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Homogenization of the variable-speed wave equation

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ABSTRACT

The existence of traveling waves in strongly inhomogeneous media is reviewed in the framework of the one-dimensional linear wave equation with a variable speed. Such solutions are found by using a homogenization, in which the variable-coefficient wave equation is transformed to a constant-coefficient Klein–Gordon equation. This transformation exists if and only if the spatial variations of the variable speed satisfy a constraint expressed by a second-order ordinary differential equation with two arbitrary parameters. All solutions of the constraint are found in explicit form, and our results obtained by this systematic procedure include many previous results found in the literature. Further, we show that the wave equation under the same constraint on the variable speed admits a two-parameter Lie group of nontrivial commuting point symmetries.

1. Introduction

The linear wave equation is a widely used model in mathematical physics. However, explicit solutions for this equation when the wave speed varies spatially can only be obtained in certain special cases [2,4,6,9,20,23]. These solutions are nevertheless useful to describe wave propagation and reflection in a strongly inhomogeneous medium, and to test numerical algorithms.

If the speed of the wave equation changes slowly in space, then asymptotic solutions of the wave equation can be found using the well-known WKB method; see the texts [1,3,7,15,18]. The main feature of the WKB approximation is that the basic monochromatic wave has the same structure as a traveling wave in the wave equation with constant coefficients, but its amplitude and phase vary slowly in space. On the other hand, it has often been assumed that such traveling waves do not exist in a strongly inhomogeneous medium due to internal reflections along the wave path [8,12]. In the framework of the WKB approximation, these internal reflections are exponentially small with respect to the small parameter characterizing the slow variation of the medium relative to a typical wavelength (see, for instance, [3,7]), whereas in a strongly inhomogeneous medium, the internal reflections could be quite large.

For certain special cases of the spatial dependence of the wave speed, solutions of the variable-coefficient wave equation having the form of a unidirectional traveling wave have been found in various physical contexts [14,22,24,25], including surface and internal waves in fluids [11,13].

In this paper, we re-examine the existence of such traveling waves in the one-dimensional linear wave equation with a spatially-variable wave speed $c(x)$, written in the form

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left[c^2(x)\frac{\partial u}{\partial x}\right] = 0. \tag{1}$$

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This form arises for instance for surface water waves, when expressed in terms of the wave elevation. We shall also consider another version of the wave equation (which follows for instance from the electro-magnetic Maxwell equations in one dimension)

$$\frac{\partial^2 v}{\partial t^2} - c^2(x)\frac{\partial^2 v}{\partial x^2} = 0. \tag{2}$$

The two equations (Eqs. (1) and (2)) are related and the transformation between the two equations is not unique. For instance, $u = \partial v/\partial x$ is one possible relation, while another one follows from the wave system

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial t} = c^2(x)\frac{\partial u}{\partial x}. \tag{3}$$

We will show that traveling waves of the wave equations (Eqs. (1) and (2)) exist for different constraints on the wave speed $c(x)$. In view of the relations between the two equations, this fact implies that two different sets of traveling waves exist in each of the wave equations and in the wave system (Eq. (3)).

The main idea to find traveling waves in an inhomogeneous medium is based on the possibility to find a change of variables which reduces the variable-coefficient wave equations (Eqs. (1) and (2)) to a constant-coefficient Klein–Gordon equation

$$\frac{\partial^2 W}{\partial T^2} - \frac{\partial^2 W}{\partial X^2} + \kappa W = 0, \tag{4}$$

where (X,T,W) are new variables and κ is a constant parameter. Explicit solutions of the Klein–Gordon equation (Eq. (4)) describing traveling waves can be found and these explicit solutions become traveling waves of the wave equations (Eqs. (1) and (2)).

The technique behind the derivation of the constant-coefficient Klein–Gordon equation from the variable-coefficient wave equation has been known since the classical works of P.S. Laplace [16] and S. Lie [17]. Lie groups of point symmetries and similarity solutions for the wave equation (Eq. (2)) and the wave system (Eq. (3)) were considered by Bluman and Kumei [6] and reproduced in the handbook ([10], Section 12.2). It is surprising, however, that nontrivial point symmetries of the wave equation (Eq. (1)) arise for a different constraint on $c(x)$ and differ from those of the other wave equation (Eq. (2)). Moreover, although many examples of such transformations for the linear and nonlinear wave equations can be found in the literature, see for instance the works of Bluman [4], Bluman and Cheviakov [5], and Ibragimov and Rudenko [14], as far as we are aware, no general transformation of the wave equation (Eq. (1)) to the Klein–Gordon equation (Eq. (4)) has been reported in literature. Although the method for finding such transformations is well known, we thought that it would be useful to find and tabulate all such transformations in a single article. Hence, in this paper, we characterize all possible constraints on $c(x)$ that enable existence of nontrivial point symmetries, and all possible constraints on $c(x)$ that enable reductions to the Klein–Gordon equation (Eq. (4)) in the context of the wave equation (Eq. (1)).

Our article is structured as follows. Section 2 gives a classification of point symmetries of the wave equation (Eq. (1)). Section 3 describes the transformation of the wave equation (Eq. (1)) to the Klein–Gordon equation (Eq. (4)). Section 4 discusses the various profiles of $c(x)$ that admit a homogenization. Similar results for the wave equation (Eq. (2)) are described in Section 5. Section 6 concludes the article with a brief discussion of traveling waves in inhomogeneous media.

