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Rogue periodic waves of the focusing nonlinear Schrödinger equation

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Rogue periodic waves stand for rogue waves on a periodic background. The nonlinear Schrödinger equation in the focusing case admits two families of periodic wave solutions expressed by the Jacobian elliptic functions *dn* and *cn*. Both periodic waves are modulationally unstable with respect to long-wave perturbations. Exact solutions for the rogue periodic waves are constructed by using the explicit expressions for the periodic eigenfunctions of the Zakharov–Shabat spectral problem and the Darboux transformations. These exact solutions generalize the classical rogue wave (the so-called Peregrine's breather). The magnification factor of the rogue periodic waves is computed as a function of the elliptic modulus. Rogue periodic waves constructed here are compared with the rogue wave patterns obtained numerically in recent publications.

1. Introduction

Nonlinear waves in fluids and optics are modelled by the nonlinear Schrödinger (NLS) equation in many physical situations. The same model is also relevant to describe frequent occurrence of gigantic waves on the ocean's surface [1] and in optical fibres [2]. Such gigantic waves bear the name of rogue waves: these waves appear from nowhere and disappear without a trace [3]. From a physical perspective, the rogue waves emerge on the background of modulationally

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unstable nonlinear waves, e.g. constant waves, periodic waves or quasi-periodic spatially temporal patterns [4–6].

In what follows, we take the focusing NLS equation in the normalized form

$$\mathrm{i}u_t + u_{xx} + 2|u|^2 u = 0. \tag{1.1}$$

The NLS equation (1.1) appears as a compatibility condition of the following Lax pair of linear equations on $\varphi \in \mathbb{C}^2$:

$$\varphi_x = U\varphi, \quad U = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix} \tag{1.2}$$

and

$$\varphi_t = V\varphi, \quad V = \mathrm{i}\begin{pmatrix} 2\lambda^2 + |u|^2 & u_x + 2\lambda u \\ \bar{u}_x - 2\lambda\bar{u} & -2\lambda^2 - |u|^2 \end{pmatrix}, \tag{1.3}$$

where $\bar{u}$ is the conjugate of u. The first equation (1.2) is usually referred to as the Zakharov–Shabat spectral problem with the spectral parameter λ, whereas the second equation (1.3) determines the time evolution of the eigenfunctions of the Zakharov–Shabat spectral problem.

The *classical rogue wave* up to the translations in the (x, t) plane is given by the exact rational solution of the NLS equation (1.1)

$$u(x, t) = \left[1 - \frac{4(1 + 4\mathrm{i}t)}{1 + 4x^2 + 16t^2}\right] \mathrm{e}^{2\mathrm{i}t}. \tag{1.4}$$

As $|t| + |x| \to \infty$, the rogue wave (1.4) approaches the constant wave background $u_0(x, t) = \mathrm{e}^{2\mathrm{i}t}$. At the origin $(x, t) = (0, 0)$, the rogue wave reaches the maximum at $|u(0, 0)| = 3$, from which the *magnification factor* of the constant wave background is defined to be $M_0 = 3$. The rogue wave (1.4) was derived by Peregrine [7] and is sometimes referred to as *Peregrine's breather*. More complicated rational solutions for rogue waves in the NLS equation (1.1) were constructed by applications of the multi-fold Darboux transformations [8–10].

It is relatively less studied on how to construct rogue waves on the non-constant background. Several recent publications offer different computational tools in the context of rogue waves on the background of periodic or two-phase solutions.

Computations of rogue waves on the periodic background were performed for the first time in [11], where solutions of the Zakharov–Shabat spectral problem were computed numerically and these approximations were substituted into the onefold Darboux transformation. As the spectral parameter was selected at random in [11] without connection to the band-gap spectrum of the Zakharov–Shabat spectral problem, the resulting wave patterns do not single out a localized rogue wave on the periodic background.

Computations of rogue waves on the two-phase background were achieved in [12] with a more accurate numerical scheme. The authors constructed numerical solutions of the Zakharov–Shabat spectral problem for particular branch points obtained also numerically, after which the onefold Darboux transformation was used. The resulting wave patterns are periodic both in space and time with a rogue wave placed at the origin and these patterns matched well with experimental data for rogue waves in fluids [12].

General two-phase solutions of the focusing NLS equation (1.1) were analysed in [13] and in [14,15]. Although the wave patterns for the general two-phase solutions are quasi-periodic both in space and time, some parts of the quasi-periodic pattern look like rogue waves on the periodic background. In particular, the magnification factor of a rogue wave was computed as a ratio between the maximal amplitude and the mean value of the two-phase solution [14,15].

Integrable turbulence and rogue waves were observed numerically in [5] during the modulational instability of the *dn*-periodic waves. In particular, the magnification factor of a rogue wave arising as a result of two-soliton collisions was observed to be two, in agreement with recent results of [16] obtained in the context of the focusing modified Korteweg–de Vries (KdV) equation. The rogue wave at the time of their maximal elevation was observed to have a quasi-rational profile similar to that of the Peregrine's breather [5].

The purpose of our work is to obtain *exact analytical solutions* for the rogue waves on the periodic background, which we name here as *rogue periodic waves*. We show how to compute exactly the branch points in the band-gap spectrum of the Zakharov–Shabat problem associated with the periodic background, how to represent analytically the periodic and non-periodic solutions of the Zakharov–Shabat problem, and how to generate accurately the rogue periodic waves by means of the onefold or twofold Darboux transformations.

The standing periodic wave solutions to the focusing NLS equation (1.1) can be represented in the form

$$u(x,t) = U(x)\,\mathrm{e}^{\mathrm{i}ct}, \tag{1.5}$$

where the periodic function U satisfies the following second-order equation:

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} + 2|U|^2U = cU. \tag{1.6}$$

The second-order equation (1.6) can be integrated to yield the following first-order invariant:

$$\left|\frac{\mathrm{d}U}{\mathrm{d}x}\right|^2 + |U|^4 = c|U|^2 + d. \tag{1.7}$$

Here c and d are real-valued constants, whereas U may be complex-valued. In addition to the standing periodic wave solutions (1.5), there exist travelling periodic wave solutions with non-trivial dependence of the wave phase (e.g. in [17]). For simplicity of our presentation, we only consider rogue waves on the standing periodic waves (1.5) with real U.

There are two particular families of the periodic wave solutions in the focusing NLS equation (1.1) expressed by the Jacobian elliptic functions *dn* and *cn* [18]. The positive-definite *dn*-periodic waves are given by

$$U(x) = \mathrm{dn}(x;k), \quad c = 2-k^2, \quad d = -(1-k^2), \quad k \in (0,1), \tag{1.8}$$

whereas the sign-indefinite *cn*-periodic waves are given by

$$U(x) = k\,\mathrm{cn}(x;k), \quad c = 2k^2-1, \quad d = k^2(1-k^2), \quad k \in (0,1). \tag{1.9}$$

In both cases, the periodic waves are even and centred at the point $x=0$ thanks to the translational invariance of the NLS equation (1.1) in x. The parameter $k \in (0,1)$ is elliptic modulus and in the limit $k \to 1$; both solutions converge to the normalized NLS soliton

$$U(x) = \mathrm{sech}(x), \quad c = 1, \quad d = 0. \tag{1.10}$$

In the limit $k \to 0$, the *dn* wave converges to the constant wave background $u_0(x,t) = \mathrm{e}^{2\mathrm{i}t}$, whereas the *cn* wave converges to the zero background.

Spectral stability of the periodic waves in the focusing NLS equation was investigated in detail [17] (see also [19,20]). It was found that both *dn*- and *cn*-periodic waves are modulationally unstable with respect to the long-wave perturbations (see review in [21]). The rogue periodic waves constructed in our work are related to the modulational instability of the two periodic waves with respect to the long-wave perturbations; see numerical experiments in [5] for *dn*-periodic waves. The rogue waves are related to the modulational instability of the periodic waves because they are constructed at the branch points λ of the band-gap spectrum of the Zakharov–Shabat spectral problem (1.2) in the unstable domain with $\mathrm{Re}(\lambda) > 0$; see also [6,12].

We will adopt the following definition of a rogue wave on the periodic background. For a given periodic wave $u_{\mathrm{per}}(x,t) = U(x)\,\mathrm{e}^{\mathrm{i}ct}$, we say that the new solution u is a *rogue periodic wave* if it is different from an orbit of the periodic wave u_{per} generated by translational and phase invariance of the NLS equation (1.1) but

$$\inf_{x_0,\alpha_0\in\mathbb{R}} \sup_{x\in\mathbb{R}} |u(x,t) - U(x-x_0)\,\mathrm{e}^{\mathrm{i}\alpha_0}| \to 0 \quad \text{as } t \to \pm\infty. \tag{1.11}$$

This definition corresponds to the common understanding of rogue waves as the waves that appear from nowhere and disappear without a trace as the time evolves [3].

