FOR ε_m AS $m \to \infty$

By using the asymptotic solution (S16) with $\mu_1 = 1$, together with (S19), and assuming $m^2 = \mathcal{O}(\varepsilon)$ as $\varepsilon \to \infty$,

$$\begin{aligned} L_m^- &= -\frac{\varepsilon}{4}\partial_z^2 + \tan\left(\frac{2z}{\sqrt{\varepsilon}}\right)\frac{\sqrt{\varepsilon}}{2}\partial_z + \frac{m^2}{\cos^2\left(\frac{2z}{\sqrt{\varepsilon}}\right)} + \varepsilon g^2(z) - \mu \\ &= -\frac{\varepsilon}{4}\Big\{\partial_z^2 - \frac{4m^2}{\varepsilon} + 2\mathrm{sech}^2(z)\left[1 + \frac{2 - 2z\tanh(z)}{\sqrt{\varepsilon}}\right] \\ &\quad + \mathcal{O}\left(\frac{1}{\varepsilon}\right)\Big\}. \end{aligned} \tag{S30}$$

The spectrum of $\mathcal{L}_- = -\partial_z^2 - 2\,\mathrm{sech}^2(z)$ includes a simple negative eigenvalue at -1 with the eigenfunction spanned by $\mathrm{sech}(z)$ and the continuous spectrum at $[0, \infty)$. By using perturbation theory for an isolated eigenvalue, we obtain that L_m^- has a zero eigenvalue if and only if

$$\begin{aligned} \frac{4m^2}{\varepsilon} &= 1 + \frac{4}{\sqrt{\varepsilon}}\frac{\int_{\mathbb{R}}(1 - z\tanh(z))\mathrm{sech}^4(z)dz}{\int_{\mathbb{R}}\mathrm{sech}^2(z)dz} + \mathcal{O}\left(\frac{1}{\varepsilon}\right) \\ &= 1 + \frac{2}{\sqrt{\varepsilon}} + \mathcal{O}\left(\frac{1}{\varepsilon}\right), \end{aligned} \tag{S31}$$

which yields the quantization formula for bifurcations at $\{\varepsilon_m\}_{m=2}^\infty$. From the above, $4m^2 = (\sqrt{\varepsilon_m} + 1)^2 - 1 + \mathcal{O}(1)$,

$$\sqrt{\varepsilon_m} + 1 = \sqrt{4m^2 + 1 - \mathcal{O}(1)},$$

which leads to

$$\varepsilon_m \approx 4m(m-1) + \mathcal{O}(1), \quad \text{as } m \to \infty. \tag{S32}$$

In view of the inequality $0 < \varepsilon_m < \varepsilon_{m+1}$ for $m \geq 2$, the asymptotic formula (S32) implies that the lowest eigenvalue of L_m^- crosses 0 at $\varepsilon = \varepsilon_m$ for every $m \geq 2$.

NUMERICAL METHOD FOR DARK SOLITONS

We define $\tilde{f} \equiv \tilde{f}(\theta) = f(\theta)\sqrt{\varepsilon}$ for solutions of (S1),

$$\mu\tilde{f} = -\frac{d^2\tilde{f}}{d\theta^2} - \cot(\theta)\frac{d\tilde{f}}{d\theta} + \tilde{f}^3, \tag{S33}$$

subject to the normalization

$$\int_0^\pi d\theta\cos(\theta)|\tilde{f}(\theta)|^2 = \varepsilon. \tag{S34}$$

The problem is reformulated as for a given μ, we obtain the solution $\tilde{f}(\theta)$ from (S33) and define ε from (S34). Since we are looking for dark solitons, we solve (S33) just from $\theta = 0$ to $\theta = \pi/2$ and then take the odd continuation of $\tilde{f}(\theta)$ from $\theta = \pi/2$ to $\theta = \pi$. It is a two-point boundary-value problem with the boundary conditions $\tilde{f}'(0) = 0$ and $\tilde{f}(\pi/2) = 0$.