2. Point symmetries of the wave equation (Eq. (1))

Let us rewrite the wave equation (Eq. (1)) in the form

$$A(x, u_x, u_{xx}, u_{tt}) := u_{tt} - \left(c^2(x)u_x\right)_x = 0. \tag{5}$$

We classify here all point symmetries generated by the infinitesimal operator

$$M = \tau(t,x)\frac{\partial}{\partial t} + \xi(t,x)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{6}$$

following the classical algorithm in the text by Olver [19]. Compared to the general algorithm, we have here a simplified operator (Eq. (6)) thanks to the fact that the wave equation (Eq. (5)) is linear in u. Prolongations of the infinitesimal generator (Eq. (6)) to the first and second orders are computed explicitly by

$$\begin{aligned}
\eta_x^{(1)} &= \eta_x + u_x(\eta_u - \xi_x) - u_t\tau_x,\\
\eta_t^{(1)} &= \eta_t + u_t(\eta_u - \tau_t) - u_x\xi_t,\\
\eta_{xx}^{(2)} &= \eta_{xx} + u_x(2\eta_{ux} - \xi_{xx}) + (u_x)^2\eta_{uu} - u_t\tau_{xx} + u_{xx}(\eta_u - 2\xi_x) - 2u_{xt}\tau_x,\\
\eta_{tt}^{(2)} &= \eta_{tt} + u_t(2\eta_{ut} - \tau_{tt}) + (u_t)^2\eta_{uu} - u_x\xi_{tt} + u_{tt}(\eta_u - 2\tau_t) - 2u_{xt}\xi_t,
\end{aligned}$$

so that the second-order prolongation of the symmetry generator is given by

$$M^{(2)} = M + \eta_x^{(1)}\frac{\partial}{\partial u_x} + \eta_t^{(1)}\frac{\partial}{\partial u_t} + \eta_{xx}^{(2)}\frac{\partial}{\partial u_{xx}} + \eta_{tt}^{(2)}\frac{\partial}{\partial u_{tt}}. \tag{7}$$

No terms containing u_{xt} are included in Eq. (7) because these terms are not included in the main equation (Eq. (5)). The generator equation $M^{(2)}A(x,u_x,u_{xx},u_{tt})=0$ results in a polynomial expression in terms of u_x, u_t, u_{xx}, u_{tx}, u_{tt}, where u_{tt} can be eliminated using Eq. (5). The terms u_t^2 and u_x^2 are removed if and only if $\eta_{uu}=0$, so that we may write

$$\eta(t,x,u) = d(t,x) + \gamma(t,x)u.$$

The term with $d(t,x)\neq 0$ generates an infinite-dimensional group of symmetries due to the linear superposition principle for the linear wave equation ([19], Section 2.4), and so we can set $d(t,x)\equiv 0$ without loss of generality. Hence, we rewrite $M^{(2)}A(x,u_x,u_{xx},u_{tt})=0$ as a system of five equations

$$\begin{aligned}
&O(u_{xx}) \quad \tau_t = \xi_x - \frac{c'}{c}\xi,\\
&O(u_{tx}) \quad \xi_t = c^2\tau_x,\\
&O(u_x) \quad \xi_{tt} = \left(c^2\xi_x\right)_x - 2c^2\gamma_x - 4cc'\tau_t - 2(cc')'\xi,\\
&O(u_t) \quad \tau_{tt} = \left(c^2\tau_x\right)_x + 2\gamma_t,\\
&O(u) \quad \gamma_{tt} = \left(c^2\gamma_x\right)_x.
\end{aligned}$$

Eliminating τ_{tt} from the first two equations and comparing the result with the fourth equation, we find that

$$\gamma = -\frac{c'(x)}{2c(x)}\xi + \gamma_0,$$

where γ_0 is an arbitrary constant. The term with $\gamma_0\neq 0$ generates a one-dimensional group of symmetries due to scaling invariance of the linear wave equation ([19], Section 2.4]), and so we can set $\gamma_0\equiv 0$ without loss of generality. Eliminating ξ_{tt} from the first two equations and comparing the result with the third equation, we find the same condition on γ. Finally, substituting γ to the last fifth equation and expressing ξ_{tt} from the other equations, we obtain a nontrivial constraint on ξ in terms of c:

$$c\left(2cc'' + (c')^2\right)\xi_x + \left(c^2c''' - (c')^3\right)\xi = 0.$$

Using a substitution $\xi(x,t)=c(x)\Xi(x,t)$, we separate the variables and rewrite the constraint on $c(x)$ in the equivalent form

$$\frac{cc''' + 2c'c''}{2cc'' + (c')^2} = -\frac{\Xi_x}{\Xi}\equiv b(x), \tag{8}$$

where $b(x)$ is an arbitrary function. As a result, we obtain

$$\Xi(x,t) = a(t)exp\left(-\int_0^x b(x')dx'\right),$$

where $a(t)$ is also an arbitrary function. The system of two first-order equations on ξ and τ is now written in the symmetric form

$$\Xi_t = c\tau_x, \quad \tau_t = c\Xi_x.$$

Using the explicit form for $\Xi(x,t)$, we obtain

$$\tau = \tau_0(x) - c(x)b(x)exp\left(-\int_0^x b(x')dx'\right)\int_0^t a(t')dt',$$

where $\tau_0(x)$ is an arbitrary function, and an additional constraint

$$a'(t) + c(x)\left(c(x)b'(x) + c'(x)b(x) - c(x)b^2(x)\right)\int_0^t a(t')dt' = c(x)\tau_0'(x)exp\left(\int_0^x b(x')dx'\right). \tag{9}$$

Separating variables in the constraint (9), we find that there exist a constant $\lambda\in\mathbb{R}$ such that

$$c^2b'(x) + cc'b - c^2b^2 + \lambda = 0, \tag{10}$$

while $\tau_0(x)$ and $a(t)$ are found from the system

$$c(x)\tau_0'(x)exp\left(\int_0^x b(x')dx'\right) = a'(0),\ a'(t) - \lambda\int_0^t a(t')dt' = a'(0).$$

Here $\tau_0(0)$, $a(0)$ and $a'(0)$ generate a three-parameter group of symmetries, where the term with $\tau_0(0)\neq 0$ is related to the time-invariance of the wave equation (Eq. (5)) ([19], Section 2.4). The nontrivial two-parameter group of symmetries generated by the terms with $a(0)$, $a'(0)\neq 0$ exists under the constraint on $c(x)$ given by systems (8) and (10).