Our work relies on the analytical algorithm introduced recently in the context of periodic waves in the focusing modified KdV equation [22]. First, by using the algebraic technique based on nonlinearization of the Lax pair [23], we obtain the explicit expressions for the branch points of the band-gap spectrum in the Zakharov–Shabat spectral problem (1.2) associated with the *dn*- and *cn*-periodic waves. Only one periodic eigenfunction of the Zakharov–Shabat spectral problem exists at each branch point. For each periodic eigenfunction, we construct the second, linearly independent solution to the linear system (1.2)–(1.3), which is not periodic but linearly growing in (x, t). Finally, substituting non-periodic solutions to the linear system (1.2)–(1.3) into the onefold and twofold Darboux transformations [24] yields the rogue periodic waves in the sense of definition (1.11).

The paper is organized as follows. Section 2 reports construction of the periodic eigenfunctions of the linear system (1.2)–(1.3). Section 3 describes construction of the rogue periodic waves. Section 4 concludes the paper with further discussions.

2. Periodic eigenfunctions of the Lax pair

The algebraic technique based on the nonlinearization of the Lax pair was introduced in [23]. It was implemented for the linear system (1.2)–(1.3) in [25,26]. Here, we use this algebraic technique for a novel purpose of constructing the explicit expressions for periodic eigenfunctions of the Zakharov–Shabat spectral problem associated with the periodic wave solutions (1.8) and (1.9).

(a) Nonlinearization of the Lax pair

We introduce the following constraint [25,26]:

$$u = p_1^2 + \bar{q}_1^2, \tag{2.1}$$

between the potential u and a particular non-zero solution $\varphi = (p_1, q_1)^{\mathrm{T}}$ of the linear system (1.2)–(1.3) for $\lambda = \lambda_1$, where $\lambda_1 \in \mathbb{C}$ is fixed arbitrarily.

Substituting (2.1) into the spectral problem (1.2) yields a finite-dimensional Hamiltonian system in complex variables

$$\frac{\mathrm{d}p_1}{\mathrm{d}x} = \frac{\partial H}{\partial q_1} \quad \text{and} \quad \frac{\mathrm{d}q_1}{\mathrm{d}x} = -\frac{\partial H}{\partial p_1}, \tag{2.2}$$

which is associated with the real-valued Hamiltonian function

$$H = \lambda_1 p_1 q_1 + \bar{\lambda}_1 \bar{p}_1 \bar{q}_1 + \tfrac{1}{2}(p_1^2 + \bar{q}_1^2)(\bar{p}_1^2 + q_1^2). \tag{2.3}$$

Two constants of motion exist for the system (2.2)–(2.3)

$$F_0 = \mathrm{i}(p_1 q_1 - \bar{p}_1 \bar{q}_1) \tag{2.4}$$

and

$$F_1 = \lambda_1 p_1 q_1 + \bar{\lambda}_1 \bar{p}_1 \bar{q}_1 + \tfrac{1}{2}(|p_1|^2 + |q_1|^2)^2. \tag{2.5}$$

Indeed, F_0 is constant in x due to the following cancellation:

$$\begin{aligned}\frac{\mathrm{d}F_0}{\mathrm{d}x} &= \mathrm{i}q_1[\lambda_1 p_1 + (p_1^2 + \bar{q}_1^2)q_1] + \mathrm{i}p_1[-\lambda_1 q_1 - (\bar{p}_1^2 + q_1^2)p_1] \\ &\quad - \mathrm{i}\bar{q}_1[\bar{\lambda}_1 \bar{p}_1 + (\bar{p}_1^2 + q_1^2)\bar{q}_1] - \mathrm{i}\bar{p}_1[-\bar{\lambda}_1 \bar{q}_1 - (p_1^2 + \bar{q}_1^2)\bar{p}_1] = 0,\end{aligned}$$

whereas F_1 is constant in x because it is related to the constant values of H and F_0 as follows:

$$H - F_1 = \tfrac{1}{2}[|p_1|^4 + p_1^2 q_1^2 + \bar{p}_1^2 \bar{q}_1^2 + |q_1|^4] - \tfrac{1}{2}[|p_1|^4 + 2|p_1|^2|q_1|^2 + |q_1|^4] = -\tfrac{1}{2}F_0^2. \tag{2.6}$$

Substituting (2.1) into the time-evolution system (1.3) yields another Hamiltonian system

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = \frac{\partial K}{\partial q_1} \quad \text{and} \quad \frac{\mathrm{d}q_1}{\mathrm{d}t} = -\frac{\partial K}{\partial p_1}, \tag{2.7}$$

associated with the real-valued Hamiltonian function

$$K = \mathrm{i}\Big[2\lambda_1^2 p_1 q_1 - 2\bar{\lambda}_1^2 \bar{p}_1 \bar{q}_1 + |p_1^2 + \bar{q}_1^2|^2 (p_1 q_1 - \bar{p}_1 \bar{q}_1) \\ + (\lambda_1 p_1^2 - \bar{\lambda}_1 \bar{q}_1^2)(\bar{p}_1^2 + q_1^2) + (p_1^2 + \bar{q}_1^2)(\lambda_1 q_1^2 - \bar{\lambda}_1 \bar{p}_1^2)\Big]. \tag{2.8}$$

In the derivation of system (2.7)–(2.8) from system (1.3), we have used the constraint (2.1) and the following constraint:

$$u_x = 2p_1(\lambda_1 p_1 + (p_1^2 + \bar{q}_1^2) q_1) - 2\bar{q}_1(\bar{\lambda}_1 \bar{q}_1 + (p_1^2 + \bar{q}_1^2)\bar{p}_1) \\ = 2(\lambda_1 p_1^2 - \bar{\lambda}_1 \bar{q}_1^2) + 2(p_1^2 + \bar{q}_1^2)(p_1 q_1 - \bar{p}_1 \bar{q}_1), \tag{2.9}$$

which follows from the differentiation of (2.1) and the substitution of (2.2)–(2.3).

The two quantities F_0 and F_1 given by (2.4) and (2.5) are constants of motion for the system (2.7)–(2.8). Indeed, F_0 is constant in t due to the following cancelation:

$$\begin{aligned}\frac{\mathrm{d}F_0}{\mathrm{d}t} &= -q_1[(2\lambda_1^2 + |u|^2)p_1 + (u_x + 2\lambda_1 u)q_1] - p_1[-(2\lambda_1^2 + |u|^2)q_1 + (\bar{u}_x - 2\lambda_1 \bar{u})p_1] \\ &\quad - \bar{q}_1[(2\bar{\lambda}_1^2 + |u|^2)\bar{p}_1 + (\bar{u}_x + 2\bar{\lambda}_1 \bar{u})\bar{q}_1] - \bar{p}_1[-(2\bar{\lambda}_1^2 + |u|^2)\bar{q}_1 + (u_x - 2\bar{\lambda}_1 u)\bar{p}_1] \\ &= -[\bar{u}u_x + u\bar{u}_x + 2u(\lambda_1 q_1^2 - \bar{\lambda}_1 \bar{p}_1^2) + 2\bar{u}(\bar{\lambda}_1 \bar{q}_1^2 - \lambda_1^2 p_1^2)] \\ &= 0,\end{aligned}$$

where the last identity follows by (2.9) and its complex conjugate. To prove that F_1 is constant in t, it is sufficient to prove that H is constant in t, owing to the relation (2.6) between H, F_0 and F_1. To do so, we introduce the complex Poisson bracket in $\mathbb{C}^2$ associated with the symplectic structures of the systems (2.2)–(2.3) and (2.7)–(2.8)

$$\{f, g\} := \frac{\partial f}{\partial p_1}\frac{\partial g}{\partial q_1} - \frac{\partial f}{\partial q_1}\frac{\partial g}{\partial p_1} + \frac{\partial f}{\partial \bar{p}_1}\frac{\partial g}{\partial \bar{q}_1} - \frac{\partial f}{\partial \bar{q}_1}\frac{\partial g}{\partial \bar{p}_1}.$$

Then, it follows from (2.3), (2.8) and (2.9) that

$$\begin{aligned}\{H, K\} &= \mathrm{i}[(\lambda_1 q_1 + \bar{u}p_1)((2\lambda_1^2 + |u|^2)p_1 + (u_x + 2\lambda_1 u)q_1) \\ &\quad + (\lambda_1 p_1 + u q_1)((\bar{u}_x - 2\lambda_1 \bar{u})p_1 - (2\lambda_1^2 + |u|^2)q_1) \\ &\quad - (\bar{\lambda}_1 \bar{q}_1 + u\bar{p}_1)((2\bar{\lambda}_1^2 + |u|^2)\bar{p}_1 + (\bar{u}_x + 2\bar{\lambda}_1 \bar{u})\bar{q}_1) \\ &\quad - (\bar{\lambda}_1 \bar{p}_1 + \bar{u}\bar{q}_1)((u_x - 2\bar{\lambda}_1 u)\bar{p}_1 - (2\bar{\lambda}_1^2 + |u|^2)\bar{q}_1)] \\ &= \mathrm{i}[\lambda_1(u_x q_1^2 + \bar{u}_x p_1^2) - \bar{\lambda}_1(\bar{u}_x \bar{q}_1^2 + u_x \bar{p}_1^2) + (\bar{u}u_x + \bar{u}_x u)(p_1 q_1 - \bar{p}_1 \bar{q}_1)] \\ &= \mathrm{i}[u_x(\lambda_1 q_1^2 - \bar{\lambda}_1 \bar{p}_1^2 + \bar{u}(p_1 q_1 - \bar{p}_1 \bar{q}_1)) + \bar{u}_x(\lambda_1 p_1^2 - \bar{\lambda}_1 \bar{q}_1^2 + u(p_1 q_1 - \bar{p}_1 \bar{q}_1))] \\ &= 0.\end{aligned}$$

As H and K commute, it follows that H is constant in t and K is constant in x.

