Rogue waves on the double-periodic background in the focusing nonlinear Schrödinger equation

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The double-periodic solutions of the focusing nonlinear Schrödinger equation have been previously obtained by the method of separation of variables. We construct these solutions by using an algebraic method with two eigenvalues. Furthermore, we characterize the Lax spectrum for the double-periodic solutions and analyze rogue waves arising on the background of such solutions. Magnification of the rogue waves is studied numerically.

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I. INTRODUCTION

Rogue waves are commonly defined as gigantic waves appearing from nowhere and disappearing without trace. They are frequently seen on the ocean's surface [1] and in optical fibers [2]. The appearance of rogue waves can be related to the modulation instability of the wave background [3]. Formation of particular rogue waves such as Akhmediev breathers, Peregrine solution, and Kuznetsov-Ma breathers have been modeled from different initial data such as local condensates [4], multi-soliton gases [5,6], and periodic perturbations [7,8]. Experimental observations of rogue waves have been confirmed both in hydrodynamical and optical laboratories [9,10] (see recent review in Ref. [11]). Statistical analysis of rogue waves has recently been developed [12] (see also Refs. [5,7]).

Rogue waves and modulation instability of the wave background are commonly modeled by the focusing nonlinear Schrödinger (NLS) equation, which we take in the following dimensionless form:

$$i\psi_t + \tfrac{1}{2}\psi_{xx} + |\psi|^2\psi = 0. \tag{1}$$

This fundamental model is rich of many exact solutions due to its integrability discovered in Ref. [13].

A number of important new results were recently obtained in the mathematical theory of rogue waves. Universal behavior of the modulationally unstable constant background was studied asymptotically in Ref. [14]. The finite-gap method was employed to relate the unstable modes on the constant background with the occurrence of rogue waves [15,16]. Rogue waves of infinite order were constructed in Ref. [17] based on recent developments in the inverse scattering method [18]. Rogue waves of the soliton superposition were studied asymptotically in the limit of many solitons [19,20].

Many wave patterns are periodic in space and time variables. Simplest traveling wave solutions have a space-periodic and time-independent profile of $|\psi|$. Modulational instability of traveling wave solutions was analyzed in many explicit details [21,22]. Numerical experiments showed formation of rogue waves from modulationally unstable traveling waves [8]. Exact solutions for rogue waves on the traveling wave background were constructed in our previous work [23,24] by using an algebraic method with one eigenvalue. Similar exact solutions for rogue waves were constructed in Ref. [25].

There exist other exact solutions of the focusing NLS equation (1) for which $|\psi|$ is periodic both in space and in time. These double-periodic solutions describe spatial-temporal wave patterns and are referred to as *the double-periodic background*. Exact representations of such solutions were constructed in Ref. [26] by separating the variables and reducing the NLS equation (1) to the first-order quadratures. Two particular families are given by the following rational functions of Jacobian elliptic functions sn, cn, and dn:

$$\psi(x,t) = k\frac{\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa) + i\sqrt{1+k}\,\mathrm{sn}(t;k)\mathrm{dn}(\sqrt{1+k}x;\kappa)}{\sqrt{1+k}\,\mathrm{dn}(\sqrt{1+k}x;\kappa) - \mathrm{dn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa)}e^{it}, \quad \kappa = \frac{\sqrt{1-k}}{\sqrt{1+k}} \tag{2}$$

and

$$\psi(x,t) = \frac{\mathrm{dn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa) + i\sqrt{k(1+k)}\,\mathrm{sn}(t;k)}{\sqrt{1+k} - \sqrt{k}\,\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa)}e^{ikt}, \quad \kappa = \frac{\sqrt{1-k}}{\sqrt{2}}, \tag{3}$$

where $k \in (0,1)$. It follows from (2) and (3) that

$$|\psi(x,t)| = |\psi(x+L,t)| = |\psi(x,t+T)|, \quad (x,t) \in \mathbb{R}^2, \tag{4}$$

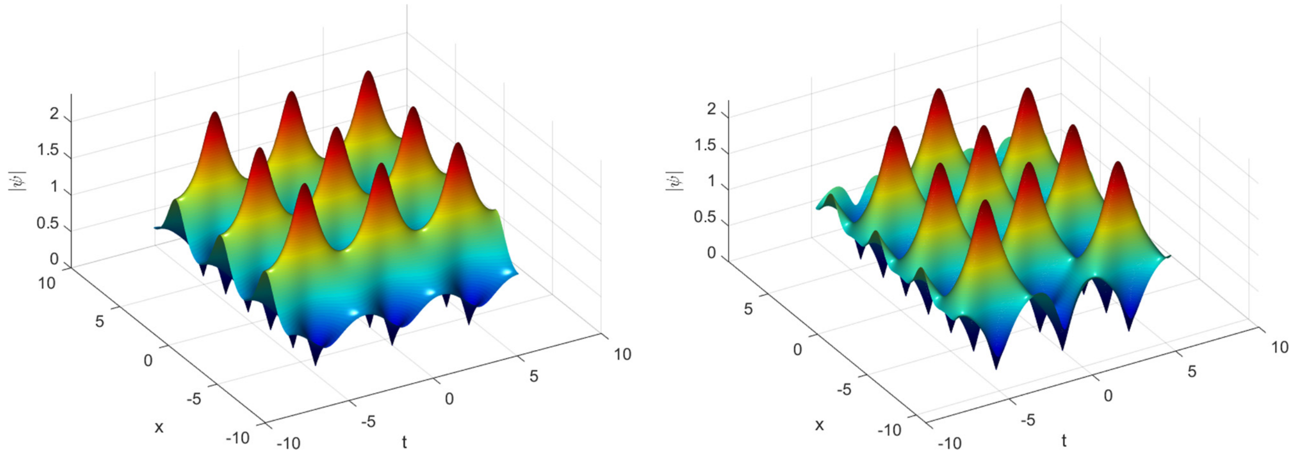

FIG. 1. Solution surface for the double-periodic solution (2) with $k = 0.9$ (left) and the double-periodic solution (3) with $k = 0.8$ (right).

with the fundamental periods $L = \frac{4K(\kappa)}{\sqrt{1+k}}$ and $T = 2K(k)$ for (2) and $L = 2\sqrt{2}K(\kappa)$ and $T = 4K(k)$ for (3), where $K(k)$ denotes the complete elliptic integral of the first kind with the elliptic modulus parameter k.

Figure 1 shows the double-periodic solution (2) for $k = 0.9$ (left) and the double-periodic solution (3) for $k = 0.8$ (right). Note that the solution (2) has periodically repeated maxima at the same positions, whereas the solution (3) has periodically alternated maxima in a chess pattern.

The exact solutions (2) and (3) were used to describe transformations of continuous waves into trains of pulses [27]. These solutions were also constructed for the Hirota equation and other higher-order NLS equations [28]. Rogue waves on the background of the double-periodic solution (2) were constructed numerically in Ref. [29] by using numerical approximations of eigenfunctions of the Lax spectrum and the onefold Darboux transformation.

Experimental observations of the double-periodic solutions (2) and (3) were reported in hydrodynamical experiments [30]. These solutions are relevant for dynamics of perturbations of the Akhmediev breathers, the latter solutions follow from (2) and (3) in the limit $k \to 1$:

$$\psi(x,t) = \frac{\cos(\sqrt{2}x) + i\sqrt{2}\sinh(t)}{\sqrt{2}\cosh(t) - \cos(\sqrt{2}x)} e^{it}. \quad (5)$$

Akhmediev breathers are periodic in x and homoclinic in t, perturbing them with perturbations of different signs produce the double-periodic patterns modelled by the solutions (2) and (3).

The main purpose of this work is to construct rogue waves on the double-periodic background in a closed analytical form and to study their magnification factors. We use a general method of nonlinearization of Lax equations (originally proposed in Ref. [31] and developed for the NLS equation in Refs. [32,33]) and apply it to the case of two eigenvalues compared to the case of one eigenvalue considered in Refs. [23,24]. Although the exact solutions for rogue waves in the focusing NLS equation are more complicated than the corresponding solutions for the modified Korteweg–de Vries equation (which have been analyzed in Refs. [34,35]), efficient computational methods are developed to visualize the Lax spectrum associated with the double-periodic solutions, the admissible eigenvalues, and rogue waves on the double-periodic background. We also study magnification of such rogue waves that depends on parameters of the double-periodic background.

The conventional definition of the rogue wave's magnification factor is *the ratio of the maximal value of the wave amplitude to the average value of the wave background*. The wave is considered to be rogue if the ratio exceeds the double factor [36]. In a series of mathematical papers, the magnification factors were analyzed for general quasiperiodic solutions of the focusing NLS equation (1) given by the Riemann Θ functions in the case of genus two [37] and of arbitrary genus [38,39]. The double-periodic solutions (2) and (3) appear to be particular cases of the Riemann Θ functions of genus two [40,41]; however, the rogue waves constructed in our paper correspond to degenerate cases of the Riemann Θ functions of genus three. We show numerically that the magnification factors for these rogue waves exceed the triple factor and that these rogue waves represent isolated waves on the double-periodic background on the (x,t) plane.

On the other hand, if the magnification factor is defined as *the ratio of the maximal value of the wave amplitude to the maximal value of the wave background*, then we show analytically that the magnification factor does not exceed the triple factor achieved by the Peregrine solution on the constant background. Moreover, we show numerically for the rogue waves constructed here that the magnification factor defined relative to the maximal value of the wave background does not exceed the double factor. This difference between two definitions of the magnification factor is important for experimental observations of the rogue waves, for which the double-periodic solutions are thought to be the transient stage before formation of random waves.

The paper is organized as follows. An algebraic method with two eigenvalues is developed in Sec. II, where we derive a general fourth-order Lax-Novikov equation for the NLS equation (1). In Sec. III, we recover the double-periodic solutions (2) and (3) from reduction of the fourth-order Lax-Novikov equation to the third-order Lax-Novikov equation and its integration in first-order quadratures. We also compute eigenvalues of the Lax spectrum analytically and the spectral bands of the Lax spectrum numerically. Rogue waves on the double-periodic background are constructed in Sec. IV. We show numerically that the rogue waves are isolated in the (x,t) plane and plot the solution surfaces for the rogue waves. The magnification factor is investigated with respect to parameters of the rogue wave solutions and we show numerically that it exceeds the triple factor. Section V contains conclusions and further directions of study. Appendices A and B give technical details on how the analytical expressions (2) and (3) are deduced from the general solution of the third-order Lax-Novikov equation.

II. ALGEBRAIC METHOD WITH TWO EIGENVALUES

A solution $\psi = u$ to the NLS equation (1) is a compatibility condition ($\varphi_{xt} = \varphi_{tx}$) of the following pair of linear equations on $\varphi \in \mathbb{C}^2$:

$$\varphi_x = U(\lambda,u)\varphi, \quad U(\lambda,u) = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix} \tag{6}$$

and

$$\varphi_t = V(\lambda,u)\varphi, \quad V(\lambda,u) = i\begin{pmatrix} \lambda^2 + \frac{1}{2}|u|^2 & \frac{1}{2}u_x + \lambda u \\ \frac{1}{2}\bar{u}_x - \lambda\bar{u} & -\lambda^2 - \frac{1}{2}|u|^2 \end{pmatrix}, \tag{7}$$

where $\bar{u}$ is the conjugate of u and $\lambda \in \mathbb{C}$ is a spectral parameter.

The procedure of computing a new solution $\psi = \hat{u}$ to the NLS equation (1) from another solution $\psi = u$ is well known [23–25]. Let $\varphi = (p_1, q_1)^t$ be any nonzero solution to the linear equations (6) and (7) for a fixed value $\lambda = \lambda_1$. The new solution is given by the onefold Darboux transformation,

$$\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)p_1\bar{q}_1}{|p_1|^2 + |q_1|^2}. \tag{8}$$

If u is a double-periodic solution, then $\hat{u}$ may represent a rogue wave on the double-periodic background. The main question is which value λ_1 to fix and which nonzero solution φ to the linear equations (6) and (7) to take. We show that the admissible values of λ_1 for the double-periodic solutions (2) and (3) are defined by the algebraic method with two eigenvalues. The latter is a particular case of a more general method of nonlinearization of the linear equations on φ, see Refs. [32,33].