Set $\lambda=\mu^2$. A general solution of the equation for $a(t)$ is given by

$$a(t) = a(0)cosh(\mu t) + \frac{a'(0)}{\mu}sinh(\mu t), \tag{11}$$

and it is analytically continued from $\mu\in\mathbb{R}$ to $\mu\in\mathbb{C}$ including the point $\mu=0$. Let us introduce new variables

$$z = \int_0^x \frac{dx'}{c(x')}, \quad b(x) = \frac{B(z)}{c(x)}. \tag{12}$$

Then, Eq. (10) reduces to

$$B'(z) = B^2-\mu^2,$$

which admits an explicit solution

$$B(z) = \mu\frac{\nu + \mu + (\nu-\mu)e^{2\mu z}}{\nu + \mu-(\nu-\mu)e^{2\mu z}}, \tag{13}$$

where $\nu=B(0)$ is an arbitrary constant. This equation is also analytically continued from $\mu\in\mathbb{R}$ to $\mu\in\mathbb{C}$ including $\mu=0$. Substituting this solution to Eq. (8), we rewrite it in the explicit form

$$\frac{d}{dx}log\left(2cc'' + (c')^2\right) = \frac{2cc''' + 4c'c''}{2cc'' + (c')^2} = \frac{2B(z)}{c}.$$

This equation can be integrated so that the constraint on $c(x)$ can be rewritten as

$$2cc'' + (c')^2 = \eta exp\left(2\int_0^z B(z')dz'\right) = \frac{4\eta\mu^2}{[(\nu + \mu)e^{-\mu z}-(\nu-\mu)e^{\mu z}]^2}, \tag{14}$$

where η is an integration constant. Thus, we have obtained the five-parameter constraint on $c(x)$, which is expressed by the second-order differential equation (Eq. (14)) with three arbitrary parameters ν, μ, and η. When $\mu\to 0$, the constraint degenerates into the equation

$$2cc'' + (c')^2 = \frac{\eta}{(1-\nu z)^2}. \tag{15}$$

Substituting $b(x)$ back to the operator M in Eq. (6) and using the variable z instead of x, we obtain the infinitesimal generator of the two-parameter symmetry group in the explicit form

$$M = \frac{a(0)}{2\mu}M_1 + \frac{a'(0)}{2\mu^2}M_2, \tag{16}$$

where

$$\begin{aligned} M_1 = &-\left[(\nu + \mu)e^{-\mu z} + (\nu-\mu)e^{\mu z}\right]sinh\,(\mu t)\frac{\partial}{\partial t} + \left[(\nu + \mu)e^{-\mu z}-(\nu-\mu)e^{\mu z}\right]cosh(\mu t)\frac{\partial}{\partial z} \\ &-\frac{1}{2}c(x)\left[(\nu + \mu)e^{-\mu z}-(\nu-\mu)e^{\mu z}\right]cosh(\mu t)u\frac{\partial}{\partial u} \end{aligned}$$

and

$$\begin{aligned} M_2 = &-\left[(\nu + \mu)e^{-\mu z} + (\nu-\mu)e^{\mu z}\right]cosh\,(\mu t)\frac{\partial}{\partial t} + \left[(\nu + \mu)e^{-\mu z}-(\nu-\mu)e^{\mu z}\right]sinh(\mu t)\frac{\partial}{\partial z} \\ &-\frac{1}{2}c(x)\left[(\nu + \mu)e^{-\mu z}-(\nu-\mu)e^{\mu z}\right]sinh(\mu t)u\frac{\partial}{\partial u}. \end{aligned}$$

In the singular limit $\mu\to 0$, a renormalization of the infinitesimal generator is needed to obtain a limit

$$\tilde{M} = \lim_{\mu\to 0}\left(M + \frac{\nu a'(0)}{\mu^2}\frac{\partial}{\partial t}\right) = a(0)\,\tilde{M}_1 + a'(0)\,\tilde{M}_2, \tag{17}$$

where

$$\tilde{M}_1 = -\nu t\frac{\partial}{\partial t} + (1-\nu z)\frac{\partial}{\partial z} - \frac{1}{2}c(x)(1-\nu z)u\frac{\partial}{\partial u}$$

and

$$\tilde{M}_2 = \left(z - \frac{1}{2}\nu t^2\right)\frac{\partial}{\partial t} + t(1-\nu z)\frac{\partial}{\partial z} - \frac{1}{2}c(x)t(1-\nu z)u\frac{\partial}{\partial u}.$$

The existence of the nontrivial two-parameter symmetry group mean that if $u(x,t)$ is a solution of the wave equation (Eq. (5)), then $\tilde{u}(\tilde{x},\tilde{t})$ is also a solution of the same equation in new coordinates $(\tilde{t},\tilde{x},\tilde{u})$, which are obtained from solutions of the differential equations

$$\frac{d\tilde{t}}{d\epsilon} = \tau(\tilde{t},\tilde{x}), \quad \frac{d\tilde{x}}{d\epsilon} = \xi(\tilde{t},\tilde{x}), \quad \frac{d\tilde{u}}{d\epsilon} = -\frac{c'(\tilde{x})}{2c(\tilde{x})}\xi(\tilde{t},\tilde{x})\tilde{u}, \tag{18}$$

for $\epsilon>0$, starting with initial conditions $\tilde{t}|_{\epsilon=0}=t$, $\tilde{x}|_{\epsilon=0}=x$, and $\tilde{u}|_{\epsilon=0}=u$. The last two equations in system (18) admit the exact solution

$$\sqrt{c(\tilde{x})}\,\tilde{u}(\tilde{x},\tilde{t}) = \sqrt{c(x)}u(x,t), \tag{19}$$

where $\tilde{x} = \tilde{x}(x,t,\epsilon)$ and $\tilde{t} = \tilde{t}(x,t,\epsilon)$. Similarly, the first two equations of system (18) admit the exact solution

$$\frac{sinh(\mu\tilde{t})}{sinh(\mu t)} = \frac{(\nu+\mu)e^{-\mu\tilde{z}} - (\nu-\mu)e^{\mu\tilde{z}}}{(\nu+\mu)e^{-\mu z} - (\nu-\mu)e^{\mu z}}, \tag{20}$$

where

$$z = \int_0^x dx'/c(x') \quad \text{and} \quad \tilde{z} = \int_0^{\tilde{x}} d\tilde{x}'/c(\tilde{x}').$$