Let us summarize this first step of our computational algorithm. We have obtained two commuting Hamiltonian systems (2.2)–(2.3) and (2.7)–(2.8) on the eigenfunction (p_1, q_1) of the linear system (1.2)–(1.3) associated with the eigenvalue λ_1 and the potential u related to (p_1, q_1) by the algebraic constraint (2.1). In the next step, we obtain differential constraints on the potential u from the integrability scheme for the Hamiltonian system (2.2)–(2.3). One differential constraint is already obtained in (2.9), which can be written in the equivalent form

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2(\bar{\lambda}_1 p_1^2 - \lambda_1 \bar{q}_1^2) + 2(p_1^2 + \bar{q}_1^2)(\lambda_1 - \bar{\lambda}_1 + p_1 q_1 - \bar{p}_1 \bar{q}_1), \tag{2.10}$$

where the ordinary derivatives are used for convenience and the time dependence is also assumed. We will obtain other differential constraints on u, which resemble the second-order equation (1.6) and its first-order invariant (1.7) for the periodic waves (1.8) and (1.9). From here, we will conclude that the differential constraints are satisfied if u is the periodic wave of the NLS equation (1.1) given by (1.5). As u is a compatibility condition of the linear system (1.2)–(1.3), we do not have to deal with the commuting Hamiltonian system (2.7)–(2.8), as the time evolution of (p_1, q_1) can be deduced from the algebraic constraint (2.1) and the conserved quantities F_0 and F_1 in (2.4) and (2.5).

We note here that the extension of the constraint (2.1) is possible with several solutions of the linear system (1.2)–(1.3) for distinct values of λ [25,26]. This multifunction construction is related to the multiphase (quasi-periodic) solutions of the NLS equation (1.1) expressed by the Riemann's Theta function [27]. It remains open due to higher computational difficulties to obtain rogue waves on the background of multiphase solutions.

(b) Differential constraints on the potential u

Hamiltonian system (2.2)–(2.3) is a compatibility condition for the Lax equation

$$\frac{\mathrm{d}}{\mathrm{d}x}W(\lambda) = [Q(\lambda), W(\lambda)], \quad \lambda \in \mathbb{C}, \tag{2.11}$$

where

$$Q(\lambda) = \begin{pmatrix} \lambda & p_1^2 + \bar{q}_1^2 \\ -\bar{p}_1^2 - q_1^2 & -\lambda \end{pmatrix} \quad \text{and} \quad W(\lambda) = \begin{pmatrix} W_{11}(\lambda) & W_{12}(\lambda) \\ \overline{W_{12}(-\bar{\lambda})} & -\overline{W_{11}(-\bar{\lambda})} \end{pmatrix} \tag{2.12}$$

with

$$W_{11}(\lambda) = 1 - \frac{p_1 q_1}{\lambda - \lambda_1} + \frac{\bar{p}_1 \bar{q}_1}{\lambda + \bar{\lambda}_1}$$

and

$$W_{12}(\lambda) = \frac{p_1^2}{\lambda - \lambda_1} + \frac{\bar{q}_1^2}{\lambda + \bar{\lambda}_1}.$$

In particular, the (1,2) entry of the Lax equation (2.11) is rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}W_{12}(\lambda) = 2\lambda W_{12}(\lambda) - 2(p_1^2 + \bar{q}_1^2)W_{11}(\lambda). \tag{2.13}$$

We rewrite $W_{11}(\lambda)$ and $W_{12}(\lambda)$ in terms of u and constants of motion F_0 and F_1 by using relations (2.1), (2.4), (2.5) and (2.10). Some routine computations yield the following explicit expressions:

$$\begin{aligned} W_{11}(\lambda) &= 1 - \frac{\lambda(p_1 q_1 - \bar{p}_1 \bar{q}_1) + \bar{\lambda}_1 p_1 q_1 + \lambda_1 \bar{p}_1 \bar{q}_1}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)} \\ &= 1 + \frac{\mathrm{i}F_0(\lambda - \lambda_1 + \bar{\lambda}_1) + (1/2)F_0^2 - F_1 + (1/2)|u|^2}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)} \end{aligned}$$

and

$$\begin{aligned} W_{12}(\lambda) &= \frac{\lambda(p_1^2 + \bar{q}_1^2) + \bar{\lambda}_1 p_1^2 - \lambda_1 \bar{q}_1^2}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)} \\ &= \frac{(\lambda - \lambda_1 + \bar{\lambda}_1 + \mathrm{i}F_0)u + (1/2)u_x}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)}. \end{aligned}$$

Substituting these expressions for $W_{11}(\lambda)$ and $W_{12}(\lambda)$ into equation (2.13) yields the following differential constraint on u:

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2\mathrm{i}(F_0 + \mathrm{i}\lambda_1 - \mathrm{i}\bar{\lambda}_1)\frac{\mathrm{d}u}{\mathrm{d}x} + 2|u|^2u = 4\left(|\lambda_1|^2 + F_1 - \frac{1}{2}F_0^2 + \mathrm{i}F_0(\lambda_1 - \bar{\lambda}_1)\right)u. \tag{2.14}$$

This equation is to be compared with the second-order differential equation (1.6). To obtain the first-order invariant (1.7), we consider the determinant of $W(\lambda)$. As is well known [28],

- det $W(\lambda)$ has simple poles at $\lambda = \lambda_1$ and $\lambda = -\bar{\lambda}_1$;
- det $W(\lambda)$ is independent of x and t as it is related to the integrals of motion F_0 and F_1 for the Hamiltonian systems (2.2)–(2.3) and (2.7)–(2.8).

These two properties are verified with the following explicit computation:

$$\begin{aligned}\det W(\lambda) &= -W_{11}(\lambda)\overline{W_{11}(-\bar{\lambda})} - W_{12}(\lambda)\overline{W_{12}(-\bar{\lambda})}\\ &= -1 + \frac{2p_1q_1}{\lambda-\lambda_1} - \frac{2\bar{p}_1\bar{q}_1}{\lambda+\bar{\lambda}_1} + \frac{(|p_1|^2+|q_1|^2)^2}{(\lambda-\lambda_1)(\lambda+\bar{\lambda}_1)}\\ &= -\frac{(\lambda-\lambda_1)(\lambda+\bar{\lambda}_1)+2\mathrm{i}(\lambda-\lambda_1+\bar{\lambda}_1)F_0-2F_1}{(\lambda-\lambda_1)(\lambda+\bar{\lambda}_1)}.\end{aligned}$$

On the other hand, as $\overline{W_{11}(-\bar{\lambda})} = W_{11}(\lambda)$, we can also use the explicit expressions for $W_{11}(\lambda)$ and $W_{12}(\lambda)$ and rewrite det $W(\lambda)$ in the following form:

$$\begin{aligned}\det W(\lambda) = &-\left[1+\frac{\mathrm{i}F_0(\lambda-\lambda_1+\bar{\lambda}_1)+(1/2)F_0^2-F_1+(1/2)|u|^2}{(\lambda-\lambda_1)(\lambda+\bar{\lambda}_1)}\right]^2\\ &+\frac{[(\lambda-\lambda_1+\bar{\lambda}_1+\mathrm{i}F_0)u+(1/2)u_x][(\lambda-\lambda_1+\bar{\lambda}_1+\mathrm{i}F_0)\bar{u}-(1/2)\bar{u}_x]}{(\lambda-\lambda_1)^2(\lambda+\bar{\lambda}_1)^2}.\end{aligned}$$

The representation above has double poles at $\lambda = \lambda_1$ or $\lambda = -\bar{\lambda}_1$, which are identically zero due to the properties of det $W(\lambda)$. Removing the double poles at $\lambda = \lambda_1$ or $\lambda = -\bar{\lambda}_1$ yields the following two differential constraints on u:

$$\begin{aligned}&\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 + |u|^4 - 2(\mathrm{i}F_0+\bar{\lambda}_1)(\bar{u}u_x - \bar{u}_x u)\\ &\quad + 2(3F_0^2 - 2F_1 - 2\mathrm{i}F_0\bar{\lambda}_1 - 2\bar{\lambda}_1^2)|u|^2 + (F_0^2 + 2\mathrm{i}F_0\bar{\lambda}_1 - 2F_1)^2 = 0\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 + |u|^4 - 2(\mathrm{i}F_0-\lambda_1)(\bar{u}u_x - \bar{u}_x u)\\ &\quad + 2(3F_0^2 - 2F_1 + 2\mathrm{i}F_0\lambda_1 - 2\lambda_1^2)|u|^2 + (F_0^2 - 2\mathrm{i}F_0\lambda_1 - 2F_1)^2 = 0.\end{aligned}$$

