Rogue periodic waves of the mKdV equation

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Abstract

Traveling periodic waves of the modified Korteweg–de Vries (mKdV) equation are considered in the focusing case. By using one-fold and two-fold Darboux transformations, we construct explicitly the rogue periodic waves of the mKdV equation expressed by the Jacobian elliptic functions *dn* and *cn* respectively. The rogue *dn*-periodic wave describes propagation of an algebraically decaying soliton over the *dn*-periodic wave, the latter wave is modulationally stable with respect to long-wave perturbations. The rogue *cn*-periodic wave represents the outcome of the modulation instability of the *cn*-periodic wave with respect to long-wave perturbations and serves for the same purpose as the rogue wave of the nonlinear Schrödinger equation (NLS), where it is expressed by the rational function. We compute the magnification factor for the *cn*-periodic wave of the mKdV equation and show that it remains the same as in the small-amplitude NLS limit for all amplitudes. As a by-product of our work, we find explicit expressions for the periodic eigenfunctions of the AKNS spectral problem associated with the *dn* and *cn* periodic waves of the mKdV equation.

1 Introduction

Simplest models for nonlinear waves in fluids such as the nonlinear Schrödinger equation (NLS), the Korteweg–de Vries equation (KdV), and the modified Korteweg–de Vries equation (mKdV) have many things in common. First, they appear to be integrable by using the inverse scattering transform method for the same AKNS (Ablowitz–Kaup–Newell–Segur) spectral problem [1]. Second, there exist asymptotic transformations of one nonlinear evolution equation to another nonlinear evolution equation, e.g. from defocusing NLS to KdV and from KdV and focusing mKdV to the defocusing and focusing NLS respectively [29].

Modulation instability of periodic waves in the focusing NLS equation has been a paramount concept in the modern nonlinear physics [30]. More recently, spectral instability of the periodic waves expressed by the elliptic functions *cn* and *dn* has been investigated in the focusing NLS [14] (see also [19, 21]). Regarding periodic waves in the focusing mKdV equation, it was found that the *dn*-periodic waves are modulationally stable with respect to the long-wave perturbations, whereas the *cn*-periodic waves are modulationally unstable [3, 4] (see also [13]).

The outcome of the modulation instability of the periodic waves in the focusing NLS equation is the emergence of the localized spatially-temporal patterns on the background of the unstable periodic waves (see review in [5]). Such localized spatially-temporal patterns are known under the generic name of *rogue waves*. Explicit expressions for the rogue waves in the NLS equation have been obtained by using available algebraic constructions such as the multi-fold Darboux transformations

[2, 15]. In the simplest setting, the rogue waves are expressed as rational solutions of the NLS equation. For example, if the focusing NLS equation is set in the form

$$i\psi_t + \psi_{xx} + 2(|\psi|^2 - 1)\psi = 0, \tag{1.1}$$

then the basic rogue wave up to the translations in (x,t) is given by

$$\psi(x,t) = 1 - \frac{4(1+4it)}{1+4x^2+16t^2}. \tag{1.2}$$

As $|t|+|x| \to \infty$, the rogue wave (1.2) approaches the constant wave background $\psi_0(x,t) = 1$. On the other hand, at $(x,t) = (0,0)$, the rogue wave reaches the maximum at $|\psi(0,0)| = 3$, from which we define the *magnification factor* of the constant wave background to be $M_0 = 3$.

Rogue waves over non-uniform backgrounds (e.g., the periodic waves or the two-phase solutions) were addressed very recently in the context of the focusing NLS (1.1), by numerically implementing the Bäcklund transformation for the periodic waves [22] and the two-phase solutions [6]. Further analytical work to characterize the general two-phase solutions of the NLS can be found in [28].

In the present work, we address the rogue waves in the focusing mKdV equation written in the normalized form

$$u_t + 6u^2 u_x + u_{xxx} = 0. \tag{1.3}$$

Some particular rational and trigonometric solutions of the mKdV were recently constructed in connection to rogue waves [11]. In comparison with [11], the scope of our work is to obtain the rogue periodic waves expressed by the Jacobian elliptic functions and to investigate the connection of the rogue periodic waves in the small-amplitude limit to the rogue wave (1.2). In particular, we shall compute explicitly the magnification factor for the rogue periodic waves.

There are two particular periodic wave solutions of the mKdV. One solution is strictly positive and is given by the *dn* elliptic function. The other solution is sign-indefinite and is given by the *cn* elliptic function. Up to the translations in (x,t) as well as a scaling transformation, the positive solution is given by

$$u_{\rm dn}(x,t) = {\rm dn}(x-ct;k), \quad c = c_{\rm dn}(k) := 2-k^2, \tag{1.4}$$

whereas the sign-indefinite solution is given by

$$u_{\rm cn}(x,t) = k{\rm cn}(x-ct;k), \quad c = c_{\rm cn}(k) := 2k^2-1. \tag{1.5}$$

In both cases, $k \in (0,1)$ is elliptic modulus, which gives two different asymptotic limits. In the limit $k \to 0$, we obtain

$$u_{\rm dn}(x,t) \to 1, \quad c_{\rm dn} \to 2, \tag{1.6}$$

and

$$u_{\rm cn}(x,t) \sim k\cos(x+t), \quad c_{\rm cn} \to -1, \tag{1.7}$$

the latter asymptotic representation is understood in the sense of the Stokes expansion. As is well-known [18], the mKdV equation can be reduced asymptotically to the NLS equation in the small-amplitude limit when $k \to 0$. In this limit, the *cn*-periodic wave of the mKdV is reduced to the constant wave background ψ_0 of the NLS, which is modulationally unstable with respect to the long-wave perturbations. Hence, the rogue *cn*-periodic wave for the mKdV generalizes the rogue wave (1.2) of the NLS expressed by the rational function.

In the limit $k \to 1$, both Jacobian elliptic functions (1.4) and (1.5) converges to the normalized mKdV soliton

$$u_{\rm dn}(x,t), u_{\rm cn}(x,t) \to u_{\rm soliton}(x,t) = {\rm sech}(x-t), \quad c_{\rm dn}, c_{\rm cn} \to 1. \tag{1.8}$$

Very recently, the rogue waves of the mKdV built from a superposition of slowly interacting nearly identical solitons were considered numerically [25] and analytically [26]. It was found in both studies that the magnification factor of the rogue waves built from two nearly identical solitons is exactly two.

In our work, we compute the rogue periodic waves for the *dn* and *cn* Jacobian elliptic functions with the following method. First, by using the algebraic technique based on the nonlinearization of the Lax pair [7], we consider the AKNS spectral problem associated with the Jacobian elliptic functions and obtain the explicit expressions for the eigenvalues λ with $\mathrm{Re}(\lambda) > 0$ and the associated periodic eigenfunctions. These eigenvalues correspond to the branch points of the continuous bands, when the AKNS spectral problem with the periodic potentials is considered on the real line with the help of the Floquet–Bloch transform [5]. For each periodic eigenfunction, we construct the second, linearly independent solution of the AKNS spectral problem, which is not periodic but linearly growing in (x, t). The latter eigenfunction is expressed by using integrals of the Jacobian elliptic functions and hence it is not explicit. Finally, by using the one-fold and two-fold Darboux transformations [20] with the nonperiodic eigenfunctions of the AKNS spectral problem, we obtain the rogue periodic waves. Although the resulting solutions are not explicit, we prove that these solutions approach the *dn* and *cn* periodic waves as $|t| + |x| \to \infty$.

Figure 1 shows the rogue *dn*-periodic wave for $k = 0.5$ (left) and $k = 0.99$ (right). This wave corresponds to the propagation of an algebraic soliton of the mKdV [24] over the *dn*-periodic wave. Hence, it is not a proper rogue wave which appears from nowhere and disappear without a trace. The latter wave does not exist because the *dn*-periodic wave is modulationally stable with respect to the long-wave perturbations [4, 3].

Figure 2 shows the rogue *cn*-periodic wave for $k = 0.5$ (left) and $k = 0.99$ (right). We can clearly see a proper rogue wave appearing from nowhere and disappearing without a trace as time evolves. The rogue *cn*-periodic wave describes modulation instability of the *cn*-periodic wave in the mKdV with respect to the long-wave perturbations.

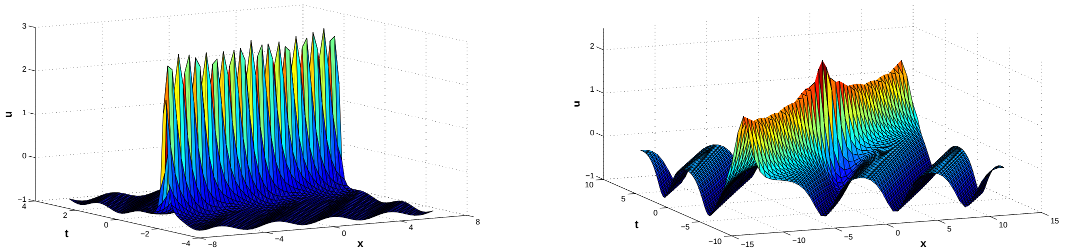

Figure 1: The rogue *dn*-periodic wave of the mKdV for $k = 0.5$ (left) and $k = 0.99$ (right).

The magnification factors for the rogue periodic waves can be computed in the explicit form:

$$M_{\mathrm{dn}}(k) = 2 + \sqrt{1 - k^2}, \quad M_{\mathrm{cn}}(k) = 3, \quad k \in [0, 1]. \tag{1.9}$$

It is remarkable that the magnification factor $M_{\mathrm{cn}}(k) = 3$ is independent of the wave amplitude in agreement with $M_0 = 3$ for the rogue wave (1.2) thanks to the small-amplitude asymptotic limit (1.7). At the same time $M_{\mathrm{dn}}(k) \in [2, 3]$ and $M_{\mathrm{dn}}(k) \to 3$ as $k \to 0$ due to the fact that the limit (1.6) of the mKdV (1.3) gives the same potential to the AKNS spectral problem as the constant wave background $\psi_0(x, t) = 1$ of the NLS equation (1.1).

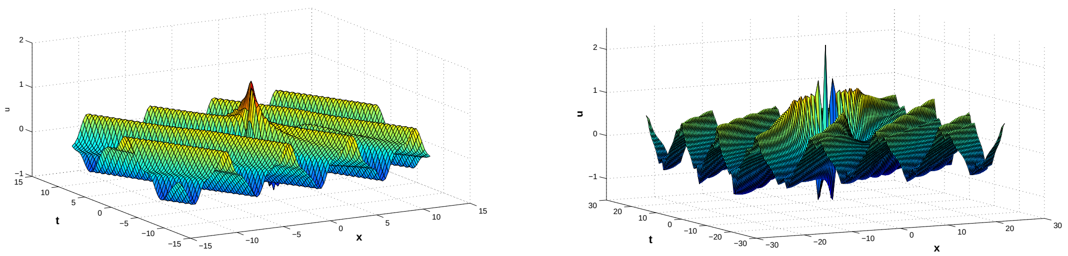

Figure 2: The rogue *cn*-periodic wave of the mKdV for $k = 0.5$ (left) and $k = 0.99$ (right).