We shall now check how the existence of the nontrivial point symmetries M_1 and M_2 is related to the possibility of the homogenization of the variable-coefficient wave equation (Eq. (1)), that is a reduction of this equation to the constant-coefficient Klein–Gordon equation (Eq. (4)). The constant-coefficient equation admits two commuting point symmetries corresponding to translations in X and T and the existence of two commuting point symmetries is coordinate-independent. Therefore, let us compute the commutator of the two nontrivial point symmetries M_1 and M_2 and set it equal to zero. Under this additional constraint, the transformation to the Klein–Gordon equation is possible and it is given by the transformation which brings the two commuting symmetries M_1 and M_2 to ∂_X and ∂_T.

Computing $[M_1,M_2]$ by using a differential function $A(t,x,u)$, we obtain

$$(M_1M_2 - M_2M_1)A = -4\mu\left(\nu^2-\mu^2\right)A_t.$$

Therefore, $[M_1,M_2]=0$ if $\nu=\pm\mu$ (the case $\mu=0$ is singular and needs special investigation). If $\nu=\pm\mu$, then $B(z)=\pm\mu=$const and the wave speed $c(x)$ satisfies the constraint

$$2cc'' + (c')^2 = \eta e^{2\mu z}, \tag{21}$$

which is expressed by a second-order differential equation with two arbitrary parameters η and μ. In the singular case $\mu=0$, the point symmetries $\tilde{M}_1$ and $\tilde{M}_2$ do not commute for any value of ν as the symmetry $\frac{\partial}{\partial t}$ occurs at the order higher than $\tilde{M}_2$. However, the commutator of $\tilde{M}_1$ and $\frac{\partial}{\partial t}$ is

$$\left(\tilde{M}_1\frac{\partial}{\partial t} - \frac{\partial}{\partial t}\tilde{M}_1\right)A = \nu A_t,$$

and it vanishes if and only if $\nu=0$.

3. Transformation to the Klein–Gordon equation

We shall now obtain the transformation of the variable-coefficient wave equation (Eq. (1)) to the constant-coefficient Klein–Gordon equation (Eq. (4)) explicitly, using a general method from the recent work of Ibragimov and Rudenko ([14], Section 4.1). In other words, we derive a general constraint on $c(x)$ that enables transformation of the wave equation

$$u_{tt} - \left(c^2(x)u_x\right)_x = 0. \tag{22}$$

to the Klein–Gordon equation

$$w_{\alpha\alpha} - w_{\beta\beta} + \kappa w = 0 \tag{23}$$

where (α,β,w) are new coordinates and $\kappa \in \mathbb{R}$ is an arbitrary parameter.

Let us introduce the functions $\alpha = \alpha(x,t)$ and $\beta = \beta(x,t)$. Using the chain rule for $u = u(x,t) = u(\alpha,\beta)$, we reduce Eq. (22) to the form

$$\left(\alpha_t^2 - c^2\alpha_x^2\right)u_{\alpha\alpha} + \left(\beta_t^2 - c^2\beta_x^2\right)u_{\beta\beta} + 2\left(\alpha_t\beta_t - c^2\alpha_x\beta_x\right)u_{\alpha\beta} + \left(\alpha_{tt} - c^2\alpha_{xx} - 2cc'(x)\alpha_x\right)u_\alpha + \left(\beta_{tt} - c^2\beta_{xx} - 2cc'(x)\beta_x\right)u_\beta = 0.$$

Without loss of generality (modulo of a scaling transformation of (α,β) and rotations in the (α,β)-plane), we can set

$$\alpha_t\beta_t = c^2\alpha_x\beta_x, \quad \alpha_t^2 - c^2\alpha_x^2 = c^2\beta_x^2 - \beta_t^2,$$

which reduce necessarily to $\alpha_t = c\beta_x$ and $\beta_t = c\alpha_x$, or equivalently to

$$\alpha_{tt} = c^2\alpha_{xx} + cc'(x)\alpha_x, \quad \beta_{tt} = c^2\beta_{xx} + cc'(x)\beta_x.$$

Given these constraints on α and β, we can rewrite the equation for $u(\alpha,\beta)$ in the equivalent form

$$u_{\alpha\alpha} - u_{\beta\beta} - \frac{cc'(x)\alpha_x}{(\alpha_t^2 - c^2\alpha_x^2)} \quad u_\alpha - \frac{cc'(x)\beta_x}{(\alpha_t^2 - c^2\alpha_x^2)}u_\beta = 0.$$

Let us next introduce $u(\alpha,\beta) = \gamma(\alpha,\beta)w(\alpha,\beta)$ and choose $\gamma(\alpha,\beta)$ to remove the terms proportional to w_α and w_β. These constraints are

$$\gamma_\alpha = \frac{cc'(x)\alpha_x\gamma}{2(\alpha_t^2 - c^2\alpha_x^2)}, \quad \gamma_\beta = -\frac{cc'(x)\beta_x\gamma}{2(\alpha_t^2 - c^2\alpha_x^2)}.$$