Let us represent $\lambda_1 = \alpha + \mathrm{i}\beta$ with $\alpha, \beta \in \mathbb{R}$. Subtracting one differential constraint from the other, one yields the following simpler constraint on u:

$$\bar{u}\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}\bar{u}}{\mathrm{d}x} = 2\mathrm{i}(2\beta - F_0)|u|^2 + 2\mathrm{i}F_0(F_0^2 + 2F_0\beta - 2F_1). \tag{2.15}$$

Substituting this constraint in either of the two differential constraints above yields another equivalent differential constraint on u

$$\begin{aligned}&\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 + |u|^4 + 2(F_0^2 - 2F_1 + 4F_0\beta - 2\alpha^2 - 2\beta^2)|u|^2\\ &\quad + (F_0^2 + 2F_0\beta - 2F_1)(5F_0^2 - 2F_0\beta - 2F_1) - 4F_0^2\alpha^2 = 0.\end{aligned} \tag{2.16}$$

The latter equation is to be compared with the first-order invariant (1.7).

We note here that the differential constraints (2.14)–(2.16) are more general than the differential equations (1.6) and (1.7). In particular, the constraints can be used to address travelling periodic wave solutions of the NLS equation (1.1) with a non-trivial dependence of the wave phase [17]. The corresponding straightforward extension is omitted for the sake of clarity.

(c) *dn*- and *cn*-periodic waves

Let us connect the differential equations (1.6) and (1.7) for the periodic waves (1.8) and (1.9) with the differential constraints (2.14)–(2.16). Both periodic waves give zero in the left-hand side of equation (2.15). Hence, we obtain the following relations:

$$F_0 = 2\beta, \quad \beta(F_1 - 4\beta^2) = 0. \tag{2.17}$$

Comparing coefficients in (1.6) and (1.7) with the coefficients in (2.14) and (2.16) yields

$$c = 4(\alpha^2 - 5\beta^2 + F_1) \quad \text{and} \quad d = 4(4\alpha^2\beta^2 - (F_1 - 4\beta^2)(F_1 - 8\beta^2)), \tag{2.18}$$

where the relations (2.17) have been taken into account. The second equation in (2.17) can be satisfied with two choices: either $\beta = 0$ or $\beta \neq 0$ and $F_1 = 4\beta^2$. Both choices are relevant for the periodic waves (1.8) and (1.9).

If $\beta = 0$, then relations (2.17) yield $F_0 = 0$, whereas relations (2.18) yield

$$c = 4(\alpha^2 + F_1) \quad \text{and} \quad d = -4F_1^2. \tag{2.19}$$

As $d < 0$, we can only compare these expressions for (c, d) with those for the *dn*-periodic wave in (1.8). This yields the following expressions for F_1 and $\lambda_1 = \alpha$ in terms of the elliptic modulus $k \in (0, 1)$:

$$F_1 = \pm\tfrac{1}{2}\sqrt{1 - k^2} \quad \text{and} \quad \lambda_1^2 = \tfrac{1}{4}[2 - k^2 \mp 2\sqrt{1 - k^2}].$$

The expressions for λ_1 give two real eigenvalues in the right half-plane

$$\lambda_\pm := \tfrac{1}{2}(1 \pm \sqrt{1 - k^2}) \tag{2.20}$$

and two symmetric eigenvalues $-\lambda_\pm$ in the left half-plane.

If $\beta \neq 0$, then relations (2.17) yield $F_0 = 2\beta$ and $F_1 = 4\beta^2$, whereas relations (2.18) yield

$$c = 4(\alpha^2 - \beta^2) \quad \text{and} \quad d = 16\alpha^2\beta^2. \tag{2.21}$$

As $d > 0$, we can only compare these expressions for (c, d) with those for the *cn*-periodic wave in (1.9). This yields the following expression for $\lambda_1 = \alpha + i\beta$ in terms of the elliptic modulus k:

$$\lambda_1^2 = \tfrac{1}{4}[2k^2 - 1 \pm 2ik\sqrt{1 - k^2}].$$

The expressions for λ_1 give the eigenvalue in the first quadrant

$$\lambda_I := \tfrac{1}{2}[k + i\sqrt{1 - k^2}] \tag{2.22}$$

and three symmetric eigenvalues $\bar{\lambda}_I$, $-\lambda_I$ and $-\bar{\lambda}_I$ in the other three quadrants.

(d) Periodic eigenfunctions

We complete the last step of the algorithm and obtain identities for the periodic eigenfunctions of the Zakharov–Shabat spectral problem (1.2) associated with the periodic wave u. These identities arise due to the constraints imposed on the periodic wave u and the eigenfunction (p_1, q_1). In particular, relations (2.1), (2.4) and (2.9) set up the following linear system for $(p_1^2, \bar{q}_1^2)$:

$$p_1^2 + \bar{q}_1^2 = u \quad \text{and} \quad \lambda_1 p_1^2 - \bar{\lambda}_1 \bar{q}_1^2 = \tfrac{1}{2}u_x + iuF_0.$$

As $\lambda_1 = \alpha + i\beta$ and $F_0 = 2\beta$, we can obtain the squared eigenfunctions explicitly as follows:

$$p_1^2 = \frac{2\lambda_1 u + u_x}{2(\lambda_1 + \bar{\lambda}_1)} \quad \text{and} \quad \bar{q}_1^2 = \frac{2\bar{\lambda}_1 u - u_x}{2(\lambda_1 + \bar{\lambda}_1)}. \tag{2.23}$$

In what follows, it will be useful to separate the time dependence from the periodic wave $u(x,t) = U(x)\,\mathrm{e}^{\mathrm{i}ct}$. Then, representation (2.23) implies the following time dependence of the periodic eigenfunction (p_1, q_1):

$$p_1(x,t) = P_1(x)\,\mathrm{e}^{\mathrm{i}ct/2} \quad \text{and} \quad q_1(x,t) = Q_1(x)\,\mathrm{e}^{-\mathrm{i}ct/2}. \tag{2.24}$$

As U is real, the squared complex eigenfunctions are expressed by

$$P_1(x)^2 = \frac{2\lambda_1 U(x) + U'(x)}{2(\lambda_1 + \bar{\lambda}_1)} \quad \text{and} \quad \bar{Q}_1(x)^2 = \frac{2\bar{\lambda}_1 U(x) - U'(x)}{2(\lambda_1 + \bar{\lambda}_1)}. \tag{2.25}$$

For the *dn*-periodic waves (1.8), we have $U(x) = \mathrm{dn}(x;k)$, $\beta = 0$ and $\alpha = \lambda_+$ given by (2.20). As $F_0 = 0$ and $H = F_1 = -\frac{1}{2}\sqrt{1-k^2}$ in this case, the representations (2.3) and (2.4) yield $4\lambda_+ p_1 q_1 = 4\lambda_+ \bar{p}_1 \bar{q}_1 = -|u|^2 - \sqrt{1-k^2}$, whereas the representation (2.5) yields $(|p_1|^2 + |q_1|^2)^2 = |u|^2$. The previous two relations can be rewritten explicitly as

$$P_1(x)Q_1(x) = -\frac{1}{4\lambda_+}[U(x)^2 + \sqrt{1-k^2}] \tag{2.26}$$

and

$$P_1(x)^2 + Q_1(x)^2 = U(x). \tag{2.27}$$

It follows from (2.25) with $\lambda_1 = \lambda_+$ that the squared eigenfunctions P_1^2 and Q_1^2 are real. Then it follows from (2.27) with $U(x) = \mathrm{dn}(x;k) > 0$ that P_1 and Q_1 are real.