In the soliton limit (1.8), $M_{\rm dn}(k) \to 2$ as $k \to 1$ in agreement with the recent results in [25, 26]. Indeed, the rogue *dn*-periodic wave degenerates as $k \to 1$ to the two-soliton solutions constructed of two nearly identical solitons. Such solutions are constructed by the one-fold Darboux transformation from the one-soliton solutions, when the eigenfunction of the AKNS spectral problem is nondecaying (exponentially growing) [23]. Therefore, the magnification factor is $M_{\rm dn}(1) = 2$ due to the weak interaction between two nearly identical solitons. On the other hand, $M_{\rm cn}(1) = 3$ because the rogue *cn*-periodic wave is built from the two-fold Darboux transformation, hence it degenerates as $k \to 1$ to the three-soliton solutions constructed of three nearly identical solitons [26].

The paper is organized as follows. Section 2 gives details of the periodic eigenfunctions of the AKNS spectral problem associated with the *dn* and *cn* Jacobian elliptic functions. Section 3 presents the general N-fold Darboux transformation for the mKdV equation and the explicit formulas for the one-fold and two-fold Darboux transformations. The non-periodic functions of the AKNS spectral problem and the rogue periodic waves of the mKdV are constructed in Section 4. Appendix A gives a proof of the N-fold Darboux transformations in the explicit form.

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2 Periodic eigenfunctions of the AKNS spectral problem

The mKdV equation (1.3) is obtained as a compatibility condition of the following Lax pair of two linear equations for the vector $\varphi = (\varphi_1, \varphi_2)^t$:

$$\varphi_x = U(\lambda, u)\varphi, \quad U(\lambda, u) = \left(\begin{array}{cc} \lambda & u \\ -u & -\lambda \end{array} \right), \tag{2.1}$$

and

$$\varphi_t = V(\lambda, u)\varphi, \quad V(\lambda, u) = \left(\begin{array}{cc} -4\lambda^3 - 2\lambda u^2 & -4\lambda^2 u - 2\lambda u_x - 2u^3 - u_{xx} \\ 4\lambda^2 u - 2\lambda u_x + 2u^3 + u_{xx} & 4\lambda^3 + 2\lambda u^2 \end{array} \right). \tag{2.2}$$

The first linear equation (2.1) is referred to as the AKNS spectral problem as it defines the spectral parameter λ for a given potential $u(x,t)$ at a frozen time t, e.g. at $t = 0$. By using the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we can rewrite $U(\lambda, u)$ and $V(\lambda, u)$ in (2.1) and (2.2) in the form

$$U(\lambda, u) = \lambda\sigma_3 + u\sigma_3\sigma_1, \tag{2.3}$$

$$V(\lambda, u) = -(4\lambda^3 + 2\lambda u^2)\sigma_3 - 4\lambda^2 u\sigma_3\sigma_1 - 2\lambda u_x\sigma_1 - (2u^3 + u_{xx})\sigma_3\sigma_1. \tag{2.4}$$

If u is either dn or cn Jacobian elliptic functions (1.4) and (1.5), the potentials are L-periodic in x with the period $L = 2K(k)$ for dn-functions and $L = 4K(k)$ for cn-functions, where $K(k)$ is the complete elliptic integral. If the AKNS spectral problem (2.1) is considered in the space of L-periodic functions, then the admissible set for the spectral parameter λ is discrete as the AKNS spectral problem has a purely point spectrum.

In the case of periodic or quasi-periodic potentials u, the algebraic technique based on the nonlinearization of the Lax pair [7] (see also applications in [8, 9, 10, 16]) can be used to obtain explicit solutions for the eigenfunctions of the AKNS spectral problem related to the particular eigenvalues λ with $\mathrm{Re}(\lambda) > 0$. Below we simplify the general method in order to obtain particular solutions of the AKNS spectral problem for the periodic waves in the focusing mKdV equation (1.3). The following two propositions represent the explicit expressions for eigenvalues and periodic eigenfunctions of system (2.1) and (2.2) related to the travelling periodic wave solution of the mKdV.

Proposition 1. *Let u be a travelling wave solution of the mKdV equation (1.3) satisfying*

$$\frac{d^2u}{dx^2} + 2u^3 = cu, \quad \left(\frac{du}{dx}\right)^2 + u^4 = cu^2 + d, \tag{2.5}$$

where c and d are real constants parameterized by

$$c = 4\lambda_1^2 + 2E_0, \quad d = -E_0^2 \tag{2.6}$$

with possibly complex λ_1 and E_0. Then, there exists a solution $\varphi = (\varphi_1, \varphi_2)^t$ of the AKNS spectral problem (2.1) with $\lambda = \lambda_1$ such that

$$\varphi_1^2 + \varphi_2^2 = u, \quad \varphi_1^2 - \varphi_2^2 = \frac{1}{2\lambda_1}\frac{du}{dx}, \quad 4\lambda_1\varphi_1\varphi_2 = E_0 - u^2. \tag{2.7}$$

In particular, if u is periodic in x, then φ is periodic in x.

Proof. Following the pioneer idea in [7], we set $u = \varphi_1^2 + \varphi_2^2$ and consider a nonlinearization of the AKNS spectral problem (2.1) given by the Hamiltonian system

$$\frac{d\varphi_1}{dx} = \frac{\partial H}{\partial \varphi_2}, \quad \frac{d\varphi_2}{dx} = -\frac{\partial H}{\partial \varphi_1}, \tag{2.8}$$

which is related to the Hamiltonian function

$$H(\varphi_1, \varphi_2) = \frac{1}{4}(\varphi_1^2 + \varphi_2^2)^2 + \lambda_1\varphi_1\varphi_2 = \frac{1}{4}E_0, \tag{2.9}$$

where E_0 is constant in x. It follows from (2.9) that $4\lambda_1\varphi_1\varphi_2 = E_0 - u^2$. Also note that

$$\frac{du}{dx} = 2\left(\varphi_1\frac{d\varphi_1}{dx} + \varphi_2\frac{d\varphi_2}{dx}\right) = 2\lambda_1(\varphi_1^2 - \varphi_2^2),$$

so that all three equations in (2.7) are satisfied by the construction.

Let us introduce

$$Q(\lambda) = \begin{pmatrix} \lambda & \varphi_1^2 + \varphi_2^2 \\ -\varphi_1^2 - \varphi_2^2 & -\lambda \end{pmatrix}, \quad W(\lambda) = \begin{pmatrix} W_{11}(\lambda) & W_{12}(\lambda) \\ W_{12}(-\lambda) & -W_{11}(-\lambda) \end{pmatrix},$$

with

$$\begin{aligned} W_{11}(\lambda) &= 1 - \frac{\varphi_1\varphi_2}{\lambda - \lambda_1} + \frac{\varphi_1\varphi_2}{\lambda + \lambda_1}, \\ W_{12}(\lambda) &= \frac{\varphi_1^2}{\lambda - \lambda_1} + \frac{\varphi_2^2}{\lambda + \lambda_1}. \end{aligned}$$

Then, one can check directly that the Lax equation

$$\frac{d}{dx} W(\lambda) = Q(\lambda)W(\lambda) - W(\lambda)Q(\lambda),$$

is satisfied for every $\lambda \in \mathbb{C}$ if and only if (φ_1, φ_2) satisfies (2.8). In particular, the $(1,2)$-entry in the above relations yields the equation

$$\frac{d}{dx} W_{12}(\lambda) = 2\lambda W_{12}(\lambda) - 2(\varphi_1^2 + \varphi_2^2) W_{11}(\lambda) \tag{2.10}$$

and the representation

$$W_{12}(\lambda) = \frac{1}{\lambda^2 - \lambda_1^2}\left[\lambda(\varphi_1^2 + \varphi_2^2) + \lambda_1(\varphi_1^2 - \varphi_2^2)\right] =: \frac{(\lambda - \mu)(\varphi_1^2 + \varphi_2^2)}{a(\lambda)}, \tag{2.11}$$

with

$$a(\lambda) := \lambda^2 - \lambda_1^2 \quad \text{and} \quad \mu := -\lambda_1 \frac{\varphi_1^2 - \varphi_2^2}{\varphi_1^2 + \varphi_2^2} = -\frac{1}{2u}\frac{du}{dx}. \tag{2.12}$$

In addition, we note that

$$\det[W(\lambda)] = -[W_{11}(\lambda)]^2 - W_{12}(\lambda)W_{21}(\lambda) = -\frac{b(\lambda)}{a(\lambda)}$$

with

$$b(\lambda) := \lambda^2 - \lambda_1^2 - 4\lambda_1\varphi_1\varphi_2 - (\varphi_1^2 + \varphi_2^2)^2 = \lambda^2 - \lambda_1^2 - E_0.$$

Since $W_{12}(\lambda)$ has a simple zero at $\lambda = \mu$, then

$$[W_{11}(\mu)]^2 = \frac{b(\mu)}{a(\mu)}. \tag{2.13}$$

By substituting (2.11), (2.13), and

$$\frac{d}{dx} W_{12}(\mu) = -\frac{(\varphi_1^2 + \varphi_2^2)}{a(\mu)}\frac{d\mu}{dx}$$

to equation (2.10) and squaring it, we obtain the closed equation on μ:

$$\frac{1}{4}\left(\frac{d\mu}{dx}\right)^2 = a(\mu)b(\mu) = (\mu^2 - \lambda_1^2)(\mu^2 - \lambda_1^2 - E_0). \tag{2.14}$$

Substituting the representation (2.12) yields

$$u^2\left(\frac{d^2u}{dx^2}\right)^2 - 2u\left(\frac{du}{dx}\right)^2\left[\frac{d^2u}{dx^2} - 2(E_0+2\lambda_1^2)u\right] = 16\lambda_1^2(E_0+\lambda_1^2)u^4. \tag{2.15}$$

Let u be a solution of the differential equations (2.5) with parameters c and d. Substituting (2.5) to (2.15) yields the relations (2.6) between (c,d) and (λ_1,E_0). Hence, the constraint (2.15) is fulfilled if u satisfies (2.5) with parameters (c,d) satisfying (2.6). □