Using the chain rule, we obtain

$$\gamma_t = 0, \quad \gamma_x = -\frac{c'(x)\gamma}{2c},$$

so that $\gamma(x) = c^{-1/2}(x)$ without loss of generality. Note that this constraint on γ corresponds exactly to condition (19) obtained from the symmetry computations. After the substitution of $u = \gamma w$ into the equation above, we obtain the Klein–Gordon equation (Eq. (23)) with

$$\kappa = \frac{\gamma_{\alpha\alpha} - \gamma_{\beta\beta}}{\gamma} - \frac{cc'(x)\left(\alpha_x\gamma_\alpha + \beta_x\gamma_\beta\right)}{\gamma(\alpha_t^2 - c^2\alpha_x^2)}.$$

Using the chain rule again, we find that

$$\gamma_{tt} - c^2\gamma_{xx} - cc'(x)\gamma_x = \left(\alpha_t^2 - c^2\alpha_x^2\right)\left(\gamma_{\alpha\alpha} - \gamma_{\beta\beta}\right).$$

Since $\gamma(x) = c^{-1/2}(x)$, we find that

$$\kappa = \frac{2cc'' + (c')^2}{4(\alpha_t^2 - c^2\alpha_x^2)}.$$

Let us now introduce $z = \int_0^x dx'/c(x')$, such that the wave equations for $\alpha = \alpha(z,t)$ and $\beta = \beta(z,t)$ simplify to the form

$$\alpha_{tt} - \alpha_{zz} = 0, \quad \beta_{tt} - \beta_{zz} = 0,$$

with the relation $\alpha_t = \beta_z$. Solving the wave equations, we obtain that

$$\alpha(z,t) = f(z-t) + g(z+t), \quad \beta(z,t) = -f(z-t) + g(z+t),$$

where f and g are arbitrary functions. As a result,

$$\kappa = -\frac{2cc'' + (c')^2}{16f'(z-t)g'(z+t)}.$$

Since κ is a constant and $c(x)$ is t-independent, we have

$$\frac{\partial}{\partial t}\left(f'(z-t)g'(z+t)\right) = f'(z-t)g''(z+t) - f''(z-t)g'(z+t) = 0.$$

Separating variables, we obtain

$$f'(z-t) = f_0 e^{\mu(z-t)}, \quad g'(z+t) = g_0 e^{\mu(z+t)}$$

for some constants (f_0, g_0) and μ. As a result, the constraint on $c(x)$ is rewritten in the explicit form

$$2cc'' + (c')^2 = -4\delta e^{2\mu z}, \tag{24}$$

where $\delta = 4\kappa f_0 g_0$ is a new constant. This constraint corresponds exactly to constraint (21) that follows from the existence of two nontrivial commuting point symmetries (Eq. (16)) with the correspondence $\eta = -4\delta = -16\kappa f_0 g_0$.

If $\mu = 0$, constraint (24) degenerates to the form

$$2cc'' + (c')^2 = \text{const}. \tag{25}$$

Differentiation in x and multiplication by $c(x)$ give

$$\frac{d}{dx}\left(c^2 c''\right) = 0. \tag{26}$$

If $\mu = 0$, the transformation of the variable-coefficient wave equation (Eq. (22)) to the constant-coefficient Klein–Gordon equation (Eq. (23)) can be obtained by a simpler transformation, which follows from the fact that the nontrivial point symmetry $\tilde{M}_1$ commutes with $\frac{\partial}{\partial t}$. Let us seek the solution of the wave equation (Eq. (22)) in the form

$$u(x,t) = A(x)W(\xi(x),t), \tag{27}$$

where $A(x)$ and $\xi(x)$ are unknown functions. Substituting Eq. (27) into Eq. (22), we obtain

$$A\left(W_{tt} - c^2(\xi')^2 W_{\xi\xi}\right) = \left(A\left(c^2\xi'\right) + 2c^2A'\xi'\right)W_\xi + \left(c^2A'\right)W. \tag{28}$$

To have constant coefficients in front of the second derivative terms, we set $c\xi' = 1$, so that

$$\xi(x) = \int_0^x \frac{dx'}{c(x')} \equiv z.$$

To have a zero coefficient in front of W_ξ, we determine $A(x)$ by $A(x) = A_0 c^{-1/2}$, where A_0 is constant. As a result, we obtain the Klein–Gordon equation

$$W_{tt} - W_{zz} + \kappa W = 0, \tag{29}$$

where

$$\frac{d^2}{dx^2}\left(c^{3/2}\right) = \frac{3\kappa}{c^{1/2}}. \tag{30}$$

After the differentiation, we see that this equation is equivalent to constraint (25).

4. Solutions of the constraint on *c*(*x*)

We shall now discuss the solution set of constraint (24) on the wave speed $c(x)$ of the wave equation (Eq. (22)). Let us introduce

$$c(x) = Q^2(z), \quad z = \int_0^x \frac{dx'}{c(x')}.$$

Then $Q(z)$ satisfies the linear equation

$$Q''(z) + \delta e^{2\mu z} Q(z) = 0,$$

which reduces, if $\mu \neq 0$, to the Bessel equation

$$Q''(t) + \frac{1}{t} Q'(t) + sQ(t) = 0, \quad s = \frac{\delta}{\mu^2}, \quad t = e^{\mu z}. \tag{31}$$

Solutions of the Bessel equation (Eq. (31)) are expressed through the Bessel functions of the zero order J_0, Y_0, I_0, and K_0. For $s>0$ ($\delta>0$), the general solution is

$$Q(t) = A_1 J_0\left(s^{1/2} t\right) + A_2 Y_0\left(s^{1/2} t\right), \tag{32}$$

and for $s<0$ ($\delta<0$), it is

$$Q(t) = B_1 I_0\left(|s|^{1/2} t\right) + B_2 K_0\left(|s|^{1/2} t\right), \tag{33}$$

where (A_1,A_2) and (B_1,B_2) are arbitrary constants. Because of the constraint $Q(z)=c^{1/2}(x)$, only the positive definite solutions (Eqs. (32) and (33)) are allowed. Although the domain of solutions (32) and (33) is $t\in\mathbb{R}_+$ ($z\in\mathbb{R}$), the domain of definition for $c(x)$ is found from the transformation

$$x = \int_0^z Q^2(z) dz = \frac{1}{\mu} \int_1^t Q^2(t) \frac{dt}{t}.$$

Since $J_0^2, Y_0^2 = O(t^{-1})$ as $t\to\infty$ and $J_0(t)=O(1)$, $Y_0(t)=O(\log(t))$ as $t\to 0$, the first solution (Eq. (32)) corresponds to a semi-axis $(-\infty, x_\infty]$ in the domain for $c(x)$. Fig. 1 demonstrates the profile $c(x)$ obtained from Eq. (32) when $(A_1,A_2)=(1,0)$ and $(A_1,A_2)=(0,1)$.