A. Nonlinearization of the linear equations

Let $\varphi = (p_1, q_1)^t$ and $\varphi = (p_2, q_2)^t$ be two nonzero solutions of the linear equations (6) and (7) with fixed $\lambda = \lambda_1$ and $\lambda = \lambda_2$ such that $\lambda_1 + \bar{\lambda}_1 \neq 0$, $\lambda_2 + \bar{\lambda}_2 \neq 0$, $\lambda_1 + \bar{\lambda}_2 \neq 0$, and $\lambda_1 \neq \lambda_2$. The following notations will be useful in this work:

$$\mathsf{p} := (p_1, p_2, \bar{q}_1, \bar{q}_2)^t,$$
$$\mathsf{q} := (q_1, q_2, -\bar{p}_1, -\bar{p}_2)^t,$$

and

$$\Lambda := \mathrm{diag}(\lambda_1, \lambda_2, -\bar{\lambda}_1, -\bar{\lambda}_2).$$

Following Refs. [32,33], we introduce the following relation between the solution u to the NLS equation (1) and the squared eigenfunctions:

$$u = \langle \mathsf{p},\mathsf{p}\rangle := p_1^2 + p_2^2 + \bar{q}_1^2 + \bar{q}_2^2. \tag{9}$$

The spectral problem (6) is nonlinearized into the Hamiltonian system given by

$$\frac{dp_j}{dx} = \frac{\partial H}{\partial q_j}, \quad \frac{dq_j}{dx} = -\frac{\partial H}{\partial p_j}, \quad j = 1,2, \tag{10}$$

where

$$H = \langle \Lambda\mathsf{p},\mathsf{q}\rangle + \tfrac{1}{2}\langle \mathsf{p},\mathsf{p}\rangle\langle \mathsf{q},\mathsf{q}\rangle. \tag{11}$$

The time-evolution problem (7) is nonlinearized into another Hamiltonian system given by

$$\frac{dp_j}{dt} = \frac{\partial K}{\partial q_j}, \quad \frac{dq_j}{dt} = -\frac{\partial K}{\partial p_j}, \quad j = 1,2, \tag{12}$$

where

$$K = i\big[\langle \Lambda^2\mathsf{p},\mathsf{q}\rangle + \tfrac{1}{2}\langle \Lambda\mathsf{p},\mathsf{p}\rangle\langle \mathsf{q},\mathsf{q}\rangle + \tfrac{1}{2}\langle \mathsf{p},\mathsf{p}\rangle\langle \Lambda\mathsf{q},\mathsf{q}\rangle + \tfrac{1}{2}\langle \mathsf{p},\mathsf{q}\rangle\langle \mathsf{p},\mathsf{p}\rangle\langle \mathsf{q},\mathsf{q}\rangle\big]. \tag{13}$$

There exist four real-valued constants of motion for the Hamiltonian systems (10) and (12), hence each Hamiltonian system is completely integrable in the sense of Liouville. The four constants of motion were found in Refs. [32,33]:

$$F_0 = i\langle \mathsf{p},\mathsf{q}\rangle, \tag{14}$$

$$F_1 = \langle \Lambda\mathsf{p},\mathsf{q}\rangle + \tfrac{1}{2}\langle \mathsf{p},\mathsf{p}\rangle\langle \mathsf{q},\mathsf{q}\rangle - \tfrac{1}{2}\langle \mathsf{p},\mathsf{q}\rangle^2, \tag{15}$$

$$F_2 = i\big[\langle \Lambda^2\mathsf{p},\mathsf{q}\rangle + \tfrac{1}{2}\langle \Lambda\mathsf{p},\mathsf{p}\rangle\langle \mathsf{q},\mathsf{q}\rangle + \tfrac{1}{2}\langle \mathsf{p},\mathsf{p}\rangle\langle \Lambda\mathsf{q},\mathsf{q}\rangle - \langle \mathsf{p},\mathsf{q}\rangle\langle \Lambda\mathsf{p},\mathsf{q}\rangle\big], \tag{16}$$

$$F_3 = \langle \Lambda^3\mathsf{p},\mathsf{q}\rangle + \tfrac{1}{2}\langle \Lambda^2\mathsf{p},\mathsf{p}\rangle\langle \mathsf{q},\mathsf{q}\rangle + \tfrac{1}{2}\langle \Lambda\mathsf{p},\mathsf{p}\rangle\langle \Lambda\mathsf{q},\mathsf{q}\rangle + \tfrac{1}{2}\langle \mathsf{p},\mathsf{p}\rangle\langle \Lambda^2\mathsf{q},\mathsf{q}\rangle - \frac{1}{2}\langle \Lambda\mathsf{p},\mathsf{q}\rangle^2 - \langle \mathsf{p},\mathsf{q}\rangle\langle \Lambda^2\mathsf{p},\mathsf{q}\rangle. \tag{17}$$

Note that $H = F_1 - \frac{1}{2}F_0^2$ and $K = F_2 + F_1F_0 - \frac{1}{2}F_0^3$.

B. Fourth-order Lax-Novikov equation

In what follows, we relate the x derivatives of u with the four constants of motion (F_0, F_1, F_2, F_3). By differentiating Eq. (9) in x and using the Hamiltonian system (10) together with the first constant (14), we obtain

$$\frac{du}{dx} + 2iF_0u = 2\langle \Lambda\mathsf{p},\mathsf{p}\rangle. \tag{18}$$

By taking another derivative of Eq. (18) in x and using the first two constants (14) and (15), we obtain

$$\frac{d^2u}{dx^2} + 2|u|^2u + 2iF_0\frac{du}{dx} - 4Hu = 4\langle \Lambda^2\mathsf{p},\mathsf{p}\rangle, \tag{19}$$

where $H = F_1 - \frac{1}{2}F_0^2$ is the Hamiltonian for system (10).

By taking another derivative of Eq. (19) in x and using the first three constants (14)–(16), we obtain

$$\frac{d^3u}{dx^3} + 6|u|^2\frac{du}{dx} + 2iF_0\left(\frac{d^2u}{dx^2} + 2|u|^2u\right) - 4H\frac{du}{dx} + 8iKu = 8\langle \Lambda^3\mathsf{p},\mathsf{p}\rangle, \tag{20}$$

where $K = F_2 + F_0 H = F_2 + F_0 F_1 - \frac{1}{2}F_0^3$ is the Hamiltonian for system (12).

Finally, by taking yet another derivative of Eq. (20) in x and using all four constants (14)–(17), we obtain

$$\frac{d^4u}{dx^4} + 8|u|^2\frac{d^2u}{dx^2} + 2u^2\frac{d^2\bar{u}}{dx^2} + 4u\left|\frac{du}{dx}\right|^2 + 6\left(\frac{du}{dx}\right)^2\bar{u} + 6|u|^4u + 2iF_0\left(\frac{d^3u}{dx^3} + 6|u|^2\frac{du}{dx}\right) - 4H\left(\frac{d^2u}{dx^2} + 2|u|^2u\right) + 8iK\frac{du}{dx} - 16Eu = 16\langle\Lambda^4\mathsf{p},\mathsf{p}\rangle, \tag{21}$$

where

$$E = F_3 - F_0K + \tfrac{1}{2}H^2 = F_3 - F_0F_2 + \tfrac{1}{2}F_1^2 - \tfrac{3}{2}F_1F_0^2 + \tfrac{5}{8}F_0^4.$$

The system of linear algebraic equations on squared eigenfunctions p_1^2, p_2^2, $\bar{q}_1^2$, and $\bar{q}_2^2$ is fully determined by four equations, Eqs. (9), (18), (19), and (20). Therefore, the fourth-order equation (21) can be closed on u as the fourth-order Lax-Novikov equation for the hierarchy of stationary NLS equations. In order to avoid the linear algebra computations, we will use integrability of the Hamiltonian system (10) and obtain the closed fourth-order equation on u by a simple computation.

C. Integrability of the fourth-order Lax-Novikov equation

It was shown in Refs. [32,33] that the Hamiltonian system (10) arises as a compatibility condition for the Lax equation

$$\frac{d}{dx}W(\lambda) = [U(\lambda,u), W(\lambda)], \quad \lambda \in \mathbb{C}, \tag{22}$$

where $U(\lambda,u)$ is given by (6) with u given by (9) and

$$W(\lambda) = \begin{bmatrix} W_{11}(\lambda) & W_{12}(\lambda) \\ \overline{W}_{12}(-\lambda) & -\overline{W}_{11}(-\lambda)\end{bmatrix}, \tag{23}$$

with the entries

$$\begin{aligned} W_{11}(\lambda) &= 1 - \sum_{j=1}^{2}\left(\frac{p_jq_j}{\lambda-\lambda_j} - \frac{\bar{p}_j\bar{q}_j}{\lambda+\bar{\lambda}_j}\right), \\ W_{12}(\lambda) &= \sum_{j=1}^{2}\left(\frac{p_j^2}{\lambda-\lambda_j} + \frac{\bar{q}_j^2}{\lambda+\bar{\lambda}_j}\right). \end{aligned} \tag{24}$$

In order to progress further, we rewrite $W_{12}(\lambda)$ in the equivalent form:

$$W_{12}(\lambda) = \frac{S_0\lambda^3 + S_1\lambda^2 + S_2\lambda + S_3}{\lambda^4 + iA_1\lambda^3 + A_2\lambda^2 + iA_3\lambda + A_4}, \tag{25}$$

where

$$\begin{aligned} A_1 &= i(\lambda_1 + \lambda_2 - \bar{\lambda}_1 - \bar{\lambda}_2), \\ A_2 &= (\lambda_1 - \bar{\lambda}_1)(\lambda_2 - \bar{\lambda}_2) - |\lambda_1|^2 - |\lambda_2|^2, \\ A_3 &= i(\bar{\lambda}_1 - \lambda_1)|\lambda_2|^2 + i(\bar{\lambda}_2 - \lambda_2)|\lambda_1|^2, \\ A_4 &= |\lambda_1|^2|\lambda_2|^2, \end{aligned} \tag{26}$$

and

$$\begin{aligned} S_0 &= \langle\mathsf{p},\mathsf{p}\rangle, \\ S_1 &= \langle\Lambda\mathsf{p},\mathsf{p}\rangle + iA_1\langle\mathsf{p},\mathsf{p}\rangle, \\ S_2 &= \langle\Lambda^2\mathsf{p},\mathsf{p}\rangle + iA_1\langle\Lambda\mathsf{p},\mathsf{p}\rangle + A_2\langle\mathsf{p},\mathsf{p}\rangle, \\ S_3 &= \langle\Lambda^3\mathsf{p},\mathsf{p}\rangle + iA_1\langle\Lambda^2\mathsf{p},\mathsf{p}\rangle + A_2\langle\Lambda\mathsf{p},\mathsf{p}\rangle + iA_3\langle\mathsf{p},\mathsf{p}\rangle. \end{aligned} \tag{27}$$

Substituting (9), (18), (19), and (20) into these expressions yield compact expressions:

$$\begin{aligned} S_0 &= u, \\ S_1 &= \frac{1}{2}u' + icu, \\ S_2 &= \frac{1}{4}(u'' + 2|u|^2u) + \frac{i}{2}cu' + bu, \\ S_3 &= \frac{1}{8}(u''' + 6|u|^2u') + \frac{i}{4}c(u'' + 2|u|^2u) + \frac{1}{2}bu' + iau, \end{aligned} \tag{28}$$

where the prime denotes the derivative in x and we have introduced three real-valued constants:

$$\begin{aligned} c &= A_1 + F_0, \\ b &= A_2 - F_0A_1 - H, \\ a &= A_3 + F_0A_2 - HA_1 + K. \end{aligned} \tag{29}$$