Proposition 2. *Let u, $\varphi=(\varphi_1,\varphi_2)^t$, and λ_1 be the same as in Proposition 1. Then $\varphi(x-ct)$ satisfies the linear system (2.2) with $\lambda=\lambda_1$ and $u(x-ct)$.*

Proof. By using (2.5), we rewrite the first equation of system (2.2) with $\lambda=\lambda_1$ as

$$\partial_t\varphi_1 = -(4\lambda_1^3+2\lambda_1u^2)\varphi_1 - (4\lambda_1^2u+2\lambda_1u_x+cu)\varphi_2. \tag{2.16}$$

By using (2.7), we note that

$$(4\lambda_1^2+2u^2)\varphi_1 + (4\lambda_1u+2u_x)\varphi_2 = (4\lambda_1^2+2u^2+8\lambda_1\varphi_1\varphi_2)\varphi_1 = (4\lambda_1^2+2E_0)\varphi_1.$$

By using (2.6) and the first equation in system (2.1), equation (2.16) becomes

$$\partial_t\varphi_1 = -\lambda_1c\varphi_1 - cu\varphi_2 = -c\partial_x\varphi_1,$$

hence $\varphi_1(x-ct)$ is a solution of system (2.1) and (2.2) with $\lambda=\lambda_1$ and $u(x-ct)$. Similar computations hold for φ_2 by symmetry from the second equations in systems (2.1) and (2.2). □

For the dn Jacobian elliptic functions (1.4), we have $c=2-k^2$ and $d=k^2-1\le 0$. Since $u(x)>0$ for every $x\in\mathbb{R}$, the periodic eigenfunction $\varphi=(\varphi_1,\varphi_2)^t$ in Proposition 1 is real with parameters $E_0=\pm\sqrt{1-k^2}$ and

$$\lambda_1^2 = \frac{1}{4}\left[2-k^2\mp 2\sqrt{1-k^2}\right]. \tag{2.17}$$

Taking the positive square root of (2.17), we obtain two particular real points

$$\lambda_\pm(k) := \frac{1}{2}\sqrt{2-k^2\pm 2\sqrt{1-k^2}}, \tag{2.18}$$

such that $0<\lambda_-(k)<\lambda_+(k)<1$ for every $k\in(0,1)$. As $k\to 0$, we have $\lambda_-(k)\to 0$ and $\lambda_+(k)\to 1$, whereas as $k\to 1$, we have $\lambda_-(k),\lambda_+(k)\to 1/2$.

For the cn Jacobian elliptic functions (1.5), we have $c=2k^2-1$ and $d=k^2(1-k^2)\ge 0$. Since $u(x)$ is sign-indefinite, the periodic eigenfunction $\varphi=(\varphi_1,\varphi_2)^t$ in Proposition 1 is complex-valued with parameters $E_0=\pm ik\sqrt{1-k^2}$ and

$$\lambda_1^2 = \frac{1}{4}\left[2k^2-1\mp 2ik\sqrt{1-k^2}\right]. \tag{2.19}$$

Defining the square root of (2.19) in the first quadrant of the complex plane, we obtain

$$\lambda_I(k) := \frac{1}{2}\sqrt{2k^2-1+2ik\sqrt{1-k^2}}. \tag{2.20}$$

As $k\to 0$, we obtain the expansion $\lambda_I(k)=\frac{1}{2}\left(i+k+\mathcal{O}(k^2)\right)$, whereas as $k\to 1$, we obtain the expansion $\lambda_I(k)=\frac{1}{2}\left(1+i\sqrt{1-k^2}+\mathcal{O}(1-k^2)\right)$.

Figure 3 shows the spectral plane of λ with the schematic representation of the Floquet–Bloch spectrum for the *dn*-periodic wave with $k=0.75$ (left) and the *cn*-periodic wave with $k=0.75$ (right). The branch points $\lambda_\pm(k)$ and $\lambda_I(k)$ obtained in (2.18) and (2.20) are marked explicitly on the figure.

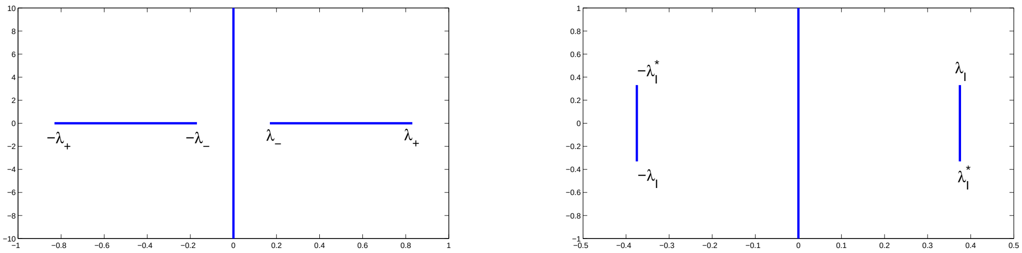

Figure 3: The spectral plane of λ for the *dn*-periodic wave with $k = 0.75$ (left) and the *cn*-periodic wave with $k = 0.75$ (right). The branch points (2.18) and (2.20) are marked explicitly.

3 One-fold, two-fold, and N-fold Darboux transformations

Here we give the explicit formulas for the one-fold and two-fold Darboux transformations for the focusing mKdV equation (1.3), as well as the general formula for the N-fold Darboux transformation. Although the formal derivation of the N-fold Darboux transformation can be found in several sources, e.g. in book [20] or original papers [17, 27], we find it useful to derive the explicit transformation formulas by using purely algebraic calculations.

By definition, we say that $T(\lambda)$ is a Darboux transformation if

$$\widetilde{\varphi} = T(\lambda)\varphi, \tag{3.1}$$

where φ satisfies (2.1)–(2.2) for a particular potential u and $\widetilde{\varphi}$ satisfies (2.1)–(2.2) for a new potential $\widetilde{u}$, which is related to u. The transformation formulas between φ and $\widetilde{\varphi}$ follow from the Darboux equations

$$\partial_x T(\lambda) + T(\lambda)U(\lambda, u) = U(\lambda, \widetilde{u})T(\lambda) \tag{3.2}$$

and

$$\partial_t T(\lambda) + T(\lambda)V(\lambda, u) = V(\lambda, \widetilde{u})T(\lambda). \tag{3.3}$$

In many derivations, e.g. in [17, 20, 27], the N-fold Darboux transformation is deduced formally from a linear system of equations imposed on the coefficients of the polynomial representation of $T(\lambda)$ without checking all the constraints arising from the Darboux equations (3.2) and (3.3). Instead of such formal computations, we would like to give a rigorous derivation of the N-fold Darboux transformation in the explicit form and show how the Darbous equations (3.2) and (3.3) are satisfied. Our derivation relies on a particular implementation of the dressing method [31, 32] which was recently reviewed in the context of the cubic NLS equation in [12].

The general N-fold Darboux transformation is given by the following theorem.

Theorem 1. *Let u be a smooth solution of the mKdV equation (1.3). Let $\varphi^{(k)} = (p_k, q_k)^t$, $1 \leq k \leq N$ be a particular smooth nonzero solution of system (2.1) and (2.2) with fixed $\lambda = \lambda_k \in \mathbb{C}\backslash\{0\}$ and potential u. Assume that $\lambda_k \neq \pm\lambda_j$ for every $k \neq j$. Let $\{\widetilde{\varphi}^{(k)}\}_{1\leq k\leq N}$ be a solution of the linear algebraic system*

$$\sigma_3\sigma_1\varphi^{(j)} = \sum_{k=1}^{N} \frac{\langle \varphi^{(j)}, \varphi^{(k)} \rangle}{\lambda_j + \lambda_k} \widetilde{\varphi}^{(k)}, \quad 1 \leq j \leq N, \tag{3.4}$$

where $\langle \varphi^{(j)}, \varphi^{(k)} \rangle := p_j p_k + q_j q_k$ *is the inner vector product. Assume that the linear system (3.4) has a unique solution. Then,* $\widetilde{\varphi}^{(k)} = (\widetilde{p}_k, \widetilde{q}_k)^t$, $1 \le k \le N$ *is a particular solution of system (2.1) and (2.2) with* $\lambda = \lambda_k$ *and the new potential* $\widetilde{u}$ *given by*

$$\widetilde{u} = u + 2\sum_{j=1}^{N} \widetilde{p}_j p_j = u - 2\sum_{j=1}^{N} \widetilde{q}_j q_j. \tag{3.5}$$

Consequently, $\widetilde{u}$ *is a new solution of the mKdV equation (1.3).*

The proof of Theorem 1 is developed in Appendix A. The following two propositions represent the one-fold and two-fold Darboux transformation formulas deduced from Theorem 1 for $N = 1$ and $N = 2$ respectively.

Proposition 3. *Let* u *be a smooth solution of the mKdV equation (1.3). Let* $\varphi = (p, q)^t$ *be a particular smooth nonzero solution of system (2.1) and (2.2) with fixed* $\lambda = \lambda_1 \in \mathbb{C}\backslash\{0\}$*. Then,*

$$\widetilde{u} = u + \frac{4\lambda_1 pq}{p^2 + q^2} \tag{3.6}$$

is a new solution of the mKdV equation (1.3).

Proof. Solving the linear system (3.4) for $\widetilde{\varphi} = (\widetilde{p}, \widetilde{q})^t$ yields

$$\widetilde{p} = \frac{2\lambda_1 q}{p^2 + q^2}, \quad \widetilde{q} = \frac{-2\lambda_1 p}{p^2 + q^2}. \tag{3.7}$$

Substituting (3.7) into (3.5) for $N = 1$ results in the transformation formula (3.6). $\square$

Proposition 4. *Let* u *be a smooth solution of the mKdV equation (1.3). Let* $\varphi^{(k)} = (p_k, q_k)^t$ *be a particular smooth nonzero solution of system (2.1) and (2.2) with fixed* $\lambda = \lambda_k \in \mathbb{C}\backslash\{0\}$ *for* $k = 1, 2$ *such that* $\lambda_1 \neq \pm\lambda_2$*. Then,*

$$\tilde{u} = u + \frac{4(\lambda_1^2 - \lambda_2^2)\left[\lambda_1 p_1 q_1 (p_2^2 + q_2^2) - \lambda_2 p_2 q_2 (p_1^2 + q_1^2)\right]}{(\lambda_1^2 + \lambda_2^2)(p_1^2 + q_1^2)(p_2^2 + q_2^2) - 2\lambda_1\lambda_2\left[4p_1 q_1 p_2 q_2 + (p_1^2 - q_1^2)(p_2^2 - q_2^2)\right]} \tag{3.8}$$

is a new solution of the mKdV equation (1.3).