On the other hand, since $I_0(t)=O(e^t)$, $K_0=O(e^{-t})$ as $t\to\infty$ and $I_0(t)=O(1)$, $K_0=O\log(t)$ as $t\to 0$, the second solution corresponds to the entire x-axis R if $B_1\neq 0$ or to a semi-axis $(-\infty, x_\infty]$ if $B_1=0$. Fig. 2 shows the profile $c(x)$ obtained from Eq. (33) when $(B_1,B_2)=(1,0)$ and $(B_1,B_2)=(0,1)$.

If $\mu=0$, the constraint is written in the form of Eq. (26). Two integrations in x lead to

$$(c')^2 + \frac{E}{c} = P, \tag{34}$$

where E and P are two arbitrary constants. Various profiles of $c(x)$ can be obtained from the quadrature (Eq. (34)).

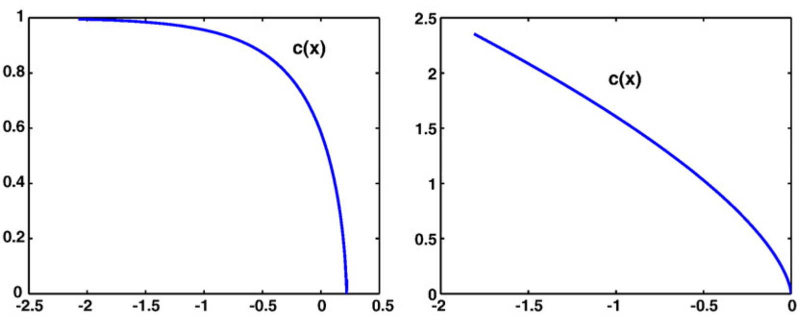

Fig. 1. Profile of $c(x)$ obtained from formula (32) with $(A_1,A_2)=(1,0)$ (left); $(A_1,A_2)=(0,1)$ (right) for $\delta=1$ and $\mu=1$.

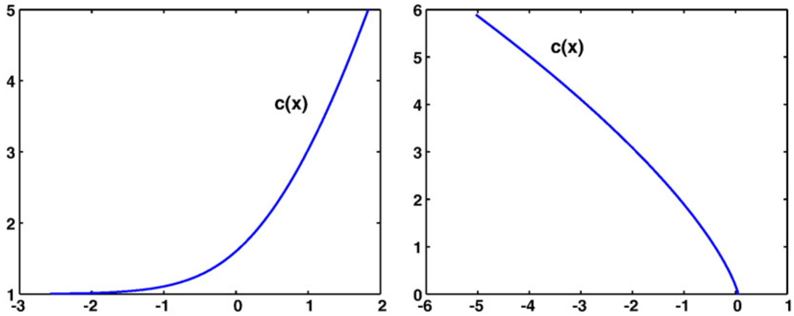

Fig. 2. Profile of $c(x)$ obtained from formula (33) with $(B_1,B_2) = (1,0)$ (left); $(B_1,B_2) = (0,1)$ (right) for $\delta = 1$ and $\mu = 1$.

If $P = 0$, the exact solution exists for $E < 0$ and is given by

$$c(x) = c_0(x - x_0)^{2/3}, \tag{35}$$

where $c_0 > 0$ and $x_0 \in \mathbb{R}$ are arbitrary constants ($E = -\frac{4}{9}c_0^3$). The exact solution (Eq. (35)) was obtained much earlier in [9] (see also recent work [11]). If $c(x)$ is given by Eq. (35), the wave equation (Eq. (22)) reduces to the constant-coefficient wave equation (Eq. (29)) with $\kappa = 0$, as follows from Eq. (30). However, the constant-coefficient wave equation is only valid on the semi-axis $x > x_0$ since $x = x_0$ is the singularity point for $c(x)$.

If $P < 0$, the exact solution of Eq. (34) exists for $E < 0$ and is given by

$$\frac{9|P|^{3/2}}{4R}(x - x_0) = arcsin\left(C^{1/2}\right) - C^{1/2}(1 - C)^{1/2}, \quad C = \frac{9|P|}{4R}c(x), \tag{36}$$

where $R > 0$ and $x_0 \in \mathbb{R}$ are arbitrary constants. Phase-plane analysis shows that the solution $c(x)$ is defined on a finite interval $[x_0, x_\infty] \subset \mathbb{R}$. The profile of $c(x)$ is shown on Fig. 3(left).

If $P > 0$, there are several solutions for $c(x)$ depending on the value of E in Eq. (34). If $E = 0$, then the solution is

$$c(x) = P^{1/2}(x - x_0), \tag{37}$$

which is also already known, see [9]. If $E < 0$, the solution is

$$\frac{9P^{3/2}}{4R}(x - x_0) = C^{1/2}(1 + C)^{1/2} - log\left(C^{1/2} + (1 + C)^{1/2}\right), \quad C = \frac{9P}{4R}c(x), \tag{38}$$

while if $E > 0$, the solution is

$$\frac{9P^{3/2}}{4R}(x - x_0) = C^{1/2}(C - 1)^{1/2} + log\left(C^{1/2} + (C - 1)^{1/2}\right), \quad C = \frac{9P}{4R}c(x), \tag{39}$$

where $R > 0$ and $x_0 \in \mathbb{R}$ are arbitrary constants. Phase-plane analysis shows again that the solution of $c(x)$ is defined on a semi-axis $[x_0, \infty)$ for $E < 0$ and on the entire axis $\mathbb{R}$ if $E > 0$. Profiles (38) and (39) are shown on Fig. 3(middle, right).