Similarly, we rewrite $W_{11}(\lambda)$ in the equivalent form:

$$W_{11}(\lambda) = \frac{\lambda^4 + iT_1\lambda^3 + T_2\lambda^2 + iT_3\lambda + T_4}{\lambda^4 + iA_1\lambda^3 + A_2\lambda^2 + iA_3\lambda + A_4}, \tag{30}$$

where

$$\begin{aligned} T_1 &= A_1 + i\langle\mathsf{p},\mathsf{q}\rangle, \\ T_2 &= A_2 - iA_1\langle\mathsf{p},\mathsf{q}\rangle - \langle\Lambda\mathsf{p},\mathsf{q}\rangle, \\ T_3 &= A_3 + iA_2\langle\mathsf{p},\mathsf{q}\rangle - A_1\langle\Lambda\mathsf{p},\mathsf{q}\rangle + i\langle\Lambda^2\mathsf{p},\mathsf{q}\rangle, \\ T_4 &= A_4 - iA_3\langle\mathsf{p},\mathsf{q}\rangle - A_2\langle\Lambda\mathsf{p},\mathsf{q}\rangle - iA_1\langle\Lambda^2\mathsf{p},\mathsf{q}\rangle - \langle\Lambda^3\mathsf{p},\mathsf{q}\rangle. \end{aligned} \tag{31}$$

Substituting (14)–(17) into these expressions yield compact expressions:

$$\begin{aligned} T_1 &= c, \\ T_2 &= b + \frac{1}{2}|u|^2, \\ T_3 &= a + \frac{1}{2}c|u|^2 - \frac{i}{4}(u'\bar{u} - u\bar{u}'), \\ T_4 &= d + \frac{1}{2}b|u|^2 + \frac{i}{4}c(u'\bar{u} - u\bar{u}') + \frac{1}{8}(u\bar{u}'' + u''\bar{u} \\ &\quad - |u'|^2 + 3|u|^4), \end{aligned} \tag{32}$$

where we have used (29) and introduced another real-valued constant:

$$d = A_4 - F_0A_3 - HA_2 - KA_1 - E. \tag{33}$$

The (1,2) component of the Lax equation (22) is written explicitly in the form:

$$\frac{d}{dx}W_{12}(\lambda) = 2\lambda W_{12}(\lambda) - 2\langle\mathsf{p},\mathsf{p}\rangle W_{11}(\lambda). \tag{34}$$

Substituting (25) and (30) with (28) and (32) into (34) yields constraints at different powers of λ. However, all constraints are satisfied identically at powers of λ^3, λ^2, and λ, whereas the constraint at λ^0, that is, $\frac{d}{dx}S_3 + 2uT_4 = 0$, is equivalent to the fourth-order Lax-Novikov equation:

$$\begin{aligned} &u'''' + 8|u|^2u'' + 2u^2\bar{u}'' + 4u|u'|^2 + 6(u')^2\bar{u} + 6|u|^4u \\ &\quad + 2ic(u''' + 6|u|^2u') + 4b(u'' + 2|u|^2u) \\ &\quad + 8iau' + 16du = 0, \end{aligned} \tag{35}$$

which provides a closed fourth-order equation on u compared to (21).

The fourth-order complex-valued Lax-Novikov equation (35) is integrable with two complex-valued constants of motion. In order to derive them, we recall from Refs. [32,33] that the determinant of $W(\lambda)$ is a constant of motion and has only simple poles at λ_1, λ_2, $-\bar{\lambda}_1$, and $-\bar{\lambda}_2$. On the other hand, it follows from (23), (25), and (30) that

$$\begin{aligned} \det W(\lambda) &= -[W_{11}(\lambda)]^2 - W_{12}(\lambda)\bar{W}_{12}(-\lambda) \\ &= -\frac{P(\lambda)}{(\lambda-\lambda_1)^2(\lambda-\lambda_2)^2(\lambda+\bar{\lambda}_1)^2(\lambda+\bar{\lambda}_2)^2}, \end{aligned} \tag{36}$$

where

$$\begin{aligned} P(\lambda) := {}& (\lambda^4 + iT_1\lambda^3 + T_2\lambda^2 + iT_3\lambda + T_4)^2 - (S_0\lambda^3 + S_1\lambda^2 \\ & + S_2\lambda + S_3)(\bar{S}_0\lambda^3 - \bar{S}_1\lambda^2 + \bar{S}_2\lambda - \bar{S}_3). \end{aligned} \tag{37}$$

Since λ_1, λ_2, $-\bar{\lambda}_1$, and $-\bar{\lambda}_2$ are roots of $P(\lambda)$, substituting (28) and (32) into (37) and evaluating at λ_1 and λ_2 yield two complex-valued constants of motion for the fourth-order Lax-Novikov equation (35). The following symmetry of roots of $P(\lambda)$ provides complex-conjugate symmetry of the two complex-valued constants of motion: *If* λ_0 *is a root of* $P(\lambda)$, *then so is* $-\bar{\lambda}_0$, *thanks to the symmetry of the coefficients in* $P(\lambda)$. As a result, $\det W(\lambda)$ has simple poles at λ_1, λ_2, $-\bar{\lambda}_1$, and $-\bar{\lambda}_2$ in the quotient given by (36).

The admissible values of the algebraic method for λ_1 and λ_2 are defined from the four pairs of roots of $P(\lambda)$ which are symmetric about the imaginary axis. Therefore, we are free to choose λ_1 and λ_2 from any of the four pairs of roots of $P(\lambda)$. In Sec. III, we adopt the algebraic method with two eigenvalues to recover the double-periodic solutions (2) and (3).

III. DOUBLE-PERIODIC WAVES

Truncation of polynomials for $W_{11}(\lambda)$ and $W_{12}(\lambda)$ by setting $S_3 = 0$ and $T_4 = 0$ yield the third-order Lax-Novikov equation studied in Ref. [37]. Since the third-order Lax-Novikov equation contains all solutions written in terms of the Riemann Θ function of genus two including the double-periodic solutions (2) and (3), we restrict our work to exploring this reduction. Setting $S_3 = 0$ and $T_4 = 0$ yields the third-order differential equation,

$$u''' + 6|u|^2u' + 2ic(u'' + 2|u|^2u) + 4bu' + 8iau = 0, \tag{38}$$

and its second-order invariant,

$$\begin{aligned} &d + \frac{1}{2}b|u|^2 + \frac{i}{4}c(u'\bar{u} - u\bar{u}') \\ &\quad + \frac{1}{8}(u\bar{u}'' + u''\bar{u} - |u'|^2 + 3|u|^4) = 0. \end{aligned} \tag{39}$$

The fourth-order equation (35) is now satisfied identically in view of (38) and (39).

Since $P(\lambda)$ is independent on x, substituting (28) and (32) into (37) and expanding in λ yields two more invariants in powers λ and λ^0:

$$2e - a|u|^2 - \frac{1}{4}c(|u'|^2 + |u|^4) + \frac{i}{8}(u''\bar{u}' - u'\bar{u}'') = 0 \tag{40}$$

and

$$\begin{aligned} &f - \frac{i}{2}a(u'\bar{u} - u\bar{u}') + \frac{1}{4}b(|u'|^2 + |u|^4) \\ &\quad + \frac{1}{16}(|u'' + 2|u|^2u|^2 - (u'\bar{u} - u\bar{u}')^2) = 0, \end{aligned} \tag{41}$$

where (e, f) are additional real-valued constants to the previous list (a, b, c, d) defined in (29) and (33). It has been verified directly that (39), (40), and (41) are constants of motion for the third-order equation (38).

With the account of (38)–(41), it follows from (37) that $P(\lambda) = \lambda^2\tilde{P}(\lambda)$, where

$$\begin{aligned} \tilde{P}(\lambda) := {}& \lambda^6 + 2ic\lambda^5 + (2b - c^2)\lambda^4 + 2i(a + bc)\lambda^3 \\ & + (b^2 - 2ac + 2d)\lambda^2 + 2i(e + ab + cd)\lambda \\ & + f + 2bd - 2ce - a^2. \end{aligned} \tag{42}$$

There exist three admissible pairs of eigenvalues found from roots of $\tilde{P}(\lambda)$, two of which can be taken as λ_1 and λ_2. In order to express the three pairs of roots of $\tilde{P}(\lambda)$ explicitly in terms of parameters (a, b, c, d, e, f), we reduce the third-order Lax-Novikov equation (38) to the first-order quadratures. This requires us to set $a = c = e = 0$, whereas the other three constants (b, d, f) are left arbitrarily.

A. Reduction to the first-order quadratures

The following transformation

$$u(x) = \tilde{u}(x)e^{-\frac{2}{3}icx} \tag{43}$$

with

$$\begin{aligned} &b = \tilde{b} - \tfrac{1}{3}c^2, \quad a = \tilde{a} + \tfrac{1}{3}bc + \tfrac{2}{27}c^3, \\ &d = \tilde{d}, \quad e = \tilde{e} - \tfrac{1}{3}cd, \quad f = \tilde{f} + \tfrac{4}{3}ce + \tfrac{2}{9}c^2d \end{aligned}$$

leaves the system (38)–(41) invariant for tilde variables and eliminates the parameter c. Hence $c = 0$ can be set without loss of generality. In addition, we set $a = e = 0$ in order to reduce the system (38)–(41) to the first-order quadratures.

It follows from (40) with $a = c = e = 0$ that

$$\frac{d}{dx}\log\left(\frac{u'}{\bar{u}'}\right) = 0 \Rightarrow \frac{u'}{\bar{u}'} = e^{2i\theta}, \tag{44}$$

where real θ is constant in x. Hence $u'(x) = e^{i\theta}Q'(x)$ with real Q and integrating it again and adding the time variable t, we obtain the following form for the solution to the NLS equation (1):

$$\psi(x,t) = [Q(x,t) + i\delta(t)]e^{i\theta(t)}, \tag{45}$$

where real δ is constant in x. The exact solution to the NLS equation (1) in the form (45) was characterized in Ref. [26] by using separation of variables. Here we characterize the same solution by using the third-order Lax-Novikov equation (38) with the remaining two conserved quantities in (39) and (41). Substituting

$$u = (Q + i\delta)e^{i\theta}, \tag{46}$$

with x-independent δ and θ into the third-order equation (38) with $a = c = 0$ yields

$$Q_{xxx} + 6(Q^2 + \delta^2)Q_x + 4bQ_x = 0. \tag{47}$$

Integrating this third-order equation in x yields the following second-order equation:

$$Q_{xx} + 2Q^3 + 6\delta^2 Q + 4bQ = A, \tag{48}$$

where A is independent of x but may depend on t. Substituting (46) and (48) into (39) with $c = 0$ yields the first-order quadrature:

$$\left(\frac{dQ}{dx}\right)^2 + F(Q) = 0, \tag{49}$$

where

$$F(Q) = Q^4 + (6\delta^2 + 4b)Q^2 - 2AQ - 3\delta^4 - 4b\delta^2 - 8d.$$

Substituting (46), (48), and (49) into (41) with $a = 0$ yields

$$A^2 + 16[(\delta^2 + b)(\delta^4 + b\delta^2 + 2d) + f] = 0. \tag{50}$$

It remains to relate A with δ, for which we use the time evolution of the NLS equation (1). Substituting (45) into (1) and separating the variables yield the following system:

$$\begin{aligned} Q_{xx} + 2(Q^2 + \delta^2 - \dot{\theta})Q - 2\dot{\delta} &= 0 \\ Q_t + (Q^2 + \delta^2 - \dot{\theta})\delta &= 0, \end{aligned} \tag{51}$$

where the dot denotes the derivative of δ and θ in t. Comparing (48) with the first equation of system (51) yields $A = 2\dot{\delta}$ and $\dot{\theta} = -2(\delta^2 + b)$. Setting $z := \delta^2$ reduces (50) with $A = 2\dot{\delta}$ to the first-order quadrature:

$$\left(\frac{dz}{dt}\right)^2 + G(z) = 0, \tag{52}$$

where

$$G(z) = 16z(z^3 + 2bz^2 + (b^2 + 2d)z + f + 2bd).$$

Let us parametrize b, d, and f as follows:

$$\begin{aligned} -2b &= z_1 + z_2 + z_3, \\ 2d + b^2 &= z_1z_2 + z_1z_3 + z_2z_3, \\ f + 2bd &= -z_1z_2z_3. \end{aligned} \tag{53}$$

In this case, the quadratures in (49) and (52) are parameterized by

$$\begin{aligned} F(Q) = Q^4 &+ 2(3z - z_1 - z_2 - z_3)Q^2 - 4\dot{\delta}Q - 3z^2 \\ &+ 2z(z_1 + z_2 + z_3) - 2(z_1z_2 + z_1z_3 + z_2z_3) + z_1^2 \\ &+ z_2^2 + z_3^2 \end{aligned} \tag{54}$$

and

$$G(z) = 16z(z - z_1)(z - z_2)(z - z_3). \tag{55}$$

The polynomial $\tilde{P}(\lambda)$ in (42) is transformed under the parametrization (53) with $a = c = e = 0$ to the form:

$$\begin{aligned} \tilde{P}(\lambda) = \lambda^6 &- (z_1 + z_2 + z_3)\lambda^4 + (z_1z_2 + z_1z_3 + z_2z_3)\lambda^2 \\ &- z_1z_2z_3, \end{aligned} \tag{56}$$

which shows that $\pm\sqrt{z_1}, \pm\sqrt{z_2}, \pm\sqrt{z_3}$ are roots of $\tilde{P}(\lambda)$.