For the *cn*-periodic waves (1.9), we have $U(x) = k\mathrm{cn}(x;k)$, $\alpha = \frac{1}{2}k$ and $\beta = \frac{1}{2}\sqrt{1-k^2}$, so that $\lambda_I = \alpha + \mathrm{i}\beta$ is given by (2.22). As $F_0 = 2\beta$, $F_1 = 4\beta^2$ and $H = 2\beta^2$, it follows from (2.3) and (2.4) that $\mathrm{Re}(p_1 q_1) = -(1/2k)|u|^2$ and $\mathrm{Im}(p_1 q_1) = -\frac{1}{2}\sqrt{1-k^2}$ so that $2kp_1q_1 = -|u|^2 - \mathrm{i}k\sqrt{1-k^2}$, which can be written explicitly as

$$P_1(x)Q_1(x) = -\frac{1}{2k}[U(x)^2 + \mathrm{i}k\sqrt{1-k^2}]. \tag{2.28}$$

On the other hand, it follows from (2.5) and (2.28) that $(|p_1|^2 + |q_1|^2)^2 = 1 - k^2 + |u|^2$, hence

$$|P_1(x)|^2 + |Q_1(x)|^2 = \mathrm{dn}(x;k). \tag{2.29}$$

Furthermore, by using $F_1 = F_0^2$, we derive another relation

$$\lambda_1 p_1 q_1 + \bar{\lambda}_1 \bar{p}_1 \bar{q}_1 + p_1^2 q_1^2 + \bar{p}_1^2 \bar{q}_1^2 + \tfrac{1}{2}(|p_1|^2 - |q_1|^2)^2 = 0,$$

which yields $(|p_1|^2 - |q_1|^2)^2 = |u|^2 - |u|^4/k^2$ due to (2.22) and (2.28). Taking the negative square root yields the relation

$$|P_1(x)|^2 - |Q_1(x)|^2 = -k\,\mathrm{sn}(x;k)\mathrm{cn}(x;k). \tag{2.30}$$

The reason why the negative square root must be taken is explained from the following argument. By using (2.25), we know that

$$|P_1(x)|^4 = \frac{1}{4k^2}[(kU(x) + U'(x))^2 + (1-k^2)U(x)^2]$$

and

$$|Q_1(x)|^4 = \frac{1}{4k^2}[(kU(x) - U'(x))^2 + (1-k^2)U(x)^2],$$

where $U'(x) = -k\mathrm{sn}(x;k)\,\mathrm{dn}(x;k)$. As $\mathrm{dn}(x;k) > 0$, we have $|P(x)| < |Q(x)|$ if $\mathrm{sn}(x;k)\mathrm{cn}(x;k) > 0$. This is true for the negative square root in (2.30) and false for the positive square root.

It follows from (2.29) and (2.30) that

$$|P_1(x)|^2 = \frac{\mathrm{dn}(x;k) - k\mathrm{sn}(x;k)\mathrm{cn}(x;k)}{2} \quad \text{and} \quad |Q_1(x)|^2 = \frac{\mathrm{dn}(x;k) + k\mathrm{sn}(x;k)\mathrm{cn}(x;k)}{2}. \tag{2.31}$$

Furthermore, it follows from (2.22) and (2.25) that

$$P_1(x)^2\bar{Q}_1(x)^2 = \tfrac{1}{4}[\mathrm{cn}^2(x;k) - \mathrm{sn}^2(x;k)\,\mathrm{dn}^2(x;k) + 2\mathrm{i}\sqrt{1-k^2}\,\mathrm{sn}(x;k)\,\mathrm{cn}(x;k)\,\mathrm{dn}(x;k)].$$

Taking the negative square root yields the following relation:

$$P_1(x)\bar{Q}_1(x) = -\frac{1}{2}\mathrm{cn}(x;k)\,\mathrm{dn}(x;k) - \frac{\mathrm{i}}{2}\sqrt{1-k^2}\,\mathrm{sn}(x;k). \tag{2.32}$$

The choice of the negative square root is explained as follows. Combining (2.28) with (2.32) yields

$$\begin{aligned} P_1(x)^2|Q_1(x)|^2 &= \frac{1}{4}(k\mathrm{cn}^3(x;k)\mathrm{dn}(x;k) - (1-k^2)\mathrm{sn}(x;k)) \\ &\quad + \frac{\mathrm{i}}{4}\sqrt{1-k^2}\,\mathrm{cn}(x;k)(\mathrm{dn}(x;k) + k\,\mathrm{sn}(x;k)\,\mathrm{cn}(x;k)), \end{aligned}$$

which coincides with the expression for $P_1(x)^2|Q_1(x)|^2$ obtained from (2.25) and (2.31). In the case of the positive square root in (2.32), the expression for $P_1(x)^2|Q_1(x)|^2$ obtained from (2.28) would be negative to the one obtained from (2.25). Thus, the negative sign in (2.32) is justified.

3. Construction of rogue periodic waves

The rogue periodic waves can be constructed with the onefold or twofold Darboux transformations involving the periodic eigenfunction (p_1, q_1) for the eigenvalue λ_1 and possibly another periodic eigenfunction (p_2, q_2) for the eigenvalue λ_2, as two eigenvalues with positive real parts were identified for each periodic wave. However, such Darboux transformations recover only trivial solutions produced from the periodic wave by means of spatial translations. To obtain non-trivial solutions which corresponds to a rogue wave on the periodic background in the sense of definition (1.11), we will obtain the non-periodic solutions to the linear system (1.2)–(1.3) for the same eigenvalue λ_1.

(a) Non-periodic solutions of the Lax pair

Let u be a periodic wave of the NLS equation (1.1) and (p_1, q_1) be the x-periodic eigenfunctions of the linear system (1.2) and (1.3) with $\lambda = \lambda_1$. Let us now construct the second, linearly independent solution of the linear system (1.2)–(1.3) with $\lambda = \lambda_1$ denoted by $(\mathfrak{p}_1, \mathfrak{q}_1)$. If λ_1 is a simple eigenvalue of the periodic spectral problem (1.2), then $(\mathfrak{p}_1, \mathfrak{q}_1)$ is not periodic in x. We set

$$\mathfrak{p}_1 = \frac{\theta - 1}{q_1} \quad \text{and} \quad \mathfrak{q}_1 = \frac{\theta + 1}{p_1}, \tag{3.1}$$

so that the Wronskian between the two linearly independent solutions (p_1, q_1) and $(\mathfrak{p}_1, \mathfrak{q}_1)$ is normalized by 2. Substituting (3.1) into (1.2) yields a first-order equation on θ

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \theta\frac{uq_1^2 - \bar{u}p_1^2}{p_1q_1} + \frac{uq_1^2 + \bar{u}p_1^2}{p_1q_1}. \tag{3.2}$$

Note that this differential equation is invariant with respect to t thanks to the representation (1.5) and (2.24). Hence we write

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \theta U\frac{Q_1^2 - P_1^2}{P_1Q_1} + U\frac{Q_1^2 + P_1^2}{P_1Q_1}, \tag{3.3}$$

where U is real for both periodic waves (1.8) and (1.9).

For the *dn*-periodic waves with $U(x)=\mathrm{dn}(x;k)$, it follows from (2.20) and (2.25) that

$$4\lambda_+[P_1(x)^2-Q_1(x)^2]=2U'(x), \tag{3.4}$$

where $\lambda_1=\lambda_+$ is used. Together with (2.26) and (2.27) for real P_1 and Q_1, we rewrite the differential relation (3.3) in the explicit form

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\theta}{U^2+\sqrt{1-k^2}}=-\frac{4\lambda_+U^2}{(U^2+\sqrt{1-k^2})^2}, \tag{3.5}$$

which can be integrated to the form

$$\theta(x,t)=[U(x)^2+\sqrt{1-k^2}]\left[-4\lambda_+\int_0^x\frac{U(y)^2}{(U(y)^2+\sqrt{1-k^2})^2}\,\mathrm{d}y+\theta_0(t)\right], \tag{3.6}$$

where θ_0 is a constant of integration in x that may depend on t.

For the *cn*-periodic waves with $U(x)=k\,\mathrm{cn}(x;k)$, it follows from (2.22) and (2.25) that

$$2k[P_1(x)^2-Q_1(x)^2]=2U'(x) \tag{3.7}$$

and

$$2k[P_1(x)^2+Q_1(x)^2]=2\lambda_I U(x), \tag{3.8}$$

where $\lambda=\lambda_I$ is used. Together with (2.28), we rewrite the differential relation (3.3) in the explicit form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\theta}{|U|^2+\mathrm{i}k\sqrt{1-k^2}}=-\frac{4\lambda_I U^2}{(U^2+\mathrm{i}k\sqrt{1-k^2})^2}, \tag{3.9}$$

which can be integrated to the form

$$\theta(x,t)=[U(x)^2+\mathrm{i}k\sqrt{1-k^2}]\left[-4\lambda_I\int_0^x\frac{U(y)^2}{(U(y)^2+\mathrm{i}k\sqrt{1-k^2})^2}\,\mathrm{d}y+\theta_0(t)\right], \tag{3.10}$$

where θ_0 is a constant of integration in x that may depend on t.

We shall now add the time dependence for the function θ. By using (2.24) and (3.1), we can write the non-periodic solutions $(\mathfrak{p}_1,\mathfrak{q}_1)$ in the form

$$\mathfrak{p}_1(x,t)=\frac{\theta(x,t)-1}{Q_1(x)}\,\mathrm{e}^{\mathrm{i}ct/2}\quad\text{and}\quad\mathfrak{q}_1(x,t)=\frac{\theta(x,t)+1}{P_1(x)}\,\mathrm{e}^{-\mathrm{i}ct/2}. \tag{3.11}$$

Substituting (3.11) into (1.3) yields the following equation on θ:

$$\frac{\partial\theta}{\partial t}=\mathrm{i}\frac{2Q_1(x)(U'(x)+2\lambda_1U(x))}{P_1(x)}.$$

By using (2.25), this equation can be further rewritten in the form

$$\frac{\partial\theta}{\partial t}=8\mathrm{i}\,\mathrm{Re}(\lambda_1)P_1(x)Q_1(x). \tag{3.12}$$

For both *dn*- and *cn*-periodic waves, we substitute either (3.6) or (3.10) into (3.12) and use either (2.26) or (2.28). Both cases yield the same equation $\theta_0'(t)=-2\mathrm{i}$ with the solution $\theta_0(t)=-2\mathrm{i}t$, where the constant of integration in t is neglected due to translational invariance of the NLS equation (1.1) with respect to t.