Proof. The linear system (3.4) is generated by the matrix A with the entries

$$A_{jk} = \frac{\langle \varphi^{(j)}, \varphi^{(k)} \rangle}{\lambda_j + \lambda_k}, \quad 1 \le j, k \le N. \tag{3.9}$$

For $N = 2$, we compute the determinant of this matrix as

$$\begin{aligned} \det(A) &= \frac{1}{4\lambda_1\lambda_2(\lambda_1+\lambda_2)^2}\left[(\lambda_1+\lambda_2)^2(p_1^2+q_1^2)(p_2^2+q_2^2) - 4\lambda_1\lambda_2(p_1p_2+q_1q_2)^2\right] \\ &= \frac{1}{4\lambda_1\lambda_2(\lambda_1+\lambda_2)^2}\left[(\lambda_1^2+\lambda_2^2)(p_1^2+q_1^2)(p_2^2+q_2^2) - 2\lambda_1\lambda_2\left(4p_1p_2q_1q_2 + (p_1^2-q_1^2)(p_2^2-q_2^2)\right)\right]. \end{aligned}$$

Solving the linear system (3.4) with Cramer's rule yields the components

$$\widetilde{p}_1 = \frac{(\lambda_1+\lambda_2)q_1(p_2^2+q_2^2) - 2\lambda_2 q_2(p_1p_2+q_1q_2)}{2\lambda_2(\lambda_1+\lambda_2)\det(A)}$$

and

$$\widetilde{p}_2 = \frac{(\lambda_1+\lambda_2)q_2(p_1^2+q_1^2) - 2\lambda_1 q_1(p_1p_2+q_1q_2)}{2\lambda_1(\lambda_1+\lambda_2)\det(A)}.$$

Substituting these formulas to the representation (3.5) with $N = 2$ and reordering the similar terms result in the transformation formula (3.8). $\square$

4 Construction of rogue periodic waves

Here we apply the Darboux transformations (3.6) and (3.8) to the Jacobian elliptic functions *dn* and *cn* in (1.4) and (1.5) in order to obtain the rogue periodic waves.

Let u be the *dn* periodic wave (1.4), whereas $\varphi = (\varphi_1, \varphi_2)^t$ be the periodic eigenfunction of the linear system (2.1) and (2.2) with $\lambda = \lambda_1$ defined by Propositions 1 and 2. Since the connection formulas (2.7) are satisfied for every $t \in \mathbb{R}$, substituting $p = \varphi_1$ and $q = \varphi_2$ into the one-fold Darboux transformation (3.6) yields another solution of the mKdV equation in the form

$$\tilde{u} = u + \frac{4\lambda_1\varphi_1\varphi_2}{\varphi_1^2 + \varphi_2^2} = \frac{E_0}{u},$$

where $E_0 = \pm\sqrt{1-k^2}$. However, since

$$\mathrm{dn}(x + K(k); k) = \frac{\sqrt{1-k^2}}{\mathrm{dn}(x;k)},$$

the new solution $\tilde{u}$ to the mKdV equation (1.3) is obtained trivially by the spatial translation of the *dn* periodic wave on the half-period $\frac{1}{2}L = K(k)$.

In order to derive the rogue *dn*-periodic solution to the mKdV equation (1.3), we shall construct the second linearly independent solution to the AKNS spectral problem (2.1) with $\lambda = \lambda_1$ and extend it to satisfy the linear system (2.2). The second solution is no longer periodic in variables (x, t). The following result represents the corresponding solution.

Proposition 5. *Let u, λ_1, E_0, and $\varphi = (\varphi_1, \varphi_2)^t$ be the same as in Proposition 1. Assume that $u(x)^2 - E_0 \neq 0$ for every x. The second linearly independent solution of the AKNS spectral problem (2.1) with $\lambda = \lambda_1$ is given by $\psi = (\psi_1, \psi_2)^t$, where*

$$\psi_1 = \frac{\theta - 1}{\varphi_2}, \quad \psi_2 = \frac{\theta + 1}{\varphi_1}, \tag{4.1}$$

and

$$\theta(x) = -4\lambda_1(u(x)^2 - E_0)\int_0^x \frac{u(y)^2}{(u(y)^2 - E_0)^2} dy. \tag{4.2}$$

In particular, if u is periodic in x, then θ grows linearly in x as $|x| \to \infty$, so that ψ_1 and ψ_2 are not periodic in x.

Proof. Since the AKNS spectral problem (2.1) is related to the traceless matrix, the Wronskian of the two linearly independent solutions $\varphi = (\varphi_1, \varphi_2)^t$ and $\psi = (\psi_1, \psi_2)^t$ is independent of x. Normalizing it by 2, we write the relation

$$\varphi_1\psi_2 - \varphi_2\psi_1 = 2,$$

from which the representation (4.1) follows with arbitrary θ. If $u^2 - E_0 \neq 0$ for every x, then $\varphi_1 \neq 0$ and $\varphi_2 \neq 0$ for every x. Substituting (4.1) to (2.1), we obtain the following scalar linear differential equation for θ:

$$\frac{d\theta}{dx} = u\theta\frac{\varphi_2^2 - \varphi_1^2}{\varphi_1\varphi_2} + u\frac{\varphi_1^2 + \varphi_2^2}{\varphi_1\varphi_2}.$$

By using relations (2.7), we rewrite it in the equivalent forms:

$$\frac{d\theta}{dx} = \theta \frac{2uu'}{u^2 - E_0} - \frac{4\lambda_1 u^2}{u^2 - E_0} \quad \Rightarrow \quad \frac{d}{dx}\left[\frac{\theta}{u^2 - E_0}\right] = -\frac{4\lambda_1 u^2}{(u^2 - E_0)^2}.$$

Integrating the last equation with the boundary condition $\theta(0) = 0$, we obtain (4.2). ☐

Proposition 6. *Let u, λ_1, E_0, $\varphi = (\varphi_1, \varphi_2)^t$, and $\psi = (\psi_1, \psi_2)^t$ be the same as in Proposition 5. Then, $\psi = (\psi_1, \psi_2)^t$ expressed by (4.1) satisfies the linear system (2.2) with $\lambda = \lambda_1$ and $u(x - ct)$ if θ is expressed by*

$$\theta(x,t) = -4\lambda_1 (u(x-ct)^2 - E_0) \left[\int_0^{x-ct} \frac{u(y)^2}{(u(y)^2 - E_0)^2} dy - t\right]. \tag{4.3}$$

Proof. By using (2.5), we rewrite the first equation of system (2.2) with $\lambda = \lambda_1$ as

$$\partial_t \psi_1 = -(4\lambda_1^3 + 2\lambda_1 u^2)\psi_1 - (4\lambda_1^2 u + 2\lambda_1 u_x + cu)\psi_2. \tag{4.4}$$

By using (2.7), (4.1), and expressing $\partial_t \varphi_2$ from the second equation of system (2.2), we obtain from (4.4):

$$\begin{aligned} \partial_t \theta &= \frac{(4\lambda_1^2 u - 2\lambda_1 u_x + cu)\varphi_1(\theta - 1)}{\varphi_2} - \frac{(4\lambda_1^2 u + 2\lambda_1 u_x + cu)\varphi_2(\theta + 1)}{\varphi_1} \\ &= -16\lambda_1^2 \varphi_1 \varphi_2 - \frac{cu}{\varphi_1 \varphi_2}\left[\theta(\varphi_2^2 - \varphi_1^2) + \varphi_1^2 + \varphi_2^2\right] \\ &= 4\lambda_1(u^2 - E_0) - c\partial_x \theta. \end{aligned}$$

Let us represent $\theta = -4\lambda_1(u^2 - E_0)\chi$ so that χ satisfies

$$\partial_t \chi = -c\partial_x \chi - 1.$$

Hence $\chi(x,t) = -t + f(x - ct)$, where f is obtained from (4.2) in the form

$$f(x) = \int_0^x \frac{u(y)^2}{(u(y)^2 - E_0)^2} dy$$

to yield the representation (4.3). Similar computations hold for ψ_2 by symmetry from the second equations in systems (2.1) and (2.2). ☐

Note that a more general solution for $\psi = (\psi_1, \psi_2)^t$ is defined arbitrary up to an addition to the first solution $\varphi = (\varphi_1, \varphi_2)^t$. However, this addition is equivalent to the arbitrary choice of the lower limit in the integral (4.3), which is then equivalent to the translation in time t. Thus, the second linearly independent solution in the form (4.1) and (4.3) is unique up to the translation in x and t.

4.1 The rogue *dn*-periodic wave

Let u be the *dn* periodic wave (1.4), whereas $\psi = (\psi_1, \psi_2)^t$ be the non-periodic solution to the linear system (2.1) and (2.2) with $\lambda = \lambda_1$ defined by Propositions 5 and 6. Recall that there exist two choices for λ_1 in (2.18). However, for the choice $\lambda_1 = \lambda_-(k)$, we have $E_0 = \sqrt{1-k^2}$ and $u(x)^2 - E_0 = 0$ for some values of x in $[-K(k), K(k)]$, therefore, the assumption of Proposition 5 is not satisfied. For the choice $\lambda_1 = \lambda_+(k)$, we have $E_0 = -\sqrt{1-k^2}$ and $u(x)^2 - E_0 > 0$ for every x, therefore, the assumption of Proposition 5 is satisfied. Substituting $p = \psi_1$ and $q = \psi_2$ given by (4.1) into the one-fold Darboux transformation (3.6) with $\lambda_1 = \lambda_+(k)$ and $E_0 = -\sqrt{1-k^2}$ yields another solution of the mKdV equation in the form

$$\tilde{u} = u + \frac{4\lambda_1\psi_1\psi_2}{\psi_1^2 + \psi_2^2} = u + \frac{4\lambda_1\varphi_1\varphi_2(\theta^2 - 1)}{(\varphi_1^2 + \varphi_2^2)(1+\theta^2) - 2(\varphi_1^2 - \varphi_2^2)\theta}.$$

By using relations (2.7) again, we finally write the new solution in the form

$$u_{\rm dn-rogue} = u_{\rm dn} + \frac{(1-\theta_{\rm dn}^2)(u_{\rm dn}^2 + \sqrt{1-k^2})}{(1+\theta_{\rm dn}^2)u_{\rm dn} - \lambda_1^{-1}\theta_{\rm dn}u_{\rm dn}'} \tag{4.5}$$

where

$$\theta_{\rm dn}(x,t) = -4\lambda_1(u_{\rm dn}(x-ct)^2 + \sqrt{1-k^2})\left[\int_0^{x-ct}\frac{u_{\rm dn}(y)^2}{(u_{\rm dn}(y)^2 + \sqrt{1-k^2})^2}dy - t\right]. \tag{4.6}$$

We refer to the exact solution (4.5)–(4.6) as the rogue *dn* periodic wave of the mKdV equation.