B. Exact solutions with elliptic functions

We obtain the exact solutions of the first-order quadratures (49) and (52) with $F(Q)$ and $G(z)$ given by (54) and (55). Since $z = \delta^2 \geqslant 0$, one of the roots $z_{1,2,3}$ must be positive, whereas the other two roots are either real or complex-conjugate.

Real roots

When the three roots $z_{1,2,3}$ are real, let us order them by $z_1 \leqslant z_2 \leqslant z_3$. From positivity of the solutions $z(t)$ of the first-order quadrature (52) with (55) it follows that $z_3 > 0$ and $z(t) \in [0, z_3]$. Proceeding as in Ref. [26], we replace formally $\dot{\delta} = 2\sqrt{(z_1 - z)(z_2 - z)(z_3 - z)}$ and factorize the quartic polynomial $F(Q)$ in (54) for $z \in [0, z_3]$ by

$$\begin{aligned} F(Q) = [&Q^2 + 2\sqrt{z_3 - z}Q + z_3 + z - z_1 - z_2 \\ &+ 2\sqrt{(z_1 - z)(z_2 - z)}][Q^2 - 2\sqrt{z_3 - z}Q + z_3 + z - z_1 \\ &- z_2 - 2\sqrt{(z_1 - z)(z_2 - z)}], \end{aligned} \tag{57}$$

so that the discriminants of the two quadratic equations are given by

$$D_\pm = z_1 + z_2 - 2z \pm 2\sqrt{(z_1 - z)(z_2 - z)}. \tag{58}$$

It follows from the discriminants (58) that all four roots of $F(Q)$ are complex valued unless $z \leqslant z_1$. Hence $z_1 > 0$ and $z(t) \in [0, z_1]$ so that the three real roots satisfy the following ordering:

$$0 \leqslant z_1 \leqslant z_2 \leqslant z_3. \tag{59}$$

The exact solution of the quadrature (52) for $z(t) \in [0, z_1]$ under the ordering (59) is given by the following explicit expression:

$$z(t) = \frac{z_1z_3\,\mathrm{sn}^2(\mu t; k)}{z_3 - z_1\mathrm{cn}^2(\mu t; k)}, \tag{60}$$

where μ and k are related to (z_1, z_2, z_3) by

$$\mu^2 = 4z_2(z_3 - z_1), \quad k^2 = \frac{z_1(z_3 - z_2)}{z_2(z_3 - z_1)}.$$

The validity of the explicit expression (60) can be verified from (52) and (55) by explicit substitution, see Ref. [35] for details. Extracting the square root from $z = \delta^2$ in such a way that $\delta(t)$ remains smooth in t yields the exact solution:

$$\delta(t) = \frac{\sqrt{z_1z_3}\,\mathrm{sn}(\mu t; k)}{\sqrt{z_3 - z_1\mathrm{cn}^2(\mu t; k)}}, \tag{61}$$

which is a smooth periodic function of t with period $T = 4K(k)/\mu$.

When $z(t) \in [0, z_1]$ under the ordering (59), the four roots of $F(Q)$ in the factorization (57) are real and can be ordered as

$$Q_4 \leqslant Q_3 \leqslant Q_2 \leqslant Q_1. \tag{62}$$

The following explicit expressions for the roots $Q_{1,2,3,4}$ were used in Ref. [26]:

$$\begin{aligned} Q_1 &= \sqrt{z_1 - z} + \sqrt{z_2 - z} + \sqrt{z_3 - z},\\ Q_2 &= -\sqrt{z_1 - z} - \sqrt{z_2 - z} + \sqrt{z_3 - z},\\ Q_3 &= -\sqrt{z_1 - z} + \sqrt{z_2 - z} - \sqrt{z_3 - z},\\ Q_4 &= \sqrt{z_1 - z} - \sqrt{z_2 - z} - \sqrt{z_3 - z}, \end{aligned} \tag{63}$$

and can be verified by the explicit computations from (57). However, since $\sqrt{z_1 - z(t)}$ is nonsmooth at $t = K(k)/\mu$, this parametrization introduces singularities in the definition of $Q(x,t)$. Similarly, one can use the parametrization (63) with $\sqrt{z_1 - z}$ replaced by $-\sqrt{z_1 - z}$ but this choice also introduces singularities in the definition of $Q(x,t)$.

In order to avoid the branch point singularities, we shall parametrize the roots $Q_{1,2,3,4}$ by using the original representation (54) in the form:

$$\begin{aligned} Q_1 &= \sqrt{z_3 - z} + \sqrt{z_1 + z_2 - 2z + \dot{\delta}/\sqrt{z_3 - z}},\\ Q_2 &= \sqrt{z_3 - z} - \sqrt{z_1 + z_2 - 2z + \dot{\delta}/\sqrt{z_3 - z}},\\ Q_3 &= -\sqrt{z_3 - z} + \sqrt{z_1 + z_2 - 2z - \dot{\delta}/\sqrt{z_3 - z}},\\ Q_4 &= -\sqrt{z_3 - z} - \sqrt{z_1 + z_2 - 2z - \dot{\delta}/\sqrt{z_3 - z}}, \end{aligned} \tag{64}$$

where all square roots stay away from zero, so that the definition of $Q(x,t)$ is smooth.

The exact solution of the quadrature (49) with

$$F(Q) = (Q - Q_1)(Q - Q_2)(Q - Q_3)(Q - Q_4)$$

for $Q(x,t) \in [Q_2, Q_1]$ is given by

$$Q(x,t) = Q_4 + \frac{(Q_1 - Q_4)(Q_2 - Q_4)}{(Q_2 - Q_4) + (Q_1 - Q_2)\mathrm{sn}^2(\nu x; \kappa)}, \tag{65}$$

where

$$\begin{aligned} 4\nu^2 &= (Q_1 - Q_3)(Q_2 - Q_4),\\ 4\nu^2\kappa^2 &= (Q_1 - Q_2)(Q_3 - Q_4), \end{aligned} \quad \Rightarrow \quad \begin{aligned} \nu^2 &= z_3 - z_1,\\ \nu^2\kappa^2 &= z_2 - z_1. \end{aligned}$$

The function Q is periodic in x with period $L = 2K(\kappa)/\nu$ and periodic in t with period $T = 4K(k)/\mu$ thanks to the T periodicity of $z(t)$ and $\dot{\delta}(t)$ in (64). Similarly, the exact solution of the quadrature (49) for $Q(x,t) \in [Q_4, Q_3]$ is given by

$$Q(x,t) = Q_2 + \frac{(Q_3 - Q_2)(Q_4 - Q_2)}{(Q_4 - Q_2) + (Q_3 - Q_4)\mathrm{sn}^2(\nu x; \kappa)}, \tag{66}$$

with the same values for ν and κ. Solution (65) and (66) have the same periodicity in x and t.

At $t = 0$, we have $\delta(0) = 0$, whereas $\theta(0) = 0$ can be chosen without loss of generality, thanks to the gauge transformation of the NLS equation (1). In this case, the real-valued function $\psi(x,0) = Q(x,0)$ in (45) coincide with the general periodic wave solution of the modified KdV equation

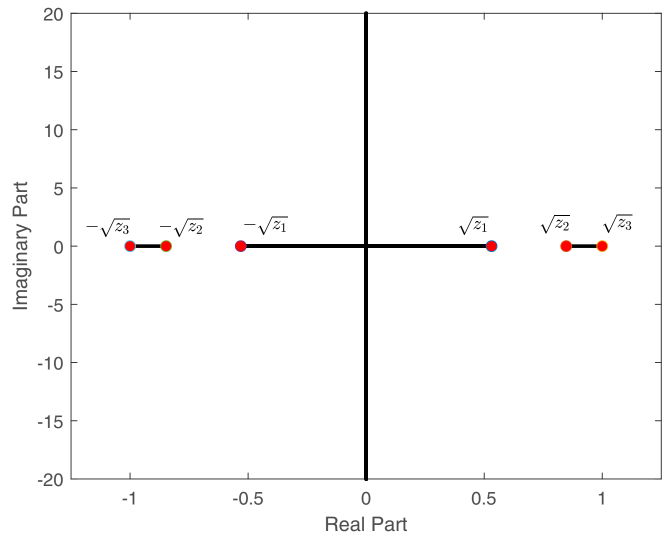

FIG. 2. Lax spectrum for the double-periodic solution (2) with $k = 0.9$. Red dots represent eigenvalues $\pm\sqrt{z_1}$, $\pm\sqrt{z_2}$, and $\pm\sqrt{z_3}$.

considered in Ref. [35]. The roots of the polynomial $\tilde{P}(\lambda)$ in (56) can then be written in the form:

$$\begin{aligned} \lambda_1^{\pm} &= \pm\sqrt{z_1} = \pm\frac{Q_1+Q_4}{2}\bigg|_{t=0},\\ \lambda_2^{\pm} &= \pm\sqrt{z_2} = \pm\frac{Q_1+Q_3}{2}\bigg|_{t=0},\\ \lambda_3^{\pm} &= \pm\sqrt{z_3} = \pm\frac{Q_1+Q_2}{2}\bigg|_{t=0}. \end{aligned} \tag{67}$$

For $t \neq 0$, the complex-valued function $\psi(x,t)$ in (45) describes a double-periodic solution to the NLS equation (1). In the special case $z_3 = z_1 + z_2 = 1$, the exact solution can be simplified to the form (2) derived in Ref. [26]. Appendix A gives technical details of the relevant transformations.

Figure 2 represents the Lax spectrum computed numerically by using the Floquet–Bloch decomposition of solutions to the spectral problem (6) with the periodic potential u in the form (2). The numerical method is described in Appendix A in our previous work [24]. The black curves represent the purely continuous spectrum whereas the red dots represent eigenvalues (67).

Complex-conjugate roots

When one root in $z_{1,2,3}$ is real and positive while the other two roots are complex-conjugate, we can introduce the parametrization:

$$z_1 \geqslant 0, \quad z_2 = \xi + i\eta, \quad z_3 = \xi - i\eta \tag{68}$$

with real-valued ξ and η. The exact solution of the quadrature (52) for $z(t) \in [0, z_1]$ under the parametrization (68) is given by

$$z(t) = \frac{z_1[1 - \mathrm{cn}(\mu t; k)]}{(1 + \zeta) + (\zeta - 1)\mathrm{cn}(\mu t; k)}, \tag{69}$$

where $\zeta > 0$, $\mu > 0$, and $k \in (0, 1)$ are parameters given by

$$\zeta = \frac{\sqrt{(z_1-\xi)^2+\eta^2}}{\sqrt{\xi^2+\eta^2}},$$
$$\mu^2 = 16\sqrt{(z_1-\xi)^2+\eta^2}\sqrt{\xi^2+\eta^2},$$
$$2k^2 = 1 + \frac{\xi(z_1-\xi)-\eta^2}{\sqrt{(z_1-\xi)^2+\eta^2}\sqrt{\xi^2+\eta^2}}.$$

We need to extract the square root from $z = \delta^2$ so that $\delta(t)$ remain smooth in t. To do so, we use the half-argument formula

$$\mathrm{sn}^2\left(\frac{1}{2}\mu t;k\right) = \frac{1-\mathrm{cn}(\mu t;k)}{1+\mathrm{dn}(\mu t;k)},$$

which yields the exact solution:

$$\delta(t) = \frac{\sqrt{z_1}\sqrt{1+\mathrm{dn}(\mu t;k)}}{\sqrt{(1+\zeta)+(\zeta-1)\mathrm{cn}(\mu t;k)}}\mathrm{sn}\left(\frac{1}{2}\mu t;k\right), \quad (70)$$

which is a smooth periodic function of t with period $T = 8K(k)/\mu$.