(b) Darboux transformation

The N-fold transformation for the NLS equation was derived and justified in [29] by using the dressing method. Adopting the present notations with $N=1$, $\lambda_1=-\mathrm{i}z_1$, $(p_1,q_1)=\sigma_3\sigma_1\bar{s}_1$, where

s_1 and z_1 were used in [29] and σ_1 and σ_3 are standard Pauli matrices, we obtain the onefold transformation in the explicit form

$$\tilde{u} = u + \frac{4\mathrm{Re}(\lambda_1) p_1 \bar{q}_1}{|p_1|^2 + |q_1|^2}. \tag{3.13}$$

The onefold transformation (3.13) is fairly well known for the NLS equation (1.1) (e.g. [30] and references therein). Note that (p_1, q_1) is any non-zero solution of the linear system (1.2)–(1.3) with $\lambda = \lambda_1$.

To obtain the twofold Darboux transformation by using the formalism of [29], we set $N = 2$, $\lambda_{1,2} = -\mathrm{i} z_{1,2}$, $(p_{1,2}, q_{1,2}) = \sigma_3 \sigma_1 \bar{s}_{1,2}$ and the transformation matrix

$$M = \begin{bmatrix} \dfrac{|p_1|^2 + |q_1|^2}{2\mathrm{Re}(\lambda_1)} & \dfrac{\bar{p}_1 p_2 + \bar{q}_1 q_2}{\bar{\lambda}_1 + \lambda_2} \\ \dfrac{p_1 \bar{p}_2 + q_1 \bar{q}_2}{\lambda_1 + \bar{\lambda}_2} & \dfrac{|p_2|^2 + |q_2|^2}{2\mathrm{Re}(\lambda_2)} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

By solving the linear system of the dressing method obtained in [29], we obtain solutions $r_{1,2}$ of the linear system (1.2)–(1.3) with $\lambda_{1,2}$ and the new potential $\tilde{u}$, where r_1, r_2 and $\tilde{u}$ are defined in the form

$$r_1 = \frac{1}{\det(M)} \begin{bmatrix} \bar{q}_2 M_{12} - \bar{q}_1 M_{22} \\ \bar{p}_1 M_{22} - \bar{p}_2 M_{12} \end{bmatrix}, \quad r_2 = \frac{1}{\det(M)} \begin{bmatrix} \bar{q}_1 M_{21} - \bar{q}_2 M_{11} \\ \bar{p}_2 M_{11} - \bar{p}_1 M_{21} \end{bmatrix}$$

and

$$\tilde{u} = u + \frac{2\Sigma}{\det(M)}, \tag{3.14}$$

with

$$\begin{aligned} \Sigma &= p_1 \bar{q}_1 M_{22} + p_2 \bar{q}_2 M_{11} - p_1 \bar{q}_2 M_{12} - p_2 \bar{q}_1 M_{21} \\ &= \frac{p_1 \bar{q}_1 (|p_2|^2 + |q_2|^2)}{2\,\mathrm{Re}(\lambda_2)} + \frac{p_2 \bar{q}_2 (|p_1|^2 + |q_1|^2)}{2\,\mathrm{Re}(\lambda_1)} - \frac{p_2 \bar{q}_1 (p_1 \bar{p}_2 + q_1 \bar{q}_2)}{\lambda_1 + \bar{\lambda}_2} - \frac{p_1 \bar{q}_2 (\bar{p}_1 p_2 + \bar{q}_1 q_2)}{\bar{\lambda}_1 + \lambda_2} \end{aligned}$$

and

$$\begin{aligned} \det(M) &= M_{11} M_{22} - M_{12} M_{21} \\ &= \frac{(|p_1|^2 + |q_1|^2)(|p_2|^2 + |q_2|^2)}{4\,\mathrm{Re}(\lambda_1)\,\mathrm{Re}(\lambda_2)} - \frac{|p_1 \bar{p}_2 + q_1 \bar{q}_2|^2}{|\lambda_1 + \bar{\lambda}_2|^2}. \end{aligned}$$

This solution was used in [29] to inspect two-soliton solutions of the NLS equation (1.1).

By using the non-periodic solutions of the linear system (1.2)–(1.3) and the Darboux transformations (3.13) and (3.14), we can finally obtain the exact solutions for the rogue periodic waves of the NLS equation (1.1) in the sense of definition (1.11).

(c) Rogue *dn*-periodic waves

Let u be the periodic wave given by (1.5) and (1.8), while (p_1, q_1) be the x-periodic eigenfunction of the linear system (1.2)–(1.3) with $\lambda = \lambda_+$ given by (2.20). Substituting (2.24), (2.26) and (2.27) into the onefold Darboux transformation (3.13) yields a new solution to the NLS equation (1.1) in the form

$$\tilde{u}(x,t) = -\frac{\sqrt{1-k^2}}{\mathrm{dn}(x;k)}\,\mathrm{e}^{\mathrm{i}ct} = -\mathrm{dn}(x + K(k); k)\,\mathrm{e}^{\mathrm{i}ct}, \tag{3.15}$$

where $K(k)$ is the complete elliptic integral and table 16.8 in [18] has been used for the half-period of the function $\mathrm{dn}(x;k)$ in x. The new solution $\tilde{u}$ is just a translation of the *dn*-periodic wave in x, hence it is not a new rogue wave in the sense of definition (1.11).

To obtain a rogue *dn*-periodic wave, we replace (p_1, q_1) in (3.13) by the non-periodic solution $(\mathfrak{p}_1, \mathfrak{q}_1)$ of the linear system (1.2)–(1.3) with $\lambda = \lambda_+$ given by (2.20). Substituting (2.26), (2.27),

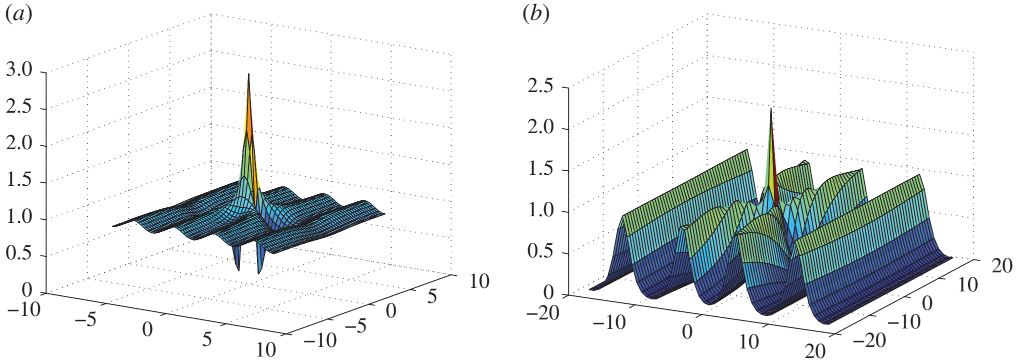

Figure 1. The rogue *dn*-periodic wave of the NLS for $k = 0.5$ (*a*) and $k = 0.999$ (*b*). (Online version in colour.)

(3.4) and (3.11) into the onefold Darboux transformation (3.13) yields a new solution to the NLS equation (1.1) in the form

$$\begin{aligned}\tilde{u}(x,t) &= \mathrm{e}^{\mathrm{i}ct}\left[U(x) - \frac{4\lambda_+(1-2\mathrm{i}\,\mathrm{Im}\,\theta(x,t)-|\theta(x,t)|^2)P_1(x)\bar{Q}_1(x)}{(|\theta(x,t)|^2+1)(|P_1(x)|^2+|Q_1(x)|^2)+2\,\mathrm{Re}\,\theta(x,t)(|Q_1(x)|^2-|P_1(x)|^2)}\right]\\ &= \mathrm{e}^{\mathrm{i}ct}\left[\mathrm{dn}(x;k) + \frac{(1-2\mathrm{i}\,\mathrm{Im}\,\theta(x,t)-|\theta(x,t)|^2)(\mathrm{dn}(x;k)^2+\sqrt{1-k^2})}{(|\theta(x,t)|^2+1)\,\mathrm{dn}(x;k)+2(1-\sqrt{1-k^2})\,\mathrm{Re}\,\theta(x,t)\mathrm{sn}(x;k)\mathrm{cn}(x;k)}\right],\end{aligned}$$

where

$$\theta(x,t) = [U(x)^2+\sqrt{1-k^2}]\left[-4\lambda_+\int_0^x \frac{U(y)^2}{(U(y)^2+\sqrt{1-k^2})^2}\,\mathrm{d}y - 2\mathrm{i}t\right]. \tag{3.16}$$

The new solution $\tilde{u}$ is no longer periodic in x. Owing to the separation of real and imaginary parts in (3.16), $|\theta(x,t)| \to \infty$ as $|x|+|t| \to \infty$ everywhere on the plane (x,t), so that

$$|\tilde{u}(x,t)| \to \mathrm{dn}(x+K(k);k) \quad \text{as } |x|+|t| \to \infty.$$

Hence, $\tilde{u}$ is a rogue *dn*-periodic wave in the sense of definition (1.11). Similarly to the computations in [22] one can show that the maximum of $|\tilde{u}(x,t)|$ occurs at $(x,t)=(0,0)$, for which we use $\theta(0,0)=0$ and obtain $|\tilde{u}(0,0)| = 2+\sqrt{1-k^2}$. As the maximum of $\mathrm{dn}(x;k)$ is one, the magnification factor of the rogue *dn*-periodic wave is $M_{\mathrm{dn}}(k) = 2+\sqrt{1-k^2}$.