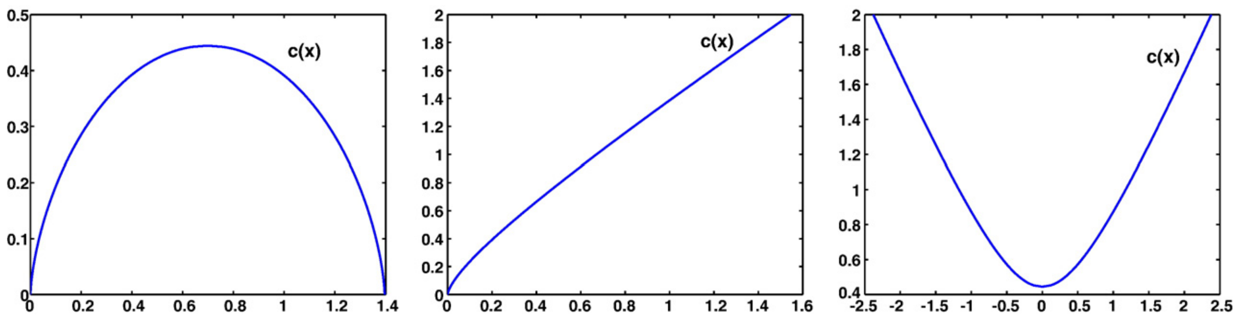

Fig. 3. Profile of $c(x)$ obtained from formula (36) (left), formula (38) (middle), and formula (39) (right) with $x_0 = 0$, $P = \pm 1$, and $R = 1$.

5. Symmetries and transformations of the wave equation (Eq. (2))

The previous results can also readily be obtained for the wave equation (Eq. (2)) and compared with the analysis of Bluman [4] and Bluman and Kumei [6]. The constraint on $c(x)$ turns out to be different from constraints (14) and (21), in spite of the correspondence between solutions of the wave equations (Eqs. (1) and (2)). We summarize here the main results for the wave equation (Eq. (2)) written in the form

$$A(x,v_{xx},v_{tt}) := v_{tt} - c^2(x)v_{xx} = 0. \tag{40}$$

Using the same algorithm as in Section 2, we rewrite the generator equation $M^{(2)}A(x,v_{xx},v_{tt}) = 0$ as an over-determined system of five equations

$$\begin{aligned} &O(v_{xx}) && \tau_t = \xi_x - \frac{c'}{c}\xi,\\ &O(v_{tx}) && \xi_t = c^2\tau_x,\\ &O(v_x) && \xi_{tt} = c^2\xi_{xx} - 2c^2\gamma_x,\\ &O(v_t) && \tau_{tt} = c^2\tau_{xx} + 2\gamma_t,\\ &O(v) && \gamma_{tt} = c^2\gamma_{xx}, \end{aligned}$$

for the infinitesimal generator

$$M = \tau(t,x)\frac{\partial}{\partial t} + \xi(t,x)\frac{\partial}{\partial x} + \gamma(t,x)v\frac{\partial}{\partial v}.$$

These equations coincide with equations (2.2a–e) in [6]. However, our parametrization of the solutions of these equations is different.

The third and fourth equations of the system are satisfied by

$$\gamma = \frac{c'(x)}{2c(x)}\xi,$$

which coincides with Eq. (2.4) in [6]. The last fifth equation gives a nontrivial constraint on ξ in the form

$$c\left(2cc'' - (c')^2\right)\xi_x + \left(c^2c''' - 2cc'c'' + (c')^3\right)\xi = 0,$$

which can be written as

$$\partial_x\left[\frac{2cc'' - (c')^2}{c^2}\xi^2\right] = 0.$$

Let $\xi = c(x)\Xi(x,t)$, so that the equation above can be rewritten in the form

$$\frac{cc'''}{2cc'' - (c')^2} = -\frac{\Xi_x}{\Xi} \equiv b(x), \tag{41}$$

where Ξ is parameterized as follows:

$$\Xi(x,t) = a(t)exp\left(-\int_0^x b(x')dx'\right).$$

Since the first and second equations of the system are identical to those in Section 2, we find a general solution for $a(t)$ and $b(t)$ in the form of Eqs. (11), (12), and (13), so that constraint (41) is rewritten in the explicit form

$$2cc'' - (c')^2 = \frac{4\eta\mu^2}{[(\nu+\mu)e^{-\mu z} - (\nu-\mu)e^{\mu z}]^2}, \tag{42}$$

where η, μ and ν are arbitrary parameters. This constraint on $c(x)$ includes all particular constraints classified in ([6], Section 2). A general solution of this constraint is constructed in Appendix A of [6].

Now we proceed with finding the nontrivial point symmetries that enable a homogenization of the wave equation (Eq. (2)) and its reduction to the Klein–Gordon equation (Eq. (4)). Repeating computations of the symmetry commutator, we obtain the same result that the symmetries commute if and only if $B(z) = \text{const}$, when constraint (42) reduces to the form

$$2cc'' - (c')^2 = \eta e^{2\mu z}, \tag{43}$$

where η and μ are two arbitrary parameters. Constraint (43) is known for the wave equation (Eq. (2)). It coincides with equation (87) in [4].