Since $z_1 \geqslant 0$ is now the largest positive root of $G(z)$, we can rewrite the factorization (57) in the equivalent form:

$$F(Q) = [Q^2+2\sqrt{z_1-z}Q+z_1+z-2\xi+2\sqrt{(\xi-z)^2+\eta^2}]$$
$$\times [Q^2-2\sqrt{z_1-z}Q+z_1+z-2\xi-2\sqrt{(\xi-z)^2+\eta^2}], \quad (71)$$

so that the discriminants of the two quadratic equations are given by

$$D_\pm = 2\xi - 2z \pm 2\sqrt{(\xi-z)^2+\eta^2}. \quad (72)$$

Note that the factorization formula (71) is formal since the sign of $\sqrt{z_1 - z}$ should be defined from smooth continuations of the roots of $F(Q)$. It follows from the discriminants (72) that two roots of $F(Q)$ are real valued, ordered as $Q_2 \leqslant Q_1$, and two roots are complex-conjugate, $Q_3 = \bar{Q}_4$.

The following explicit expressions for the roots $Q_{1,2,3,4}$ were used in Ref. [26]:

$$\begin{aligned} Q_1 &= \sqrt{z_1-z} + \sqrt{2}\sqrt{\sqrt{(\xi-z)^2+\eta^2}+(\xi-z)}, \\ Q_2 &= \sqrt{z_1-z} - \sqrt{2}\sqrt{\sqrt{(\xi-z)^2+\eta^2}+(\xi-z)}, \\ Q_3 &= -\sqrt{z_1-z} + i\sqrt{2}\sqrt{\sqrt{(\xi-z)^2+\eta^2}-(\xi-z)}, \\ Q_4 &= -\sqrt{z_1-z} - i\sqrt{2}\sqrt{\sqrt{(\xi-z)^2+\eta^2}-(\xi-z)}, \end{aligned} \quad (73)$$

however, $\sqrt{z_1 - z(t)}$ is nonsmooth at $t = 2K(k)/\mu$, which introduces singularities in the definition of $Q(x, t)$. In order to avoid the branch point singularities, we replace (73) by

$$Q_1 = \frac{\dot{\delta}}{2\sqrt{(\xi-z)^2+\eta^2}} + \sqrt{2}\sqrt{\sqrt{(\xi-z)^2+\eta^2}+(\xi-z)},$$
$$Q_2 = \frac{\dot{\delta}}{2\sqrt{(\xi-z)^2+\eta^2}} - \sqrt{2}\sqrt{\sqrt{(\xi-z)^2+\eta^2}+(\xi-z)},$$
$$Q_3 = -\frac{\dot{\delta}}{2\sqrt{(\xi-z)^2+\eta^2}} + i\sqrt{2}\sqrt{\sqrt{(\xi-z)^2+\eta^2}-(\xi-z)},$$
$$Q_4 = -\frac{\dot{\delta}}{2\sqrt{(\xi-z)^2+\eta^2}} - i\sqrt{2}\sqrt{\sqrt{(\xi-z)^2+\eta^2}-(\xi-z)}, \quad (74)$$

which is smooth in t. Let us also denote $Q_3 = \alpha + i\beta$ and $Q_4 = \alpha - i\beta$ with real-valued α and β. The exact solution of the quadrature (49) with

$$F(Q) = (Q-Q_1)(Q-Q_2)[(Q-\alpha)^2+\beta^2]$$

for $Q(x, t) \in [Q_2, Q_1]$ is given by

$$Q(x,t) = Q_1 + \frac{(Q_2-Q_1)[1-\mathrm{cn}(\nu x;\kappa)]}{1+\gamma+(\gamma-1)\mathrm{cn}(\nu x;\kappa)}, \quad (75)$$

where $\gamma > 0$, $\nu > 0$, and $\kappa \in (0, 1)$ are parameters given by

$$\gamma = \frac{\sqrt{(Q_2-\alpha)^2+\beta^2}}{\sqrt{(Q_1-\alpha)^2+\beta^2}},$$

and

$$\nu^2 = \sqrt{[(Q_1-\alpha)^2+\beta^2][(Q_2-\alpha)^2+\beta^2]},$$
$$2\kappa^2 = 1 - \frac{(Q_1-\alpha)(Q_2-\alpha)+\beta^2}{\sqrt{[(Q_1-\alpha)^2+\beta^2][(Q_2-\alpha)^2+\beta^2]}},$$

so that

$$\nu^2 = 4\sqrt{(z_1-\xi)^2+\eta^2},$$
$$2\kappa^2 = 1 - \frac{z_1-\xi}{\sqrt{(z_1-\xi)^2+\eta^2}}.$$

The function Q is periodic in x with the period $L = 4K(\kappa)/\nu$ and periodic in t with the period $T = 8K(k)/\mu$ thanks to the T periodicity of $z(t)$ and $\dot{\delta}(t)$ in (74).

At $t = 0$ it follows that $\delta(0) = 0$ while $\theta(0) = 0$ can be chosen without loss of generality. In this case, the real-valued function $\psi(x, 0) = Q(x, 0)$ in (45) coincide with the general periodic wave solution of the modified KdV equation considered in Ref. [35]. The roots of the polynomial $\tilde{P}(\lambda)$ in (56) can then be written in the form:

$$\lambda_1^\pm = \pm\sqrt{z_1} = \pm\left.\frac{Q_1+Q_2}{2}\right|_{t=0} \quad (76)$$

and

$$\lambda_2^\pm = \pm\sqrt{\xi+i\eta} = \pm\left.\left[\frac{1}{4}(Q_1-Q_2)+\frac{i}{2}\beta\right]\right|_{t=0},$$
$$\lambda_3^\pm = \pm\sqrt{\xi-i\eta} = \pm\left.\left[\frac{1}{4}(Q_1-Q_2)-\frac{i}{2}\beta\right]\right|_{t=0}. \quad (77)$$

For $t \neq 0$, the complex-valued function $\psi(x, t)$ in (45) describes a double-periodic solution to the NLS equation (1). In the special case $z_1 = 2\xi$ and $\xi^2 + \eta^2 = \frac{1}{4}$, the exact solution can be simplified to the form (3) derived in Ref. [26]. Appendix B gives technical details of the relevant transformations.

Figure 3 represents the Lax spectrum computed numerically by using the Floquet-Bloch decomposition of solutions to the spectral problem (6) with the periodic potential u in the

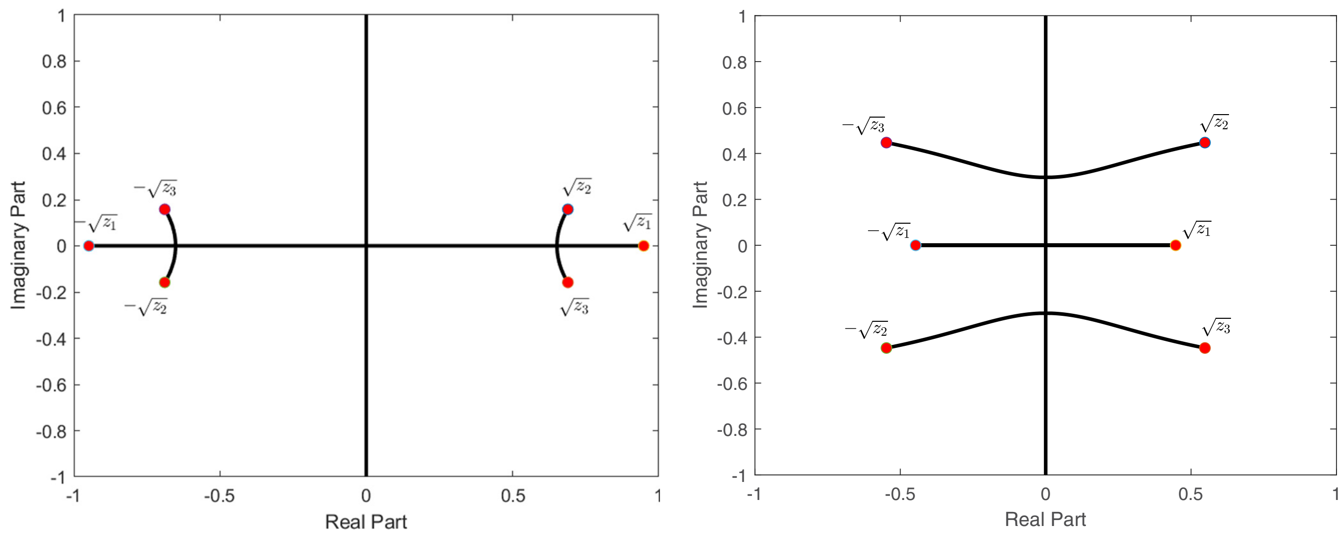

FIG. 3. Lax spectrum for the double-periodic solution (3) with $k = 0.8$ (left) and $k = 0.2$ (right). Red dots represent eigenvalues $\pm\sqrt{z_1}$, $\pm\sqrt{z_2}$, and $\pm\sqrt{z_3}$.

form (3) for two choices of $k \in (0, 1)$. The black curves represent the purely continuous spectrum, whereas the red dots represent eigenvalues (76) and (77). Different reconnections between the bands of the continuous spectrum is observed for $k = 0.8$ (left) and $k = 0.2$ (right).

IV. ROGUE WAVES ON THE DOUBLE-PERIODIC BACKGROUND

Here we characterize the squared eigenfunctions of the Lax operators in (6) and (7) in terms of the solutions u to the third-order Lax-Novikov equation (38). For each pair of admissible eigenvalues λ_1 and λ_2 among roots of the polynomial $\tilde{P}(\lambda)$ in (56), the squared eigenfunctions are double-periodic functions with the same periods as the solution u. The second, linearly independent solution of the linear equations (6) and (7) exist for the same eigenvalues and we characterize the second solution in terms of the double-periodic eigenfunctions similarly to our previous work [24]. The second solution is generally nonperiodic but linearly growing in variables (x, t).

Let us recall the representations (24), (25), and (30) for $W_{11}(\lambda)$ and $W_{12}(\lambda)$ in terms of the squared eigenfunctions and the periodic solution u. Also recall that $S_3 = T_4 = 0$ for the third-order Lax-Novikov equation (38), which we take with $a = c = e = 0$. By computing and comparing the residues of expressions (24), (25), and (30) at the simple poles $\lambda = \lambda_1$ and $\lambda = -\bar{\lambda}_1$, we obtain the explicit expressions:

$$\begin{aligned} p_1^2 &= \frac{\lambda_1}{4(\lambda_1+\bar{\lambda}_1)(\lambda_1-\lambda_2)(\lambda_1+\bar{\lambda}_2)}\left[u'' + 2|u|^2u + 4\left(b+\lambda_1^2\right)u + 2\lambda_1 u'\right], \\ q_1^2 &= \frac{\lambda_1}{4(\lambda_1+\bar{\lambda}_1)(\lambda_1-\lambda_2)(\lambda_1+\bar{\lambda}_2)}\left[\bar{u}'' + 2|u|^2\bar{u} + 4\left(b+\lambda_1^2\right)\bar{u} - 2\lambda_1 \bar{u}'\right], \\ p_1 q_1 &= -\frac{\lambda_1}{4(\lambda_1+\bar{\lambda}_1)(\lambda_1-\lambda_2)(\lambda_1+\bar{\lambda}_2)}\left[u'\bar{u} - u\bar{u}' + 2\lambda_1\left(2b + 2\lambda_1^2 + |u|^2\right)\right]. \end{aligned} \tag{78}$$

Expressions for p_2^2, q_2^2, and $p_2 q_2$ are obtained from (78) after replacing λ_1 and λ_2. There exist three admissible pairs of eigenvalues given by roots $\pm\sqrt{z_1}$, $\pm\sqrt{z_2}$, and $\pm\sqrt{z_3}$ of the polynomial $\tilde{P}(\lambda)$ in (56) and each pair of roots can be taken in place of λ_1 and λ_2.