If $k = 0$, then $u_{\rm dn}(x,t) = 1$, $\lambda_1 = 1$, $c = 2$, $\theta_{\rm dn}(x,t) = -2(x-6t)$, and

$$k = 0: \quad u_{\rm dn-rogue}(x,t) = -1 + \frac{4}{1+4(x-6t)^2}.$$

Although this expression is an analogue of the rogue wave of the NLS on the constant wave background [2, 11], it corresponds to the algebraically decaying soliton of the mKdV [24]. Hence, it should not be really referred to as the "rogue wave" of the mKdV.

If $k = 1$, then $u_{\rm dn}(x,t) = {\rm sech}(x-t)$, $\lambda_1 = \frac{1}{2}$, $c = 1$,

$$\theta_{\rm dn}(x,t) = -(x-3t){\rm sech}^2(x-t) - \tanh(x-t),$$

and

$$k = 1: \quad u_{\rm dn-rogue}(x,t) = 2{\rm sech}(x-t)\frac{1-(x-3t)\tanh(x-t)}{1+(x-3t)^2{\rm sech}^2(x-t)}.$$

in agreement with the two-soliton solutions of the mKdV for two nearly identical solitons [25, 26].

For every $k \in [0,1)$, there exists a particular line $x = c_* t$ with $c_* > c$ such that $\theta_{\rm dn}(x,t)$ given by (4.6) remains bounded as $|x| + |t| \to \infty$. This value of c_* gives the speed of the algebraically decaying soliton propagating on the *dn*-periodic wave background. For instance, if $k = 0$, then $c_* = 6 > 2 = c$. In the limit $k \to 1$, $c_* \to 3 > 1 = c$, but $\theta_{\rm dn}(x,t)$ is bounded everywhere in the limit of two-soliton solutions.

Except for the line $x = c_* t$, for every $k \in [0,1)$, the function $\theta_{\rm dn}(x,t)$ given by (4.6) grows linearly in x and t as $|x| + |t| \to \infty$, so that the representation (4.5) becomes asymptotically equivalent to

$$u_{\rm dn-rogue}(x,t) \sim -\frac{\sqrt{1-k^2}}{{\rm dn}(x-ct;k)} = -{\rm dn}(x-ct+K(k);k) = -u_{\rm dn}(x-ct+K(k)).$$

The maximal value of $u_{\rm dn-rogue}(x,t)$ as $|x|+|t| \to \infty$ except for the line $x = c_* t$ coincides with the maximal value of $u_{\rm dn}(x,t) = {\rm dn}(x-ct;k)$.

For $t = 0$, $u_{\rm dn}(x,0)$ is even in x, $\theta_{\rm dn}(x,0)$ is odd in x, hence $u_{\rm dn-rogue}(x,0)$ is even in x. The maximal value of $u_{\rm dn}(x,0)$ occurs at $u_{\rm dn}(0,0) = 1$. Since $u_{\rm dn-rogue}(x,0)$ is even in x, then $x = 0$

is an extremal point of $u_{\rm dn-rogue}(x,0)$. Moreover, $\partial_x^2 u_{\rm dn-rogue}(0,0) < 0$, which follows from the expansions $u_{\rm dn}(x,0) = 1 - \frac{1}{2}k^2x^2 + \mathcal{O}(x^4)$, $\theta_{\rm dn}(x,0) = -4\lambda_1(1+\sqrt{1-k^2})^{-1}x + \mathcal{O}(x^3)$, and

$$u_{\rm dn-rogue}(x,0) = 2 + \sqrt{1-k^2} - \left[8 - 3k^2 + 8\sqrt{1-k^2} - \frac{1}{2}k^2\sqrt{1-k^2}\right] x^2 + \mathcal{O}(x^4).$$

Hence $x = 0$ is the point of maximum of $u_{\rm dn-rogue}(x,0)$. Defining the magnification number as

$$M_{\rm dn}(k) = \frac{u_{\rm dn-rogue}(0,0)}{\max\limits_{x\in[-K(k),K(k)]} u_{\rm dn}(x,0)} = 2 + \sqrt{1-k^2},$$

we arrive to the expression in (1.9). The value $M_{\rm dn}(k)$ corresponds to the amplitude of the algebraically decaying soliton propagating on the background of the *dn*-periodic wave of unit amplitude.

4.2 The rogue *cn*-periodic wave

Let u be the *cn* periodic wave (1.5), whereas $\varphi = (\varphi_1,\varphi_2)^t$ be the periodic solution to the linear system (2.1) and (2.2) with $\lambda = \lambda_1$ defined by Propositions 1 and 2. Without loss of generality, we choose $\lambda_1 = \lambda_I(k)$, where $\lambda_I(k)$ is given by (2.20), so that $E_0 = -ik\sqrt{1-k^2}$. Since the periodic solution φ is complex, the one-fold Darboux transformation (3.6) produces a complex-valued solution to the mKdV, hence we should use the two-fold Darboux transformation (3.8).

By virtue of relations (2.7), substituting $(p_1,q_1) = (\varphi_1,\varphi_2)$ with $\lambda_1 = \lambda_I$ and $(p_2,q_2) = (\overline{\varphi}_1,\overline{\varphi}_2)$ with $\lambda_2 = \overline{\lambda}_I$ to the two-fold Darboux transformation (3.8) yields another solution of the mKdV equation in the form

$$\tilde{u} = u + \frac{4k^2(1-k^2)u}{(2k^2-1)u^2 - u^4 - k^2(1-k^2) - (u')^2} = -u,$$

where the first-order invariant in (2.5) is used in the second identity with $c = 2k^2 - 1$ and $d = k^2(1-k^2)$. Thus, the new solution $\tilde{u}$ in the two-fold transformation (3.8) is trivially related to the previous solution u if the functions (p_1,q_1) and (p_2,q_2) are periodic.

Let us now consider the non-periodic solution $\psi = (\psi_1,\psi_2)^t$ to the linear system (2.1) and (2.2) with $\lambda = \lambda_I$. The assumption of Proposition 5 is satisfied because $E_0 = -ik\sqrt{1-k^2} \neq 0$ for $k \in (0,1)$ and $u(x)^2 - E_0 \neq 0$ for every x. Therefore, the non-periodic solution ψ in Propositions 5 and 6 is well-defined. Substituting $(p_1,q_1) = (\psi_1,\psi_2)$ with $\lambda_1 = \lambda_I$ and $(p_2,q_2) = (\overline{\psi}_1,\overline{\psi}_2)$ with $\lambda_2 = \overline{\lambda}_I$ into the two-fold Darboux transformation (3.8) yields another solution of the mKdV in the form

$$\tilde{u} = u + \frac{4(\lambda_I^2 - \overline{\lambda}_I^2)\left[\lambda_I\psi_1\psi_2(\overline{\psi}_1^2 + \overline{\psi}_2^2) - \overline{\lambda}_I\overline{\psi}_1\overline{\psi}_2(\psi_1^2+\psi_2^2)\right]}{(\lambda_I^2 + \overline{\lambda}_I^2)|\psi_1^2+\psi_2^2|^2 - 2|\lambda_I|^2\left[4|\psi_1|^2|\psi_2|^2 + |\psi_1^2 - \psi_2^2|^2\right]} = u + \frac{F_1}{F_2},$$

where

$$\begin{aligned} F_1 &= 8\mathrm{Im}(\lambda_I^2)\mathrm{Im}\left[\lambda_I\varphi_1\varphi_2(1-\theta^2)[(1+\overline{\theta}^2)(\overline{\varphi}_1^2+\overline{\varphi}_2^2) - 2\overline{\theta}(\overline{\varphi}_1^2 - \overline{\varphi}_2^2)]\right], \\ F_2 &= \mathrm{Re}(\lambda_I^2)|(1+\theta^2)(\varphi_1^2+\varphi_2^2) - 2\theta(\varphi_1^2-\varphi_2^2)|^2 \\ &\quad -|\lambda_I|^2\left(4|1-\theta^2|^2|\varphi_1|^2|\varphi_2|^2 + |(1+\theta^2)(\varphi_1^2-\varphi_2^2) - 2\theta(\varphi_1^2+\varphi_2^2)|^2\right). \end{aligned}$$

By using relations (2.7) and (2.20), we finally write the new solution in the form

$$u_{\rm cn-rogue} = u_{\rm cn} + \frac{G_1}{G_2}, \tag{4.7}$$

where

$$
\begin{aligned}
G_1 &= 4k\sqrt{1-k^2}\,\mathrm{Im}\left[(u_{\rm cn}^2 + ik\sqrt{1-k^2})(1-\theta_{\rm cn}^2)[(1+\overline{\theta}_{\rm cn}^2)u_{\rm cn} - \overline{\lambda}_I^{-1}\overline{\theta}_{\rm cn}u_{\rm cn}']\right], \\
G_2 &= (1-2k^2)|(1+\theta_{\rm cn}^2)u_{\rm cn} - \lambda_I^{-1}\theta_{\rm cn}u_{\rm cn}'|^2 \\
&\quad + |1-\theta_{\rm cn}^2|^2\left[u_{\rm cn}^4 + k^2(1-k^2)\right] + |(1+\theta_{\rm cn}^2)(2\lambda_I)^{-1}u_{\rm cn}' - 2\theta_{\rm cn}u_{\rm cn}|^2,
\end{aligned}
$$

and

$$
\theta_{\rm cn}(x,t) = -4\lambda_I(u_{\rm cn}(x-ct)^2 + ik\sqrt{1-k^2})\left[\int_0^{x-ct}\frac{u_{\rm cn}(y)^2}{(u_{\rm cn}(y)^2 + ik\sqrt{1-k^2})^2}dy - t\right]. \tag{4.8}
$$

We refer to the exact solution (4.7)–(4.8) as the rogue *cn* periodic wave of the mKdV equation.

As $k \to 0$, then $u_{\rm cn}(x,t) \to 0$, $\lambda_I \to \frac{i}{2}$, $\theta_{\rm cn}(x,t) \to 0$, and $u_{\rm cn-rogue}(x,t) \to 0$. Although the limit is zero, one can derive asymptotic expansions at the order of $\mathcal{O}(k)$ which recovers the rogue wave of the NLS equation (1.1), according to the asymptotic transformation of the focusing mKdV to the focusing NLS in the small-amplitude limit [18]. The rouge *cn*-periodic wave generalizes the rogue wave (1.2) on the constant wave background.