Figure 1 illustrates the rogue *dn*-periodic waves for $k=0.5$ (*a*) and $k=0.999$ (*b*). In the small-amplitude limit $k\to 0$, the rogue *dn*-periodic wave looks like the Peregrine's breather (1.4) but the wave background is periodic rather than constant. In the soliton limit $k \to 1$, the rogue *dn*-periodic wave looks like a non-trivial interaction of the two adjacent NLS solitons (1.10). This comparison is confirmed with the limits of the magnification factor $M_{\mathrm{dn}}(k)$. As $k\to 0$, $M_{\mathrm{dn}}(k) \to M_0 = 3$, where M_0 is the magnification factor of the Peregrine's breather (1.4). As $k \to 1$, $M_{\mathrm{dn}}(k) \to 2$ for two nearly identical NLS solitons (1.10) of unit amplitude. The latter result is in agreement with the recent work [16], where it was shown in the context of the modified KdV equation that the magnification factor of the rogue waves built from N nearly identical solitons is exactly N.

Note that the onefold Darboux transformation (3.13) can be used with the periodic function (p_1, q_1) defined for $\lambda_1 = \lambda_-$ given by (2.20). However, as $U(x)^2 - \sqrt{1-k^2}$ vanishes for some $x \in [0, K(k)]$, the expression for θ becomes singular. It is apparently a technical difficulty, which can be resolved, but we leave this problem for future work.

(d) Onefold rogue *cn*-periodic waves

Let u be the periodic wave given by (1.5) and (1.9), while (p_1, q_1) be the x-periodic eigenfunction of the linear system (1.2)–(1.3) with $\lambda = \lambda_I$ given by (2.22). Substituting (2.24), (2.29) and (2.32) into

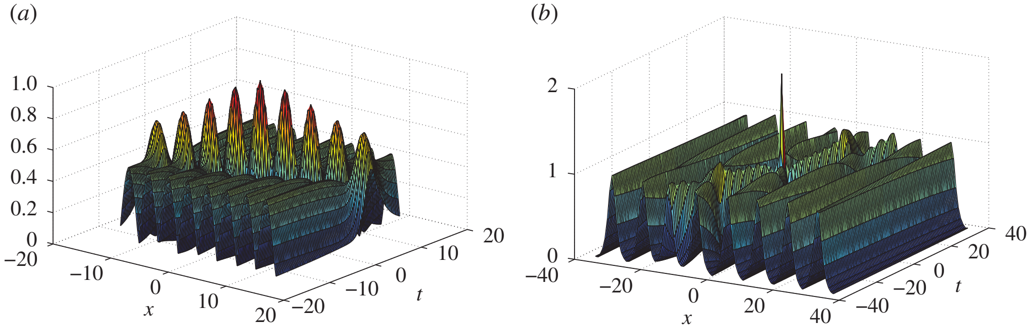

Figure 2. The onefold rogue *cn*-periodic wave of the NLS for $k = 0.5$ (*a*) and $k = 0.999$ (*b*). (Online version in colour.)

the onefold Darboux transformation (3.13) yields a new solution to the NLS equation (1.1) in the form

$$\tilde{u}(x,t) = -\frac{\mathrm{i}k\sqrt{1-k^2}\,\mathrm{sn}(x;k)}{\mathrm{dn}(x;k)}\,\mathrm{e}^{\mathrm{i}ct} = \mathrm{i}k\mathrm{cn}(x+K(k);k)\,\mathrm{e}^{\mathrm{i}ct}, \tag{3.17}$$

where table 16.8 in [18] has been used for the quarter-period of the function $\mathrm{cn}(x;k)$ in x. The new solution is just a translation of the *cn*-periodic wave u by the gauge and spatial symmetries of the NLS equation (1.1), hence it is not a rogue wave in the sense of definition (1.11).

To obtain a rogue *cn*-periodic wave, we replace (p_1, q_1) in (3.13) by the non-periodic solution $(\mathfrak{p}_1, \mathfrak{q}_1)$ of the linear system (1.2)–(1.3) with $\lambda = \lambda_I$ given by (2.22). Substituting (2.29), (2.30), (2.32) and (3.11) into the onefold Darboux transformation (3.13) yields a new solution to the NLS equation (1.1) in the form

$$\begin{aligned}\tilde{u}(x,t) &= \mathrm{e}^{\mathrm{i}ct}\left[U(x) - \frac{2k(1-2\mathrm{i}\,\mathrm{Im}\,\theta(x,t) - |\theta(x,t)|^2)P_1(x)\bar{Q}_1(x)}{(|\theta(x,t)|^2+1)(|P_1(x)|^2+|Q_1(x)|^2) + 2\,\mathrm{Re}\,\theta(x,t)(|Q_1(x)|^2-|P_1(x)|^2)}\right]\\ &= \mathrm{e}^{\mathrm{i}ct}\left[k\,\mathrm{cn}(x;k) + \frac{k(1-2\mathrm{i}\,\mathrm{Im}\,\theta(x,t) - |\theta(x,t)|^2)[\mathrm{cn}(x;k)\,\mathrm{dn}(x;k) + \mathrm{i}\sqrt{1-k^2}\mathrm{sn}(x;k)]}{(|\theta(x,t)|^2+1)\,\mathrm{dn}(x;k) + 2\,\mathrm{Re}\,\theta(x,t)k\,\mathrm{sn}(x;k)\,\mathrm{cn}(x;k)}\right],\end{aligned}$$

where

$$\theta(x,t) = [U(x)^2 + \mathrm{i}k\sqrt{1-k^2}]\left[-4\lambda_I\int_0^x \frac{U(y)^2}{(U(y)^2+\mathrm{i}k\sqrt{1-k^2})^2}\,\mathrm{d}y - 2\mathrm{i}t\right]. \tag{3.18}$$

The new solution $\tilde{u}$ is no longer periodic in x. If

$$\int_0^{4K(k)} \frac{U(y)^2(U(y)^4 - k^2(1-k^2))}{(U(y)^2+k^2(1-k^2))^2}\,\mathrm{d}y \neq 0, \tag{3.19}$$

which is satisfied at least for small k, then $|\theta(x,t)| \to \infty$ as $|x|+|t| \to \infty$ everywhere on the plane (x,t), so that

$$|\tilde{u}(x,t)| \to k|\mathrm{cn}(x+K(k);k)| \quad \text{as } |x|+|t| \to \infty.$$

Hence, $\tilde{u}$ is a rogue *cn*-periodic wave in the sense of the definition (1.11). Similarly to the computations in [22], one can show that the maximum of $|\tilde{u}(x,t)|$ occurs at $(x,t) = (0,0)$, for which we use $\theta(0,0) = 0$ and obtain $|\tilde{u}(0,0)| = 2k$. As the maximum of $\mathrm{cn}(x;k)$ is one, the magnification factor of the onefold rogue *cn*-periodic wave is $M_{\mathrm{cn}}(k) = 2$ uniformly in $k \in (0,1)$.

Figure 2 illustrates the onefold rogue *cn*-periodic waves for $k = 0.5$ (*a*) and $k = 0.999$ (*b*). In the small-amplitude limit $k \to 0$, the rogue *cn*-periodic wave looks like a propagating solitary wave; however, it is a visual illusion because the rogue wave is localized in space and time. In the soliton limit $k \to 1$, the rogue *cn*-periodic wave looks like a non-trivial interaction of the two adjacent NLS solitons (1.10) but it has a different pattern compared to the interaction of the two adjacent solitons in the rogue *dn*-periodic wave (shown in figure 1*b*). It is surprising that the magnification factor of the onefold rogue *cn*-periodic wave does not depend on the amplitude of the *cn*-periodic wave.