Let $\varphi = (p_1, q_1)^T$ be a solution of the linear equations (6) and (7) for $\lambda = \lambda_1$. The second, linearly independent solution $\varphi = (\hat{p}_1, \hat{q}_1)^T$ of the same equations is obtained in the form:

$$\hat{p}_1 = p_1\phi_1 - \frac{2\bar{q}_1}{|p_1|^2 + |q_1|^2}, \quad \hat{q}_1 = q_1\phi_1 + \frac{2\bar{p}_1}{|p_1|^2 + |q_1|^2}, \tag{79}$$

where ϕ_1 is to be determined. Wronskian between the two solutions is normalized by $p_1\hat{q}_1 - \hat{p}_1 q_1 = 2$. Substituting (79) into (6) and using (6) for $\varphi = (p_1, q_1)^T$ yield the following first-order equation for ϕ_1:

$$\frac{\partial \phi_1}{\partial x} = F := -\frac{4(\lambda_1 + \bar{\lambda}_1)\bar{p}_1\bar{q}_1}{(|p_1|^2 + |q_1|^2)^2}. \tag{80}$$

Similarly, substituting (79) into (7) and using (7) for $\varphi = (p_1, q_1)^T$ yields another equation for ϕ_1:

$$\begin{aligned} \frac{\partial \phi_1}{\partial t} = G := &-\frac{4i\left(\lambda_1^2 - \bar{\lambda}_1^2\right)\bar{p}_1\bar{q}_1}{(|p_1|^2 + |q_1|^2)^2} \\ &+ \frac{2i(\lambda_1 + \bar{\lambda}_1)\left(u\bar{p}_1^2 + \bar{u}\bar{q}_1^2\right)}{(|p_1|^2 + |q_1|^2)^2}. \end{aligned} \tag{81}$$

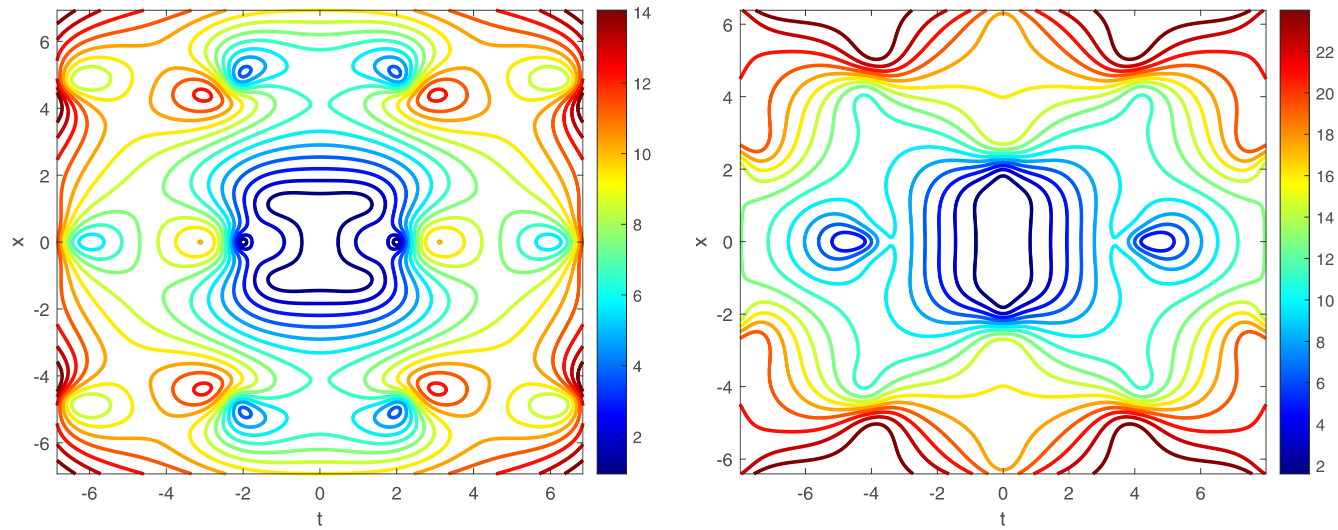

FIG. 4. The level set of $|\phi_1|$ generated for the double-periodic solution (2) with $k = 0.9$ (left) and the double-periodic solution (3) with $k = 0.8$ (right) for the first eigenvalue $\sqrt{z_1}$.

The system of first-order equations (80) and (81) is compatible in the sense $F_t = G_x$ since it is derived from the compatible Lax system (6) and (7). Therefore, it can be solved with the explicit integration formula:

$$\phi_1(x,t) = \int_{x_0}^{x} F(x',t)dx' + \int_{t_0}^{t} G(x_0,t')dt', \tag{82}$$

where (x_0, t_0) is arbitrarily fixed. Figure 4 shows contour plots of $|\phi_1|$ on the (x, t) plane generated for the double-periodic solutions with the same parameters as on Fig. 1. The choice $\lambda_1 = \sqrt{z_1}$ is used here. The value of $|\phi_1|$ increases as (x, t) deviate further away from (x_0, t_0), which is placed at the origin. We have checked that the same behavior holds for every choice of $k \in (0, 1)$ in the double-periodic solutions (2) and (3) and for other two eigenvalues $\sqrt{z_2}$ and $\sqrt{z_3}$.

By using the onefold Darboux transformation (8) with the second solution $\varphi = (\hat{p}_1, \hat{q}_1)^t$ of the linear equations (6) and (7) with $\lambda = \lambda_1$, we obtain a new solution to the NLS equation (1) in the form:

$$\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)\hat{p}_1\bar{\hat{q}}_1}{|\hat{p}_1|^2 + |\hat{q}_1|^2}. \tag{83}$$

This new solution can be rewritten in the explicit form:

$$\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)[p_1(|p_1|^2 + |q_1|^2)\phi_1 - 2\bar{q}_1][\bar{q}_1(|p_1|^2 + |q_1|^2)\bar{\phi}_1 + 2p_1]}{|p_1(|p_1|^2 + |q_1|^2)\phi_1 - 2\bar{q}_1|^2 + |\bar{q}_1(|p_1|^2 + |q_1|^2)\bar{\phi}_1 + 2p_1|^2}, \tag{84}$$

where p_1^2, q_1^2, and p_1q_1 are defined by (78). Figure 5 shows surface plots of $|\hat{u}|$ generated on the double-periodic background (2) with $k = 0.9$ for three different eigenvalues $\sqrt{z_1}$, $\sqrt{z_2}$, and $\sqrt{z_3}$ shown on Fig. 2.

Notice from Fig. 4 that $|\phi_1(x, t)| \to \infty$ as $|x| + |t| \to \infty$ everywhere on the (x, t) plane. The explicit representation (84) implies that

$$\hat{u}|_{|\phi_1|\to\infty} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)p_1\bar{q}_1}{|p_1|^2 + |q_1|^2} =: \tilde{u}. \tag{85}$$

We have verified numerically that $\tilde{u}$ is a translated version of the original double-periodic solution u due to the symmetries of the NLS equation (1):

$$|\tilde{u}(x,t)| = |u(x - \beta, t - \tau)|, \quad \beta, \tau \in \mathbb{R}. \tag{86}$$

In particular, if L and T are fundamental periods of the double-periodic solution (2) in x and t respectively, then we have found that β and τ are independent of $k \in (0, 1)$. Table I lists values of β and τ for the three eigenvalues.

In order to study magnification of the rogue wave, we compute $\hat{u}(x_0, t_0)$ for which $\phi_1(x_0, t_0) = 0$,

$$\hat{u}|_{\phi_1=0} = u - \frac{2(\lambda_1 + \bar{\lambda}_1)p_1\bar{q}_1}{|p_1|^2 + |q_1|^2} = 2u - \tilde{u}. \tag{87}$$

Figure 6 shows how $\max_{(x,t)\in\mathbb{R}^2} |\hat{u}(x, t)|$ depends on (x_0, t_0) for the three rogue waves on Fig. 5. For $\lambda_3 = \sqrt{z_3}$ (bottom panel), the maximal magnification of $|\hat{u}|$ is reached at $(x_0, t_0) = (0, 0)$, for which the argument of $\max_{(x,t)\in\mathbb{R}^2} |\hat{u}(x, t)|$ occurs at $(x, t) = (0, 0)$, where the exact formula (87) can be used. For $\lambda_1 = \sqrt{z_1}$ and $\lambda_2 = \sqrt{z_2}$ (top panel), the maximal magnification occurs at $(x_0, t_0) = (0, 0)$ and additional points, for which $\max_{(x,t)\in\mathbb{R}^2} |\hat{u}(x, t)|$ is nearly the same up to numerical errors. We have checked that the rogue waves with $(x_0, t_0) \neq (0, 0)$ look similar to those on Fig. 5 with $(x_0, t_0) = (0, 0)$ and the maximal value is still attained at the central peak near $(x, t) = (0, 0)$.

Let us now reproduce similar results for the double-periodic background (3). Figure 7 shows the surface plots of $|\hat{u}|$ for the rogue waves generated for the double-periodic

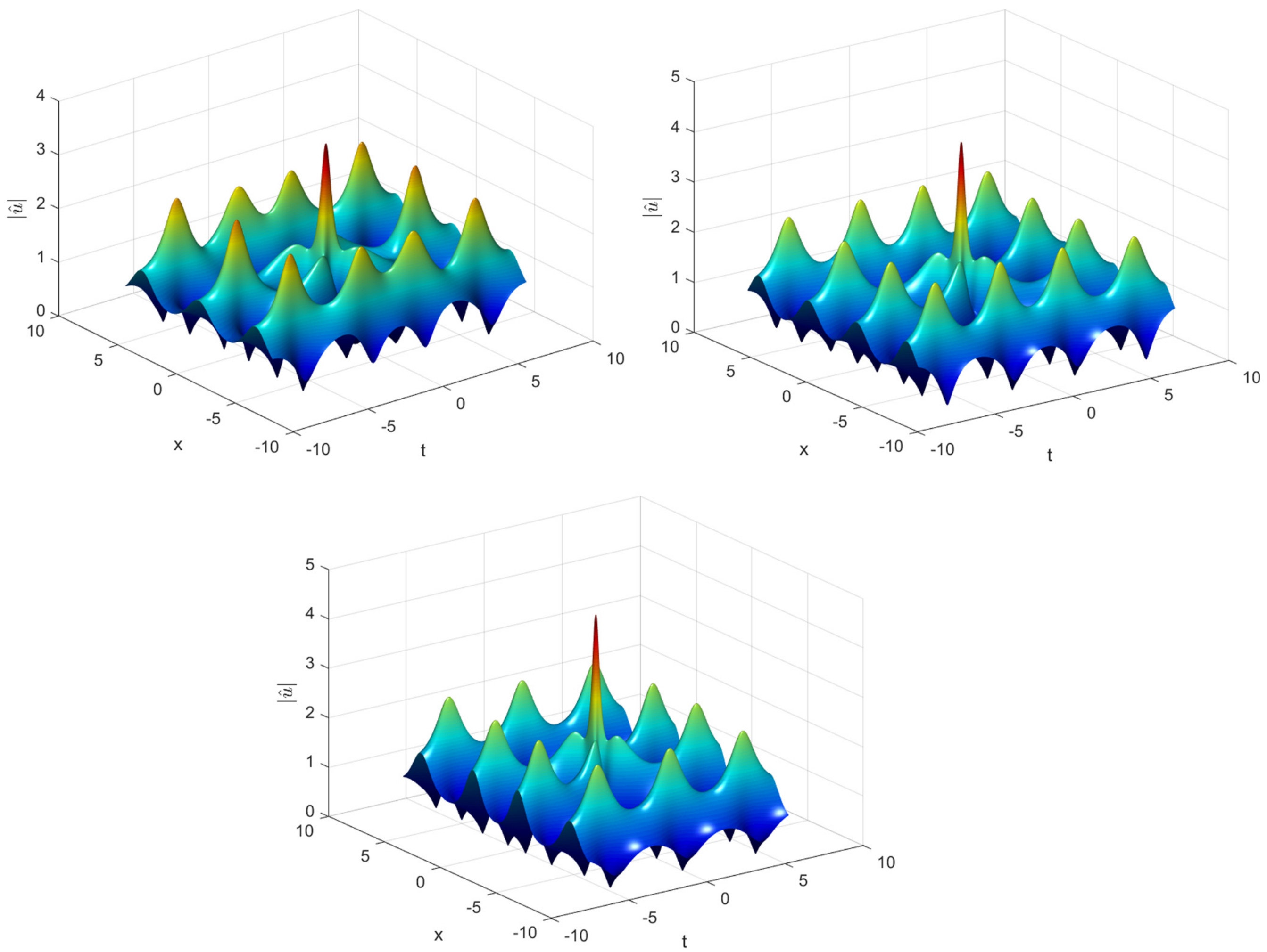

FIG. 5. Rogue waves on the background of the double-periodic solution (2) with $k = 0.9$ for eigenvalues $\lambda_1 = \sqrt{z_1}$ (top left), $\lambda_2 = \sqrt{z_2}$ (top right), and $\lambda_3 = \sqrt{z_3}$ (bottom). In all cases, we set $(x_0, t_0) = (0, 0)$.

solution (3) with $k = 0.8$ (left) and $k = 0.2$ (right) for real eigenvalue $\lambda_1 = \sqrt{z_1}$ (top) and complex eigenvalue $\lambda_2 = \sqrt{\xi + i\eta}$ (bottom) shown on Fig. 3. Rogue waves for the complex-conjugate eigenvalue $\lambda_3 = \sqrt{\xi - i\eta}$ are similar to those for $\lambda_2 = \sqrt{\xi + i\eta}$ thanks to symmetries of the NLS equation (1). Between the rogue waves for eigenvalues λ_1 and λ_2, the maximum of $|\hat{u}|$ is bigger for λ_1 if $k = 0.8$ and for λ_2 if $k = 0.2$. Comparison with Fig. 3 indicates that the rogue wave is larger if it is associated with the eigenvalue, which is most distant from the imaginary axis $\mathrm{Re}(\lambda) = 0$.