As $k \to 1$, then $u_{\rm cn}(x,t) \to \mathrm{sech}(x-t)$, $\lambda_1 \to \frac{1}{2}$, $c \to 1$, and it may first seem that the second term in (4.7) vanishes. However, $G_1 = \mathcal{O}(1-k^2)$ and $G_2 = \mathcal{O}(1-k^2)$, hence a non-trivial limit exists to yield a three-soliton solution to the mKdV with three nearly identical solitons [26].

If $k \in (0,1)$, both the real and imaginary parts of the integral in (4.8) grow linearly in x and t as $|x|+|t| \to \infty$. As a result, if $k \in (0,1)$, at least the imaginary part of $\theta_{\rm cn}(x,t)$ grows linearly in x and t as $|x|+|t| \to \infty$ everywhere on the (x,t) plane, so that the representation (4.7) becomes asymptotically equivalent to

$$
\begin{aligned}
u_{\rm cn-rogue}(x,t) &\sim u_{\rm cn}(x,t) + \frac{4k^2(1-k^2)u_{\rm cn}(x,t)}{(2k^2-1)u_{\rm cn}(x,t)^2 - (\partial_x u_{\rm cn}(x,t))^2 - u_{\rm cn}(x,t)^4 - k^2(1-k^2)} \\
&= -u_{\rm cn}(x,t),
\end{aligned}
$$

where the first-order invariant in (2.5) is used for the last identity with $c = 2k^2-1$ and $d = k^2(1-k^2)$. The maximal value of $u_{\rm cn-rogue}(x,t)$ as $|x|+|t| \to \infty$ coincides with the maximal value of $u_{\rm cn}(x,t) = k\mathrm{cn}(x-ct;k)$.

For $t=0$, $u_{\rm cn}(x,0)$ is even in x, $\theta_{\rm cn}(x,0)$ is odd in x, hence $u_{\rm cn-rogue}(x,0)$ is even in x. The maximal value of $u_{\rm cn}(x,0)$ occurs at $u_{\rm cn}(0,0)=k$. Since $u_{\rm cn-rogue}(x,0)$ is even in x, then $x=0$ is an extremal point of $u_{\rm cn-rogue}(x,0)$. Moreover, $\partial_x^2 u_{\rm cn-rogue}(0,0) < 0$, which follows from the expansions $u_{\rm cn}(x,0) = k - \frac{1}{2}kx^2 + \mathcal{O}(x^4)$, $\theta_{\rm cn}(x,0) = -4\lambda_I(k^2 - ik\sqrt{1-k^2})x + \mathcal{O}(x^3)$, and

$$
u_{\rm cn-rogue}(x,0) = 3k - \left[\frac{3}{2}k + 16k^3\right]x^2 + \mathcal{O}(x^4).
$$

Hence $x=0$ is the point of maximum of $u_{\rm cn-rogue}(x,0)$. Defining the magnification number as

$$
M_{\rm cn}(k) = \frac{u_{\rm cn-rogue}(0,0)}{\max\limits_{x\in[-2K(k),2K(k)]}|u_{\rm cn}(x,0)|} = 3,
$$

we arrive to the expression in (1.9). The magnification factor is independent of the amplitude of the *cn*-periodic wave.

A Proof of N-fold Darboux transformation

Here we prove Theorem 1 with explicit algebraic computations. The Darboux transformation matrix $T(\lambda)$ in (3.1) is sought in the following explicit form:

$$T(\lambda) = I + \sum_{k=1}^{N} \frac{1}{\lambda - \lambda_k} T_k, \quad T_k = \widetilde{\varphi}^{(k)} \otimes (\varphi^{(k)})^t \sigma_1 \sigma_3, \tag{A.1}$$

where the sign $\otimes$ denotes the outer vector product and I denotes an identity 2×2 matrix.

We note that $\varphi^{(k)} \in \ker(T_k)$ and $\widetilde{\varphi}^{(k)} \in \mathrm{ran}(T_k)$. It is assumed in Theorem 1 that $\varphi^{(k)} = (p_k, q_k)^t$, $1 \leq k \leq N$ is a particular smooth nonzero solution to system (2.1) and (2.2) with fixed $\lambda = \lambda_k \in \mathbb{C}\backslash\{0\}$ satisfying $\lambda_k \neq \pm\lambda_j$ for every $k \neq j$, whereas $\{\widetilde{\varphi}^{(k)}\}_{1 \leq k \leq N}$ is a unique solution of the linear algebraic system (3.4). Deeper in the proof, we will be able to show that $\widetilde{\varphi}^{(k)} = (\widetilde{p}_k, \widetilde{q}_k)^t$, $1 \leq k \leq N$ is a particular solution to system (2.1) and (2.2) with $\lambda = \lambda_k$ and new potential $\widetilde{u}$ given by the transformation formula (3.5).

First, let us show that the two lines in the definition (3.5) are identical. Let us define entries of the matrix A by (3.9). Each entry is finite, moreover, $A_{jk} = A_{kj}$. The linear system (3.4) can be split into two parts as follows

$$\sum_{k=1}^{N} \frac{\langle \varphi^{(j)}, \varphi^{(k)} \rangle}{\lambda_j + \lambda_k} \widetilde{p}_k = q_j, \quad \sum_{k=1}^{N} \frac{\langle \varphi^{(j)}, \varphi^{(k)} \rangle}{\lambda_j + \lambda_k} \widetilde{q}_k = -p_j. \tag{A.2}$$

Thanks to the symmetry of A, we obtain from (A.2):

$$\sum_{j=1}^{N} \widetilde{q}_j q_j = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\langle \varphi^{(j)}, \varphi^{(k)} \rangle}{\lambda_j + \lambda_k} \widetilde{p}_k \widetilde{q}_j = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\langle \varphi^{(j)}, \varphi^{(k)} \rangle}{\lambda_j + \lambda_k} \widetilde{p}_j \widetilde{q}_k = -\sum_{j=1}^{N} \widetilde{p}_j p_j. \tag{A.3}$$

This proves that the two lines in the definition (3.5) are identical. For further use, let us also derive another relation from the system (A.2):

$$\begin{aligned} \sum_{j=1}^{N} \lambda_j \widetilde{q}_j q_j - \sum_{j=1}^{N} \lambda_j \widetilde{p}_j p_j &= \sum_{j=1}^{N} \sum_{k=1}^{N} \langle \varphi^{(j)}, \varphi^{(k)} \rangle \widetilde{p}_k \widetilde{q}_j \\ &= \left(\sum_{j=1}^{N} \widetilde{p}_j p_j \right) \left(\sum_{k=1}^{N} p_k \widetilde{q}_k \right) + \left(\sum_{j=1}^{N} \widetilde{q}_j q_j \right) \left(\sum_{k=1}^{N} \widetilde{p}_k q_k \right) \\ &= \frac{1}{2} (\widetilde{u} - u) \left(\sum_{k=1}^{N} p_k \widetilde{q}_k - \sum_{k=1}^{N} \widetilde{p}_k q_k \right). \end{aligned} \tag{A.4}$$

Next, we show validity of the Darboux equation (3.2) under the transformation formula (A.1). Substituting (A.1) to (3.2) yields the following equations at the simple poles

$$\partial_x T_k + T_k U(\lambda_k, u) = U(\lambda_k, \widetilde{u}) T_k, \quad 1 \leq k \leq N, \tag{A.5}$$

and the following equation at the constant term

$$\widetilde{u} \sigma_3 \sigma_1 = u \sigma_3 \sigma_1 + \sum_{k=1}^{N} T_k \sigma_3 - \sigma_3 T_k. \tag{A.6}$$

Equation (A.6) yields (3.5) due to representation (A.1).

Let us show that equations (A.5) are satisfied if $\varphi^{(k)}$ solves (2.1) with $\lambda = \lambda_k$ and u, whereas $\widetilde{\varphi}^{(k)}$ solves (2.1) with $\lambda = \lambda_k$ and $\widetilde{u}$. Recall that $\sigma_1\sigma_3 = -\sigma_3\sigma_1$ and $\sigma_1\sigma_1 = \sigma_3\sigma_3 = I$. Substituting (A.1) to both sides of (A.5) yields

$$\begin{aligned}
&\left[\partial_x \widetilde{\varphi}^{(k)}\right] \otimes (\varphi^{(k)})^t \sigma_1\sigma_3 + \widetilde{\varphi}^{(k)} \otimes \left[\partial_x (\varphi^{(k)})^t\right] \sigma_1\sigma_3 + \widetilde{\varphi}^{(k)} \otimes (\varphi^{(k)})^t \sigma_1\sigma_3 U(\lambda_1, u) \\
&= \left[\partial_x \widetilde{\varphi}^{(k)}\right] \otimes (\varphi^{(k)})^t \sigma_1\sigma_3 + \widetilde{\varphi}^{(k)} \otimes \left[\partial_x (\varphi^{(k)})^t\right] \sigma_1\sigma_3 - \widetilde{\varphi}^{(k)} \otimes (\varphi^{(k)})^t U(\lambda_1, u)^t \sigma_1\sigma_3 \\
&= \left[\partial_x \widetilde{\varphi}^{(k)}\right] \otimes (\varphi^{(k)})^t \sigma_1\sigma_3
\end{aligned}$$

and

$$U(\lambda_1, \widetilde{u}) \widetilde{\varphi}^{(k)} \otimes (\varphi^{(k)})^t \sigma_1\sigma_3 = \left[\partial_x \widetilde{\varphi}^{(k)}\right] \otimes (\varphi^{(k)})^t \sigma_1\sigma_3,$$

hence equation (A.5) is satisfied.

We show now that if $\{\varphi^{(k)}\}_{1\leq k\leq N}$ solve (2.1) with $\{\lambda_k\}_{1\leq k\leq N}$ and u and $\{\widetilde{\varphi}^{(k)}\}_{1\leq k\leq N}$ are obtained from the linear algebraic system (3.4), then $\{\widetilde{\varphi}^{(k)}\}_{1\leq k\leq N}$ solve (2.1) with $\{\lambda_k\}_{1\leq k\leq N}$ and $\widetilde{u}$. We note from the linear system (2.1) that

$$\partial_x \langle \varphi^{(j)}, \varphi^{(k)} \rangle = (\lambda_j + \lambda_k) \langle \varphi^{(j)}, \sigma_3 \varphi^{(k)} \rangle. \tag{A.7}$$

Differentiating (3.4) in x and substituting (2.3) and (A.7) yield

$$\begin{aligned}
&\sum_{k=1}^{N} \frac{\langle \varphi^{(j)}, \varphi^{(k)} \rangle}{\lambda_j + \lambda_k} \left[\partial_x \widetilde{\varphi}^{(k)} - \lambda_k \sigma_3 \widetilde{\varphi}^{(k)} - \widetilde{u} \sigma_3\sigma_1 \widetilde{\varphi}^{(k)}\right] \\
= \;\; & (\widetilde{u} - u)\varphi^{(j)} - \sum_{k=1}^{N} \left[\langle \varphi^{(j)}, \sigma_3 \varphi^{(k)} \rangle \widetilde{\varphi}^{(k)} + \langle \varphi^{(j)}, \varphi^{(k)} \rangle \sigma_3 \widetilde{\varphi}^{(k)}\right] = 0,
\end{aligned} \tag{A.8}$$

where the last equality is due to the transformation formula (3.5). Thus, if the linear system (3.4) is assumed to admit a unique solution, then $\widetilde{\varphi}^{(k)}$ solves (2.1) with $\lambda = \lambda_k$ and $\widetilde{u}$.