The onefold rogue *cn*-periodic wave does not exist for the modified KdV equation [22], because the onefold Darboux transformation (3.13) with complex λ_1 produces a complex-valued solution of the modified KdV equation. As the NLS equation (1.1) is written for a complex-valued function u, the onefold Darboux transformation (3.13) produces a new solution to the NLS equation.

(e) Twofold rogue *cn*-periodic waves

Let us now use the twofold Darboux transformation (3.14) with $\lambda_2 = \bar{\lambda}_1$, where $\lambda_1 = \lambda_I$ is given by (2.22). The periodic eigenfunction (p_2, q_2) is related to the periodic eigenfunction (p_1, q_1) in (2.24) by the following relation:

$$p_2(x,t) = \bar{P}_1(x)\,\mathrm{e}^{\mathrm{i}ct/2} \quad \text{and} \quad q_2(x,t) = \bar{Q}_1(x)\,\mathrm{e}^{-\mathrm{i}ct/2}. \tag{3.20}$$

Substituting (2.28)–(2.30), and (2.32) into the twofold Darboux transformation (3.14) yields a new solution to the the NLS equation (1.1) in the form

$$\begin{aligned}
\tilde{u}(x,t) &= U(x)\,\mathrm{e}^{\mathrm{i}ct} + \frac{2k[(P_1\bar{Q}_1 + \bar{P}_1 Q_1)(|P_1|^2 + |Q_1|^2) - 2k\,\mathrm{Re}[(k + \mathrm{i}\sqrt{1-k^2})P_1 Q_1(\bar{P}_1^2 + \bar{Q}_1^2)]]}{(|P_1|^2 + |Q_1|^2)^2 - k^2|P_1^2 + Q_1^2|^2}\,\mathrm{e}^{\mathrm{i}ct} \\
&= \left[k\,\mathrm{cn}(x;k) + \frac{2k\,\mathrm{cn}(x;k)[k^2\,\mathrm{cn}(x;k)^2 - \mathrm{dn}(x;k)^2]}{\mathrm{dn}(x;k)^2 - k^2\,\mathrm{cn}(x;k)^2}\right]\mathrm{e}^{\mathrm{i}ct} \\
&= -k\,\mathrm{cn}(x;k)\,\mathrm{e}^{\mathrm{i}ct},
\end{aligned}$$

which is again a reflection of u by the cubic symmetry.

To obtain a rogue *cn*-periodic wave, we replace (p_1, q_1) by the non-periodic solution $(\mathfrak{p}_1, \mathfrak{q}_1)$ of the same linear system (1.2)–(1.3) with $\lambda_1 = \lambda_I$. The non-periodic solution $(\mathfrak{p}_1, \mathfrak{q}_1)$ is given by (3.11) with θ given by the same expression (3.18). For $\lambda_2 = \bar{\lambda}_1$, the non-periodic solution $(\mathfrak{p}_2, \mathfrak{q}_2)$ is given by

$$\mathfrak{p}_2(x,t) = \frac{\theta_c(x,t) - 1}{\bar{Q}_1(x)}\,\mathrm{e}^{\mathrm{i}ct/2} \quad \text{and} \quad \mathfrak{q}_2(x,t) = \frac{\theta_c(x,t) + 1}{\bar{P}_1(x)}\,\mathrm{e}^{-\mathrm{i}ct/2}, \tag{3.21}$$

where θ_c is given by

$$\theta_c(x,t) = [U(x)^2 - \mathrm{i}k\sqrt{1-k^2}]\left[-4\bar{\lambda}_I \int_0^x \frac{U(y)^2}{(U(y)^2 - \mathrm{i}k\sqrt{1-k^2})^2}\,\mathrm{d}y - 2\mathrm{i}t\right]. \tag{3.22}$$

After some lengthy computations, we obtain a new solution to the NLS equation (1.1) in the form

$$\tilde{u}(x,t) = U(x)\,\mathrm{e}^{\mathrm{i}ct} + \frac{2kN(x,t)}{D(x,t)}\,\mathrm{e}^{\mathrm{i}ct}, \tag{3.23}$$

where

$$\begin{aligned}
D &= (|P_1|^2|\theta - 1|^2 + |Q_1|^2|\theta + 1|^2)(|P_1|^2|\theta_c - 1|^2 + |Q_1|^2|\theta_c + 1|^2) \\
&\quad - k^2|P_1^2(\theta - 1)(\bar{\theta}_c - 1) + Q_1^2(\theta + 1)(\bar{\theta}_c + 1)|^2, \\
N &= P_1\bar{Q}_1(\theta - 1)(\bar{\theta} + 1)(|P_1|^2|\theta_c - 1|^2 + |Q_1|^2|\theta_c + 1|^2) \\
&\quad + \bar{P}_1 Q_1(\theta_c - 1)(\bar{\theta}_c + 1))(|P_1|^2|\theta - 1|^2 + |Q_1|^2|\theta + 1|^2) \\
&\quad - k(k + \mathrm{i}\sqrt{1-k^2})P_1 Q_1(\theta - 1)(\bar{\theta}_c + 1)(\bar{P}_1^2(\bar{\theta} - 1)(\theta_c - 1) + \bar{Q}_1^2(\bar{\theta} + 1)(\theta_c + 1)) \\
&\quad - k(k - \mathrm{i}\sqrt{1-k^2})\bar{P}_1\bar{Q}_1(\theta_c - 1)(\bar{\theta} + 1)(P_1^2(\theta - 1)(\bar{\theta}_c - 1) + Q_1^2(\theta + 1)(\bar{\theta}_c + 1)).
\end{aligned}$$

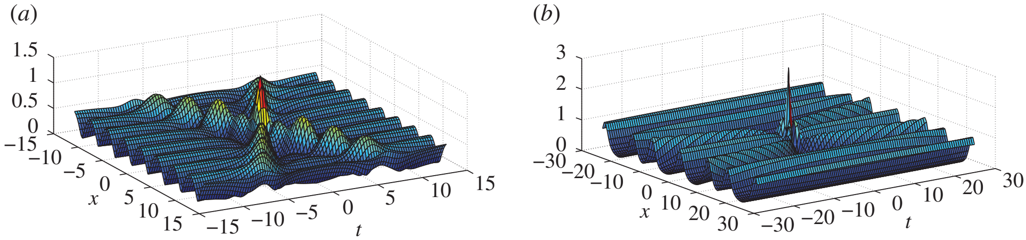

Figure 3. The twofold rogue *cn*-periodic wave of the NLS for $k = 0.5$ (*a*) and $k = 0.999$ (*b*). (Online version in colour.)

Under the same constraint (3.19), $|\theta(x,t)|, |\theta_c(x,t)| \to \infty$ as $|x| + |t| \to \infty$ everywhere on the plane (x,t), so that

$$\tilde{u}(x,t) \to -u(x,t) \quad \text{as } |x| + |t| \to \infty.$$

Hence $\tilde{u}$ is a new rogue *cn*-periodic wave in the sense of definition (1.11). By using $\theta(0,0) = \theta_c(0,0) = 0$, we obtain $\tilde{u}(0,0) = 3k$. As the maximum of $\mathrm{cn}(x;k)$ is one, the magnification factor of the twofold rogue *cn*-periodic wave is $M_{\mathrm{cn}}(k) = 3$ uniformly in $k \in (0,1)$.

Figure 3 illustrates the twofold rogue *cn*-periodic waves for $k = 0.5$ (*a*) and $k = 0.999$ (*b*). In the small-amplitude limit $k \to 0$, the rogue *cn*-periodic wave looks like two propagating solitary waves but they are again localized in space and time. In the soliton limit $k \to 1$, the rogue *cn*-periodic wave looks like a non-trivial interaction of the three adjacent NLS solitons (1.10) and these explain why the magnification factor is three, in agreement with the recent work [16]. It is still surprising that the magnification factor of the twofold rogue *cn*-periodic wave does not depend on the amplitude of the *cn*-periodic wave.

4. Further discussion

We have developed a computational algorithm of constructing rogue periodic waves in the context of the focusing NLS equation. As both *dn*- and *cn*-periodic waves are modulationally unstable, both waves exhibit rogue waves on their background which appears from nowhere and disappears without any trace. For the rogue *dn*-periodic waves, we were only able to use onefold Darboux transformation because the non-periodic solutions were obtained in the closed analytical form for only one branch point of the Zakharov–Shabat spectral problem. For the rogue *cn*-periodic waves, we were able to use both onefold and twofold Darboux transformations because the two branch points in the Zakharov–Shabat spectral problem are related to each other by complex conjugation and reflection symmetries.

These results can be developed further in view of high interest to rogue waves in the focusing NLS equation [11,12,14,15]. A relatively simple extension of these solutions would include travelling periodic waves with a non-trivial dependence of the wave phase. A more difficult problem is to extend the computational algorithm of constructing the rogue waves for Riemann's Theta functions, which represent quasi-periodic solutions including the two-phase solutions considered in [12,14,15]. These open questions are left for further studies.

Data accessibility. This work has no experimental data.

Authors' contributions. Both the authors equally contributed to writing the paper.

Competing interests. The authors have no competing interests.

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