We have checked again that $\tilde{u}$ in (85) is the translation of the original double-periodic wave u given by (86) with parameters β and τ given in Table II for every $k \in (0, 1)$.

TABLE I. Parameters of the translations (86) for the double-periodic wave (2).

Rogue wave	β	τ
$\lambda_1 = \sqrt{z_1}$	0	$T/2$
$\lambda_2 = \sqrt{z_2}$	$L/2$	$T/2$
$\lambda_3 = \sqrt{z_3}$	$L/2$	0

Figure 8 shows how $\max_{(x,t)\in\mathbb{R}^2} |\hat{u}(x,t)|$ depends on (x_0, t_0) for two rogue waves on Fig. 7 with largest magnification. The maximal magnification is reached at $(x_0, t_0) = (0, 0)$ or $(x_0, t_0) = (L/2, T/2)$. We have checked numerically that both points are equivalent to each other. This chess pattern of the maximal magnifications resembles the chess pattern of maxima in the double-periodic background (3) shown on Fig. 1 (right). Note that if $\max_{(x,t)\in\mathbb{R}^2} |\hat{u}(x,t)|$ is computed for a rogue wave with smaller magnification (e.g., for the rogue wave on the top right panel of Fig. 7), then there exist many points of the maximal magnification, which are nearly equivalent to each other.

We can finally address the question on multiplication factors for the rogue waves on the double-periodic background. Since $\tilde{u}$ is a translated version of u, it follows from (87) that

$$|\hat{u}(x_0, t_0)| \leqslant 3 \max_{(x,t)\in\mathbb{R}^2} |u(x,t)|. \tag{88}$$

Let us recall from Figs. 6 and 8 that the maximal magnification of $|\hat{u}|$ is reached at $(x_0, t_0) = (0, 0)$ and the argument of $\max_{(x,t)\in\mathbb{R}^2} |\hat{u}(x,t)|$ with $(x_0, t_0) = (0, 0)$ occurs at $(x, t) = (0, 0)$. If the magnification factor of the rogue wave is defined

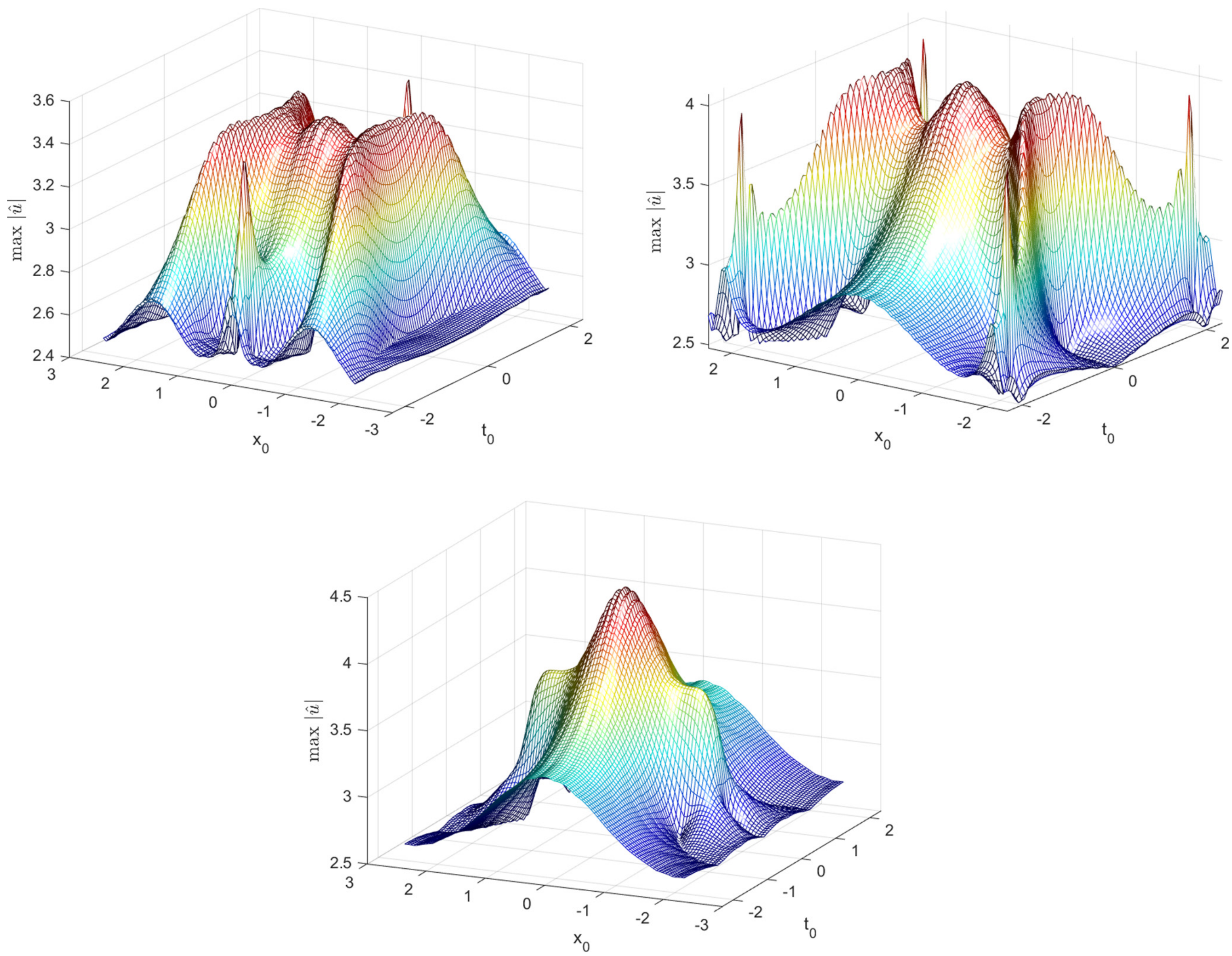

FIG. 6. Magnification given by $\max_{(x,t)\in\mathbb{R}^2} |\hat{u}(x,t)|$ versus (x_0, t_0) for rogue waves generated from the double-periodic solution (2) with $k = 0.9$ for $\lambda_1 = \sqrt{z_1}$ (top left), $\lambda_2 = \sqrt{z_2}$ (top right), and $\lambda_3 = \sqrt{z_3}$ (bottom).

by

$$M_1 := \frac{\max_{(x,t)\in\mathbb{R}^2} |\hat{u}(x,t)|}{\max_{(x,t)\in\mathbb{R}^2} |u(x,t)|}, \tag{89}$$

then it follows from (88) that $M_1 \leqslant 3$. In fact, M_1 does not exceed the double factor for all rogue waves constructed on Figs. 5 and 7, as is shown in Table III.

On the other hand, if the magnification factor is defined by

$$M_2 := \frac{\max_{(x,t)\in\mathbb{R}^2} |\hat{u}(x,t)|}{\mathrm{mean}_{(x,t)\in\mathbb{R}^2} |u(x,t)|}, \tag{90}$$

as in physical experiments (see Ref. [36]), then M_2 exceeds the triple factor for all rogue waves constructed on Figs. 5 and 7 as shown in Table III. Thus, all rogue waves constructed here correspond to physically acceptable rogue waves on the double-periodic background.

TABLE II. Parameters of the translations (86) for the double-periodic wave (3).

Rogue wave	β	τ
$\lambda_1 = \sqrt{z_1}$	$L/2$	0
$\lambda_2 = \sqrt{\xi + i\eta}$	$L/4$	$3T/4$

V. CONCLUSION

We have constructed analytically and studied numerically the rogue wave patterns appearing on the background of the double-periodic solutions to the focusing NLS equations. The analytical part relies on the algebraic method with two eigenvalues and fully characterizes eigenvalues and squared eigenfunctions of the Lax equations associated with the double-periodic solutions. The numerical part explores explicit representations of eigenfunctions of the Lax equations and the rogue waves in terms of integrals computed from the double-periodic solutions. We have argued that the properly defined magnification factor exceeds the triple value for all rogue waves appearing on the double-periodic background.

This work opens up a number of new directions in the study of rogue waves modeled by the focusing NLS equation. First, rogue waves on the continuous wave background were

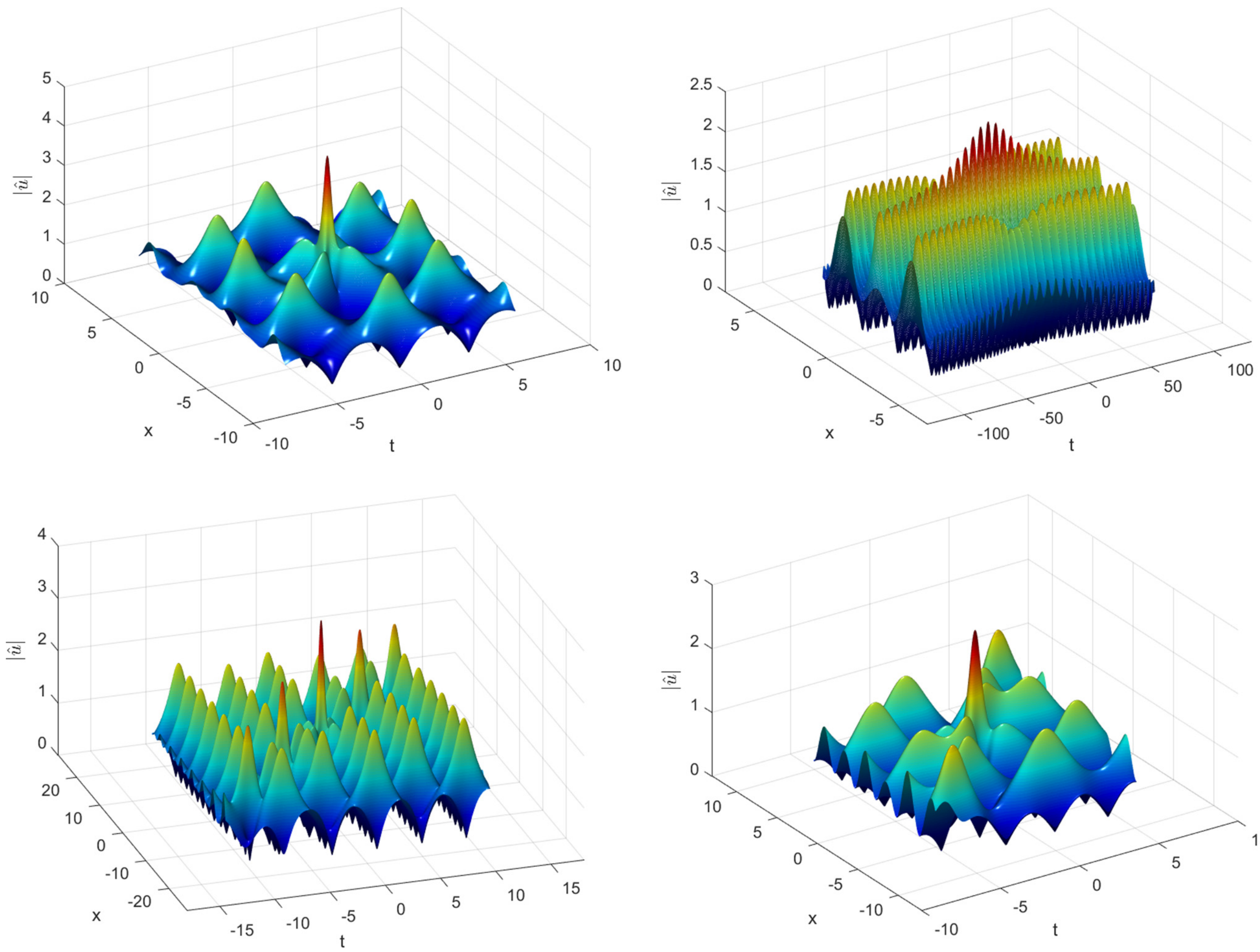

FIG. 7. Rogue waves generated on the background of the double-periodic solution (3) with $k = 0.8$ (left) and $k = 0.2$ (right) for the real eigenvalue $\lambda_1 = \sqrt{z_1}$ (top) and the complex eigenvalue $\lambda_2 = \sqrt{z_2}$ (bottom). In all cases, we set $(x_0, t_0) = (0, 0)$.