It remains to show validity of the Darboux equation (3.3) under the transformation formula (A.1). Substituting (A.1) to (3.3) yields the following equations at the simple poles

$$\partial_t T_k + T_k V(\lambda_k, u) = V(\lambda_k, \widetilde{u}) T_k, \quad 1 \leq k \leq N, \tag{A.9}$$

the same equation (A.6) at λ^2 and the following two equations at λ^1 and λ^0 respectively:

$$\widetilde{u}^2 \sigma_3 + \widetilde{u}_x \sigma_1 + 2\widetilde{u} \sum_{k=1}^{N} \sigma_3\sigma_1 T_k = u^2 \sigma_3 + u_x \sigma_1 + 2u \sum_{k=1}^{N} T_k \sigma_3\sigma_1 + 2 \sum_{k=1}^{N} \lambda_k (T_k \sigma_3 - \sigma_3 T_k) \tag{A.10}$$

and

$$\begin{aligned}
&(2\widetilde{u}^3 + \widetilde{u}_{xx})\sigma_3\sigma_1 + 2\widetilde{u}^2 \sum_{k=1}^{N} \sigma_3 T_k + 2\widetilde{u}_x \sum_{k=1}^{N} \sigma_1 T_k + 4\widetilde{u} \sum_{k=1}^{N} \lambda_k \sigma_3\sigma_1 T_k + 4 \sum_{k=1}^{N} \lambda_k^2 \sigma_3 T_k \\
&= (2u^3 + u_{xx})\sigma_3\sigma_1 + 2u^2 \sum_{k=1}^{N} T_k \sigma_3 + 2u_x \sum_{k=1}^{N} T_k \sigma_1 + 4u \sum_{k=1}^{N} \lambda_k T_k \sigma_3\sigma_1 + 4 \sum_{k=1}^{N} \lambda_k^2 T_k \sigma_3.
\end{aligned} \tag{A.11}$$

Let us show that equations (A.9) are satisfied if $\varphi^{(k)}$ solves (2.2) with $\lambda = \lambda_k$ and u, whereas $\widetilde{\varphi}^{(k)}$ solves (2.2) with $\lambda = \lambda_k$ and $\widetilde{u}$. Substituting (A.1) to both sides of (A.9) yields

$$\begin{aligned}
&\left[\partial_t \widetilde{\varphi}^{(k)}\right] \otimes (\varphi^{(k)})^t \sigma_1\sigma_3 + \widetilde{\varphi}^{(k)} \otimes \left[\partial_t (\varphi^{(k)})^t\right] \sigma_1\sigma_3 + \widetilde{\varphi}^{(k)} \otimes (\varphi^{(k)})^t \sigma_1\sigma_3 V(\lambda_1, u) \\
&= \left[\partial_t \widetilde{\varphi}^{(k)}\right] \otimes (\varphi^{(k)})^t \sigma_1\sigma_3 + \widetilde{\varphi}^{(k)} \otimes \left[\partial_t (\varphi^{(k)})^t\right] \sigma_1\sigma_3 - \widetilde{\varphi}^{(k)} \otimes (\varphi^{(k)})^t V(\lambda_1, u)^t \sigma_1\sigma_3 \\
&= \left[\partial_t \widetilde{\varphi}^{(k)}\right] \otimes (\varphi^{(k)})^t \sigma_1\sigma_3
\end{aligned}$$

and

$$V(\lambda_1, \widetilde{u}) \widetilde{\varphi}^{(k)} \otimes (\varphi^{(k)})^t \sigma_1\sigma_3 = \left[\partial_t \widetilde{\varphi}^{(k)}\right] \otimes (\varphi^{(k)})^t \sigma_1\sigma_3,$$

hence equation (A.9) is satisfied.

In order to show the validity of equation (A.10), we differentiate (A.6) in x and substitute (A.5) to obtain

$$(\widetilde{u}_x - u_x)\sigma_1 = 2\sum_{k=1}^{N} \lambda_k (T_k\sigma_3 - \sigma_3 T_k) + \widetilde{u} \sum_{k=1}^{N} (\sigma_1 T_k \sigma_3 - \sigma_3\sigma_1 T_k) + u \sum_{k=1}^{N} (\sigma_3 T_k \sigma_1 + T_k \sigma_3\sigma_1). \quad \text{(A.12)}$$

Substituting (A.12) into (A.10) yields a simplified form of the equation:

$$(\widetilde{u}^2 - u^2)\sigma_3 + \widetilde{u} \sum_{k=1}^{N} \sigma_1 T_k \sigma_3 + \sigma_3\sigma_1 T_k + u \sum_{k=1}^{N} \sigma_3 T_k \sigma_1 - T_k \sigma_3 \sigma_1 = 0. \quad \text{(A.13)}$$

Further substituting (A.6) into (A.13) yields

$$\sum_{k=1}^{N} (\sigma_1 T_k \sigma_3 + \sigma_3\sigma_1 T_k + T_k \sigma_3\sigma_1 - \sigma_3 T_k \sigma_1) = 0. \quad \text{(A.14)}$$

The validity of equation (A.14) is satisfied thanks again to equation (A.6):

$$(\widetilde{u} - u)\sigma_3 = \sum_{k=1}^{N} (T_k \sigma_3\sigma_1 - \sigma_3 T_k \sigma_1), \quad (u - \widetilde{u})\sigma_3 = \sum_{k=1}^{N} (\sigma_1 T_k \sigma_3 + \sigma_3\sigma_1 T_k). \quad \text{(A.15)}$$

Hence, equation (A.10) is satisfied.

In order to show the validity of equation (A.11), we differentiate (A.12) in x and substitute (A.5) to obtain

$$\begin{aligned}
(\tilde{u}_{xx} - u_{xx})\sigma_3\sigma_1 \;=\; & 4\sum_{k=1}^{N} \lambda_k^2 (T_k\sigma_3 - \sigma_3 T_k) + 2\tilde{u} \sum_{k=1}^{N} \lambda_k (\sigma_1 T_k \sigma_3 - \sigma_3\sigma_1 T_k) \\
& + 2u \sum_{k=1}^{N} \lambda_k (\sigma_3 T_k \sigma_1 + T_k \sigma_3\sigma_1) + \tilde{u}_x \sum_{k=1}^{N} (\sigma_3\sigma_1 T_k \sigma_3 - \sigma_1 T_k) \\
& + u_x \sum_{k=1}^{N} (T_k \sigma_1 + \sigma_3 T_k \sigma_3\sigma_1) + (\tilde{u}^2 + u^2) \sum_{k=1}^{N} (\sigma_3 T_k - T_k \sigma_3) \\
& + 2u\tilde{u} \sum_{k=1}^{N} (\sigma_3\sigma_1 T_k \sigma_1 + \sigma_1 T_k \sigma_3\sigma_1).
\end{aligned} \quad \text{(A.16)}$$

Substituting (A.6) and (A.16) into (A.11) yields a simplified form of the equation:

$$\begin{aligned}
&2\widetilde{u}\sum_{k=1}^{N}\lambda_k(\sigma_1 T_k\sigma_3+\sigma_3\sigma_1 T_k)+2u\sum_{k=1}^{N}\lambda_k(\sigma_3 T_k\sigma_1-T_k\sigma_3\sigma_1)\\
&+\widetilde{u}_x\sum_{k=1}^{N}(\sigma_3\sigma_1 T_k\sigma_3+\sigma_1 T_k)+u_x\sum_{k=1}^{N}(\sigma_3 T_k\sigma_3\sigma_1-T_k\sigma_1)+(\widetilde{u}^2-u^2)\sum_{k=1}^{N}(T_k\sigma_3+\sigma_3 T_k)\\
&+2\widetilde{u}u\sum_{k=1}^{N}(\sigma_3\sigma_1 T_k\sigma_1+\sigma_1 T_k\sigma_3\sigma_1+T_k\sigma_3-\sigma_3 T_k)=0. \qquad (A.17)
\end{aligned}$$

The last term in the left-hand side of (A.17) is identically zero thanks to equation (A.14) after multiplication by σ_1 on the right. Multiplication of equation (A.14) by σ_3 on the right allows us to group the terms containing u_x and $\widetilde{u}_x$. As a result, we rewrite (A.17) in the equivalent form

$$\begin{aligned}
&2\widetilde{u}\sum_{k=1}^{N}\lambda_k(\sigma_1 T_k\sigma_3+\sigma_3\sigma_1 T_k)+2u\sum_{k=1}^{N}\lambda_k(\sigma_3 T_k\sigma_1-T_k\sigma_3\sigma_1)\\
&+(u_x-\widetilde{u}_x)\sum_{k=1}^{N}(\sigma_3 T_k\sigma_3\sigma_1-T_k\sigma_1)+(\widetilde{u}^2-u^2)\sum_{k=1}^{N}(T_k\sigma_3+\sigma_3 T_k)=0. \qquad (A.18)
\end{aligned}$$

Multiplying (A.12) by σ_1 from the left and from the right, we obtain

$$(\widetilde{u}_x-u_x)I=2\sum_{k=1}^{N}\lambda_k(\sigma_1 T_k\sigma_3+\sigma_3\sigma_1 T_k)+\widetilde{u}\sum_{k=1}^{N}(T_k\sigma_3+\sigma_3 T_k)+u\sum_{k=1}^{N}(\sigma_1\sigma_3 T_k\sigma_1+\sigma_1 T_k\sigma_3\sigma_1)$$

and

$$(\widetilde{u}_x-u_x)I=2\sum_{k=1}^{N}\lambda_k(T_k\sigma_3\sigma_1-\sigma_3 T_k\sigma_1)+\widetilde{u}\sum_{k=1}^{N}(\sigma_1 T_k\sigma_3\sigma_1-\sigma_3\sigma_1 T_k\sigma_1)+u\sum_{k=1}^{N}(\sigma_3 T_k+T_k\sigma_3),$$

from which one can rewrite (A.18) in the equivalent form

$$(\widetilde{u}_x-u_x)(\widetilde{u}-u)I-(\widetilde{u}_x-u_x)\sum_{k=1}^{N}(\sigma_3 T_k\sigma_3\sigma_1-T_k\sigma_1)=0,$$

which is satisfied thanks to equation (A.15). Hence, equation (A.11) is satisfied.