Transformation of the wave equation (Eq. (40)) to the Klein–Gordon equation (Eq. (23)) is obtained by the same algorithm as in Section 3. Using $\gamma = c^{1/2}(x)$ in the substitution $v(x,t) = \gamma(\alpha,\beta)w(\alpha,\beta)$ for $\alpha = \alpha(x,t)$ and $\beta = \beta(x,t)$, we obtain the Klein–Gordon equation (Eq. (23)) with

$$\kappa = \frac{-2cc'' + (c')^2}{4(\alpha_t^2 - c^2\alpha_x^2)}.$$

where $\alpha(x,t)$ satisfies the same equations as in Section 3. As a result, the constraint on $c(x)$ takes the same form (Eq. (43)).

When $\mu \neq 0$, the substitution $c(x) = Q^{-2}(z)$ brings constraint (43) to the same Bessel's equation (Eq. (31)), which has explicit solutions obtained in Section 4. When $\mu = 0$, differentiation of $2\,cc'' - (c')^2 = \text{const}$ gives $c''' = 0$ with the explicit solution

$$c(x) = a + b(x - x_0)^2, \tag{44}$$

where a, b, x_0 are arbitrary parameters. Constraint (44) occurs naturally in the representation of the solution $v(x,t)$ of the wave equation (Eq. (40)) in the form $v(x,t) = A(x)W(z,t)$ with $A(x) = A_0 c^{1/2}(x)$, where A_0 is constant. As a result, we obtain the same Klein–Gordon equation (Eq. (29)) with the constraint

$$\frac{d^2}{dx^2}\left(c^{1/2}\right) = -\frac{\kappa}{c^{3/2}}. \tag{45}$$

It is clear that this constraint is equivalent to $c''' = 0$ with the general solution (Eq. (44)). When $(a,b) = (0,1)$, this solution has been obtained in the literature by several different methods [6,8,11,12,20,23–25]. Reductions to the Klein–Gordon equation (Eq. (29)) under constraint (44) were discussed in [14] in the physical context of acoustical waves in inhomogeneous media and in [13,24] in the context of internal wave beams in a stratified fluid. We note that if $a \geq 0$ and $b > 0$, the profile $c(x)$ is defined on the entire axis $\mathbb{R}$.

6. Discussions and conclusion

We have shown that, for the profile $c(x)$ satisfying constraints (21) and (43), the variable-coefficient wave equations (Eqs. (1) and (2)) can be reduced to a constant-coefficient Klein–Gordon equation. Traveling waves of these wave equations are then determined by the traveling wave solutions of the Klein–Gordon equation. Our results include many similar results already in the literature, and one of our aims was to have all such allowed cases reported in a single article.

If $\kappa = 0$ ($\eta = 0$), when constraints (21) and (43) give $c(x) = (x - x_0)^{2/3}$ and $c(x) = (x - x_0)^2$, the Klein–Gordon equation reduces further to the classical wave equation $W_{tt} - W_{zz} = 0$ which has the general solution in the form of two waves propagating in opposite directions $W(z,t) = F(z - t) + G(z + t)$, for some functions F and G. Note that this reduction was described in full by Varley and Seymour in [23].

For the specific functions $c(x)$ above for $x \in [x_0,\infty)$, the wave equation is only determined for $z \in \mathbb{R}_+$ and $z \in \mathbb{R}_-$, respectively. The boundary condition at infinity for both equations can be taken to be the Sommerfeld radiation condition

$$W_t \pm W_z \to 0 \quad \text{as} \quad z \to \pm\infty.$$

The boundary condition at $z = 0$ depends on the specific physical context. If the Sommerfeld radiation condition is applied, the wave leaves the domain in finite time, but its amplitude tends to infinity as the wave approaches this boundary. In context of the water waves on a sloping beach such a boundary condition can model wave breaking; see, for instance, [21]. But if the wave field is bounded, the boundary condition for full reflection should be applied, that is $W = 0$ at $z = 0$ and this leads to anti-symmetric reflection.

Thus, the wave dynamics of the wave equations under these constraints on $c(x)$ is essentially similar to the dynamics in a homogeneous medium: initial disturbances transform into two separated waves moving in opposite directions. Depending on the boundary condition at the point $z = 0$ the corresponding wave leaves the domain ("absorbed" at the beach), or reflects with a change in sign. The influence of the real inhomogeneity is manifested in the variations of the wave amplitude and phase; but the wave shape does not change.

If $\kappa \neq 0$, then the Klein–Gordon equation $W_{tt} - W_{zz} + \kappa W = 0$ must be used and its general solution consists of two waves propagating in opposite directions expressed by the Fourier integrals,

$$W(z,t) = \int_{-\infty}^{\infty} \hat{F}(k) e^{ikz - i\omega(k)t} dk + \int_{-\infty}^{\infty} \hat{G}(k) e^{ikz + i\omega(k)t} dk, \tag{46}$$

where $\hat{F}(k)$ and $\hat{G}(k)$ represent the Fourier spectrum of the two waves. If $\kappa = m^2 > 0$, then $\omega(k) = \sqrt{m^2 + k^2}$ and waves of any wavenumber $k \in \mathbb{R}$ propagate. If $\kappa = -m^2 < 0$, then $\omega(k) = \sqrt{k^2 - m^2}$ and waves with the wavenumbers $|k| < m$ are evanescent. In addition, under more general constraints (Eqs. (21) and (43)), the Klein–Gordon equation (Eq. (4)) may only be defined on a subset of the z-axis, so that boundary conditions will be required at the finite end points.

To conclude, we have shown that for certain profiles of the wave speed $c(x)$, the wave equations (Eqs. (1) and (2)) can be reduced to a constant-coefficient Klein–Gordon equation (Eq. (4)). Thus we are able to show the existence of traveling waves in a strongly inhomogeneous medium, for these special profiles $c(x)$, based on solutions of the Klein–Gordon equation. Two classes of solutions are obtained, and their respective domains of existence determined. The first class describes the traveling waves with a persistent shape, but with changes in amplitude and phase. These traveling waves are defined by traveling waves of the constant-coefficient wave equation. The second class describes non-reflecting dispersive traveling waves, and is found from the Klein–Gordon equation.

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