The boundary-value problem is solved by the shooting method combined with the secant method [2]. In this case, for a given μ we shoot two close values of $\tilde{f}(0)$ and propagate Eq. (S33) with the Runge-Kutta method till $\theta = \pi/2$. From the values obtained of $\tilde{f}(\pi/2)$ we can estimate a new initial shot by the secant method until we get $\tilde{f}(\pi/2) = 0$.
Results for μ =2.2 and 3.4 are shown in Fig. S2. Sweeping μ from $2 < \mu \leq 20$ can be done by continuation [3], which is performed as follows. Once we get a solution $\tilde{f}_\mu(0)$ for a given μ, we increment μ by $\delta\mu = 0.1$ and use $\tilde{f}_\mu(0)$ as an initial ansatz, and after shooting combined with the secant method, we obtain $\tilde{f}_{\mu+\delta\mu}(0)$. Subsequently, we keep incrementing μ by $\delta\mu$ and take the previous μ initial value. For $\mu > 20$, continuation is not effective. For larger values of μ, we observe from (S13) that a good approximation is given by $\tilde{f}(0) = \sqrt{\mu}$. By using this initial ansatz for each μ, it was possible to obtain the dark soliton solutions from $\mu = 2$ to $\mu = 80$. The value of ε is obtained *a posteriori* by using Eq. (S34). The numerical approximations of $f(\theta)$ and $\mu(\varepsilon)$ are shown in Fig. 1 of the main text.

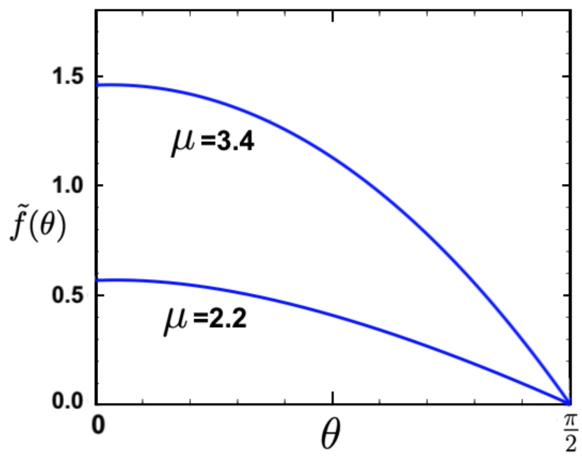

FIG. S2. Shooting method for dark solitons. The left boundary condition is $\tilde{f}'(0) = 0$. For fixed μ one shoots $\tilde{f}(0)$ and propagate Eq. (S33) till the condition $\tilde{f}(\pi/2) = 0$ is satisfied.

NUMERICAL METHOD FOR THE GPE

We consider the time-dependent evolution of the GPE for $\psi \equiv \psi(\theta, \phi, t)$, as in [4], with linear part given by

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) + \frac{1}{\sin^2\theta}\partial_\phi^2\right]\psi. \quad \text{(S35)}$$

To avoid problems at $\theta = \{0, \pi\}$, the ψ is expanded in its Fourier modes, as

$$\psi(\theta, \phi, t) = \sum_k e^{ik\phi}\psi_k(\theta, t). \quad \text{(S36)}$$

A grid of $M + 1$ points is defined, with $\theta_j = jh, j \in 0, 1, ...M$, where $h = M/\pi$. For each θ_j, $j = 1, \ldots, M-1$, ψ_k is computed by using a fast Fourier transform algorithm in the ϕ direction ($\mathcal{F}_\phi$). In the poles, where $\theta = \{0, \pi\}$ and there is no dependence on ϕ, only the mode $k = 0$ contributes. From (S36) and (S35), we have the following equation to be solved for $\psi_k \equiv \psi_k(\theta, t)$, in the interval $\theta \in (0, \pi)$ with boundary conditions $\psi_k(0, t) = \psi_k(\pi, t) = 0, \forall k \neq 0$:

$$\mathrm{i}\frac{\partial \psi_k}{\partial t} = -\left[\frac{\partial^2 \psi_k}{\partial \theta^2} + \cot\theta\frac{\partial \psi_k}{\partial \theta} - \frac{k^2}{\sin^2\theta}\psi_k\right], \quad \text{(S37)}$$

Equation (S35) is evolved one time step δt using the finite difference Crank-Nicolson method for each k, ($\mathrm{CN}_{\theta,\delta t,k}$). For $k = 0$, it is required the Neumann boundary conditions, $\partial\psi_0/\partial\theta|_{0,\pi} = 0$. We added one extra point at each boundary to implement these conditions. After the evolution, we perform the inverse Fourier transform in the ϕ direction. To include the nonlinear term $g|\psi|^2$, we employ the split-step operator technique. The complete scheme for one time step evolution is given by

$$\psi(\theta,\phi,t + \delta t) = e^{-\frac{\mathrm{i}g|\psi|^2\delta t}{2}}\mathcal{F}_\phi^{-1}\mathrm{CN}_{\theta,\delta t,k}\mathcal{F}_\phi\psi(\theta,\phi,t)e^{-\frac{\mathrm{i}g|\psi|^2\delta t}{2}}, \quad \text{(S38)}$$

where the computational operations are performed in a sequence from right to left. The numerical solutions of (S35) are shown in Fig. 3 of the main text.

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