Finally, we show that if $\{\varphi^{(k)}\}_{1\le k\le N}$ solve (2.2) with $\{\lambda_k\}_{1\le k\le N}$ and u and $\{\widetilde{\varphi}^{(k)}\}_{1\le k\le N}$ are obtained from the linear algebraic system (3.4), then $\{\widetilde{\varphi}^{(k)}\}_{1\le k\le N}$ solve (2.2) with $\{\lambda_k\}_{1\le k\le N}$ and $\widetilde{u}$. We note from the linear system (2.2) that

$$\begin{aligned}
\partial_t\langle\varphi^{(j)},\varphi^{(k)}\rangle \;=\;& -(\lambda_j+\lambda_k)\left[4(\lambda_j^2-\lambda_j\lambda_k+\lambda_k^2)+2u^2\right]\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\\
&+4(\lambda_j^2-\lambda_k^2)u\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle-2(\lambda_j+\lambda_k)u_x\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle. \qquad (A.19)
\end{aligned}$$

Differentiating (3.4) in t and substituting (2.4) and (A.19) yield

$$\sum_{k=1}^{N}\frac{\langle\varphi^{(j)},\varphi^{(k)}\rangle}{\lambda_j+\lambda_k}\Big[\partial_t\widetilde{\varphi}^{(k)}+(4\lambda_k^3+2\lambda_k\widetilde{u}^2)\sigma_3\widetilde{\varphi}^{(k)}+4\lambda_k^2\widetilde{u}\sigma_3\sigma_1\widetilde{\varphi}^{(k)}+2\lambda_k\widetilde{u}_x\sigma_1\widetilde{\varphi}^{(k)}+(2\widetilde{u}^3+\widetilde{u}_{xx})\sigma_3\sigma_1\widetilde{\varphi}^{(k)}\Big]$$
$$=\sum_{k=1}^{N}\Big[(4\lambda_j^2-4\lambda_j\lambda_k+4\lambda_k^2+2u^2)\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle+4(\lambda_k-\lambda_j)u\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle+2u_x\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\Big]\widetilde{\varphi}^{(k)}$$
$$+\sum_{k=1}^{N}\langle\varphi^{(j)},\varphi^{(k)}\rangle\Big[(4\lambda_k^2-4\lambda_k\lambda_j+4\lambda_j^2+2\widetilde{u}^2)\sigma_3\widetilde{\varphi}^{(k)}+4(\lambda_k-\lambda_j)\widetilde{u}\sigma_3\sigma_1\widetilde{\varphi}^{(k)}+2\widetilde{u}_x\sigma_1\widetilde{\varphi}^{(k)}\Big]$$
$$+2\lambda_j(u^2-\widetilde{u}^2)\sigma_1\varphi^{(j)}+4\lambda_j^2(u-\widetilde{u})\varphi^{(j)}-2\lambda_j(u_x-\widetilde{u}_x)\sigma_3\varphi^{(j)}+(2u^3+u_{xx}-2\widetilde{u}^3-\widetilde{u}_{xx})\varphi^{(j)}. \tag{A.20}$$

The terms proportional to $4\lambda_j^2$ cancel out due to the same relation (A.8). The terms proportional to $2\lambda_j$ cancel out if the following relation is true:

$$(u^2-\widetilde{u}^2)\sigma_1\varphi^{(j)}+(\widetilde{u}_x-u_x)\sigma_3\varphi^{(j)}=2\sum_{k=1}^{N}\lambda_k\Big[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\Big]$$
$$+2\widetilde{u}\sum_{k=1}^{N}\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}+2u\sum_{k=1}^{N}\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}. \tag{A.21}$$

The other λ_j-independent terms cancel out if the following relation is true:

$$(2\widetilde{u}^3+\widetilde{u}_{xx}-2u^3-u_{xx})\varphi^{(j)}=4\sum_{k=1}^{N}\lambda_k^2\Big[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\Big]$$
$$+2\widetilde{u}^2\sum_{k=1}^{N}\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}+2u^2\sum_{k=1}^{N}\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}$$
$$+4\widetilde{u}\sum_{k=1}^{N}\lambda_k\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}+4u\sum_{k=1}^{N}\lambda_k\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}$$
$$+2\widetilde{u}_x\sum_{k=1}^{N}\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}+2u_x\sum_{k=1}^{N}\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}. \tag{A.22}$$

Provided equations (A.21) and (A.22) are satisfied, the right-hand side of equation (A.20) is zero. If the linear system (3.4) is assumed to admit a unique solution, then $\widetilde{\varphi}^{(k)}$ solves (2.2) with $\{\lambda_k\}_{1\leq k\leq N}$ and $\widetilde{u}$.

Finally, we show validity of equations (A.21) and (A.22). In order to show (A.21), we first obtain the relation

$$\partial_x\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle=(\lambda_j+\lambda_k)\langle\varphi^{(j)},\varphi^{(k)}\rangle+2u\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle, \tag{A.23}$$

in addition to the relation (A.7). Then, we differentiate (A.8) in x, substitute (2.3), (A.7), and (A.23), and obtain

$$(\widetilde{u}_x-u_x)\varphi^{(j)}+(\widetilde{u}-u)u\sigma_3\sigma_1\varphi^{(j)}=2\sum_{k=1}^{N}\lambda_k\Big[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}\Big]$$
$$+\widetilde{u}\sum_{k=1}^{N}\Big[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\Big]+2u\sum_{k=1}^{N}\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}, \tag{A.24}$$

where the relation (A.8) was used to cancel the λ_j term. By using the transformation formulas (3.5), we verify that

$$(\widetilde{u}-u)\sigma_1\varphi^{(j)}=\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}-\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}\right]. \tag{A.25}$$

This allows us to simplify (A.24) to the form

$$\begin{aligned}(\widetilde{u}_x-u_x)\varphi^{(j)} &= 2\sum_{k=1}^{N}\lambda_k\left[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}\right]\\ &\quad+\widetilde{u}\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\right]\\ &\quad+u\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\right].\end{aligned} \tag{A.26}$$

Substituting (A.26) to (A.21) yields the following equation

$$\begin{aligned}(u^2-\widetilde{u}^2)\sigma_1\varphi^{(j)} &= \widetilde{u}\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}-\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\right]\\ &\quad+u\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}-\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\right].\end{aligned} \tag{A.27}$$

Thanks to the relations (A.8) and (A.25), equation (A.27) is satisfied, and so is equation (A.21).

In order to show (A.22), we first obtain the relations

$$\partial_x\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle=(\lambda_j-\lambda_k)\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle-2u\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle \tag{A.28}$$

and

$$\partial_x\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle=(\lambda_j-\lambda_k)\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle. \tag{A.29}$$

Then, we differentiate (A.26) in x, substitute (2.3), (A.7), (A.23), (A.28), and (A.29), and obtain

$$\begin{aligned}&(\widetilde{u}_{xx}-u_{xx})\varphi^{(j)}+u(\widetilde{u}_x-u_x)\sigma_3\sigma_1\varphi^{(j)}=4\sum_{k=1}^{N}\lambda_k^2\left[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\right]\\ &+2\widetilde{u}\sum_{k=1}^{N}\lambda_k\left[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}\right]+4u\sum_{k=1}^{N}\lambda_k\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\\ &+\widetilde{u}_x\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\right]\\ &+u_x\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\right]\\ &+\widetilde{u}u\sum_{k=1}^{N}\left[3\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\right]\\ &-\widetilde{u}^2\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\right]-2u^2\sum_{k=1}^{N}\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)},\end{aligned} \tag{A.30}$$

where the relation (A.26) was used to cancel the λ_j term. Substituting (A.30) into (A.22) and using (A.8) and (A.25) yield

$$2(\widetilde{u}^3-u^3)\varphi^{(j)} = (2u-\widetilde{u})(\widetilde{u}_x-u_x)\sigma_3\sigma_1\varphi^{(j)} + \widetilde{u}^2\sum_{k=1}^{N}\left[3\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}\right]$$
$$+4u^2\sum_{k=1}^{N}\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} - \widetilde{u}u\sum_{k=1}^{N}\left[3\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\right]$$
$$+2\widetilde{u}\sum_{k=1}^{N}\lambda_k\left[\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)} - \langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\right]$$
$$+4u\sum_{k=1}^{N}\lambda_k\left[\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} - \langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\right]. \tag{A.31}$$

Substituting (A.26) to (A.31) yields

$$2(\widetilde{u}-u)(\widetilde{u}^2+\widetilde{u}u+u^2)\varphi^{(j)} = 2\widetilde{u}^2\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}\right]$$
$$+2\widetilde{u}u\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)} - \langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} - 2\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}\right]$$
$$+2u^2\sum_{k=1}^{N}\left[2\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)} - \langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\right]$$
$$+4u\sum_{k=1}^{N}\lambda_k\Big[\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}$$
$$-\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)} - \langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\Big]. \tag{A.32}$$

By using the relations (A.3) and explicit computations, we obtain

$$\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)} - \langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} - 2\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}\right]$$
$$= (\widetilde{u}-u)\varphi^{(j)} - 2\left(\sum_{k=1}^{N}p_k\widetilde{q}_k - \sum_{k=1}^{N}\widetilde{p}_kq_k\right)\sigma_1\varphi^{(j)}, \tag{A.33}$$

$$\sum_{k=1}^{N}\left[2\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)} - \langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\right]$$
$$= (\widetilde{u}-u)\varphi^{(j)} + 2\left(\sum_{k=1}^{N}p_k\widetilde{q}_k - \sum_{k=1}^{N}\widetilde{p}_kq_k\right)\sigma_1\varphi^{(j)} \tag{A.34}$$

and

$$\sum_{k=1}^{N}\lambda_k\Big[\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}$$
$$-\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}-\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\Big]$$
$$= -2\left(\sum_{k=1}^{N}\lambda_k\widetilde{p}_kp_k-\sum_{k=1}^{N}\lambda_k\widetilde{q}_kq_k\right)\sigma_1\varphi^{(j)}. \tag{A.35}$$

Substituting (A.33), (A.34), and (A.35) to (A.32) cancel all terms thanks to the relations (A.4) and (A.8). Therefore, equation (A.32) is satisfied, and so is equation (A.22).

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