observed experimentally either in water tanks or in laser optics [10], hence experiments with rogue waves on the double-periodic background are feasible, particularly, in the hydrodynamical experiments [30]. Second, rogue waves on the double-periodic background could be relevant to diagnostics of rogue waves on the ocean surface and understanding random waves formed due to the modulation instability [29]. Third, it is interesting to understand how to measure the

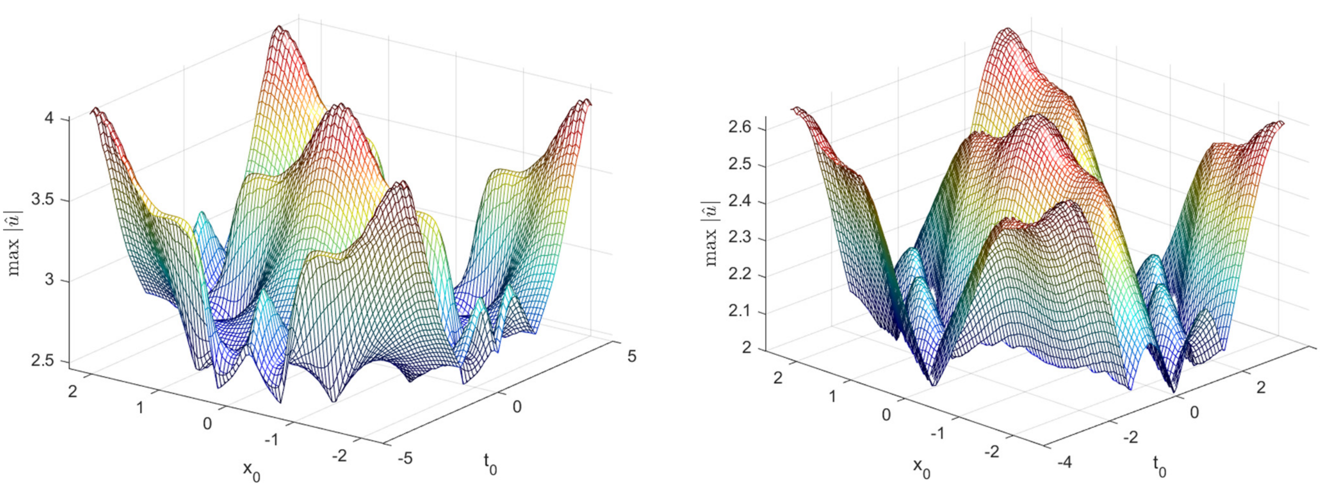

FIG. 8. Magnification given by $\max_{(x,t)\in\mathbb{R}^2} |\hat{u}(x,t)|$ versus (x_0, t_0) for rogue waves generated from the double-periodic solution (3) with $k = 0.8$ for $\lambda_1 = \sqrt{z_1}$ (left) and $k = 0.2$ for $\lambda_2 = \sqrt{\xi + i\eta}$ (right).

TABLE III. Magnification factors for the rogue waves constructed on Figs. 5 and 7.

Rogue wave	Solution	M_1 in (89)	M_2 in (90)
$\lambda_1 = \sqrt{z_1}$	(2) with $k = 0.9$	1.45	3.96
$\lambda_2 = \sqrt{z_2}$	Same	1.71	4.68
$\lambda_3 = \sqrt{z_3}$	Same	1.84	5.03
$\lambda_1 = \sqrt{z_1}$	(3) with $k = 0.8$	1.80	4.67
$\lambda_2 = \sqrt{\xi + i\eta}$	Same	1.60	4.15
$\lambda_1 = \sqrt{z_1}$	(3) with $k = 0.2$	1.58	3.55
$\lambda_2 = \sqrt{\xi + i\eta}$	Same	1.71	3.84

modulation instability gains on the double-periodic background and to compare it with the modulation instability gains for the constant and periodic wave backgrounds [21,24]. Finally, it may be interesting to develop the algebraic method further and to provide the explicit characterization of eigenvalues and squared eigenfunctions for the most general solution of the third-order and higher-order Lax-Novikov equations. The latter problem is related to characterization of parameters of the Lax-Novikov equations in terms of parameters of the Riemann Θ function of genus two and higher.

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF THE EXPLICIT SOLUTION (2) FROM (60) AND (65)

Setting $z_1 + z_2 = z_3$ and $z_3 = 1$ in the solution (60) yields $\mu = 2z_2$, $k = z_1/z_2$, and

$$z(t) = \frac{z_1 \mathrm{sn}^2(\mu t; k)}{z_2 + z_1 \mathrm{sn}^2(\mu t; k)}. \tag{A1}$$

By using the ascending Landen transformation (see 16.14 in Ref. [42]),

$$\mathrm{sn}(\mu t; k) = \frac{\mu \mathrm{sn}(t; k_0)\mathrm{cn}(t; k_0)}{\mathrm{dn}(t; k_0)}, \quad k_0 := \frac{2\sqrt{k}}{1+k} = 2\sqrt{z_1 z_2},$$

the solution (A1) is transformed to the form

$$z(t) = \frac{k_0^2 \mathrm{sn}^2(t; k_0)\mathrm{cn}^2(t; k_0)}{1 - k_0^2 \mathrm{sn}^4(t; k_0)}, \tag{A2}$$

so that we extract smoothly the square root and obtain

$$\delta(t) = \frac{k_0 \mathrm{sn}(t; k_0)\mathrm{cn}(t; k_0)}{\sqrt{1 - k_0^2 \mathrm{sn}^4(t; k_0)}}. \tag{A3}$$

In order to express $Q_{1,2}$ and $Q_{3,4}$ from $z(t)$ and $\dot{\delta}(t)$ by (64), we obtain directly

$$\sqrt{z_3 - z} = \frac{\mathrm{dn}(t; k_0)}{\sqrt{1 - k_0^2 \mathrm{sn}^4(t; k_0)}} \tag{A4}$$

and

$$\sqrt{z_1 + z_2 - 2z \pm \frac{\dot{\delta}}{\sqrt{z_3 - z}}} = \frac{\sqrt{1 \pm k_0}[1 \mp k_0 \mathrm{sn}^2(t; k_0)]}{\sqrt{1 - k_0^2 \mathrm{sn}^4(t; k_0)}}, \tag{A5}$$

where the last equality follows from (A2)–(A4) which imply

$$\begin{aligned} z_1 + z_2 - 2z \pm \frac{\dot{\delta}}{\sqrt{z_3 - z}} &= 1 - \frac{2k_0^2 \mathrm{sn}^2(t; k_0)\mathrm{cn}^2(t; k_0)}{1 - k_0^2 \mathrm{sn}^4(t; k_0)} \pm \frac{k_0[\mathrm{cn}^2(t; k_0) - \mathrm{sn}^2(t; k_0)\mathrm{dn}^2(t; k_0)]}{1 - k_0^2 \mathrm{sn}^4(t; k_0)} \\ &= \frac{(1 \pm k_0)[1 \mp k_0 \mathrm{sn}^2(t; k_0)]^2}{1 - k_0^2 \mathrm{sn}^4(t; k_0)}. \end{aligned}$$

Substituting (A4) and (A5) back into (64) yields

$$Q_{1,2} = \frac{\mathrm{dn}(t; k_0) \pm \sqrt{1 + k_0}[1 - k_0 \mathrm{sn}^2(t; k_0)]}{\sqrt{1 - k_0^2 \mathrm{sn}^4(t; k_0)}} \tag{A6}$$

and

$$Q_{3,4} = \frac{-\mathrm{dn}(t; k_0) \pm \sqrt{1 - k_0}[1 + k_0 \mathrm{sn}^2(t; k_0)]}{\sqrt{1 - k_0^2 \mathrm{sn}^4(t; k_0)}}. \tag{A7}$$

Computing parameters of the expression (65) yields $\nu = \sqrt{z_2}$ and $\kappa^2 = 1 - \frac{z_1}{z_2}$. We shall now use the descending Landen transformation (see 16.12 in Ref. [42]),

$$\mathrm{sn}(\nu x, \kappa) = \frac{(1 + \kappa_0)\mathrm{sn}(\nu_0 x; \kappa_0)}{1 + \kappa_0 \mathrm{sn}^2(\nu_0 x; \kappa_0)}, \quad \nu_0 = \frac{\sqrt{1 + k_0}}{2}, \quad \kappa_0 = \sqrt{\frac{1 - k_0}{1 + k_0}}, \tag{A8}$$

thanks to the following formulas:

$$\kappa_0 := \frac{1-\sqrt{1-\kappa^2}}{1+\sqrt{1-\kappa^2}} = \frac{\sqrt{z_2}-\sqrt{z_1}}{\sqrt{z_2}+\sqrt{z_1}} = \frac{1-\sqrt{k}}{1+\sqrt{k}} = \sqrt{\frac{1-k_0}{1+k_0}}$$

and

$$\nu_0 := \frac{\nu}{1+\kappa_0} = \frac{\sqrt{z_2}\sqrt{1+k_0}}{\sqrt{1+k_0}+\sqrt{1-k_0}} = \frac{\sqrt{1+k_0}}{2},$$

where $k_0 = \frac{2\sqrt{k}}{1+k} = 2\sqrt{z_2(1-z_2)}$ has been used. Substituting (A6) and (A7) into (65) gives

$$Q(x,t) = \frac{1}{\sqrt{1-k_0^2\mathrm{sn}^4(t;k_0)}}\frac{K_1}{K_2}, \tag{A9}$$

where

$$K_1 = 2\mathrm{dn}^2(t;k_0) + \mathrm{dn}(t;k_0)\{\sqrt{1+k_0}[1-k_0\mathrm{sn}^2(t;k_0)] + \sqrt{1-k_0}[1+k_0\mathrm{sn}^2(t;k_0)]\} + \sqrt{1-k_0^2}\big[1-k_0^2\mathrm{sn}^4(t;k_0)\big]$$
$$-(1+k_0)[1-k_0\mathrm{sn}^2(t;k_0)]^2 - 2\sqrt{1+k_0}\{1-k_0\mathrm{sn}^2(t;k_0)][\mathrm{dn}(t;k_0)+\sqrt{1-k_0}[1+k_0\mathrm{sn}^2(t;k_0)]\}\mathrm{sn}^2(\nu x;\kappa)$$

and

$$K_2 = 2\mathrm{dn}(t;k_0) - \sqrt{1+k_0}[1-k_0\mathrm{sn}^2(t;k_0)] + \sqrt{1-k_0}[1+k_0\mathrm{sn}^2(t;k_0)] + 2\sqrt{1+k_0}[1-k_0\mathrm{sn}^2(t;k_0)]\mathrm{sn}^2(\nu x;\kappa).$$

Substituting (A8) as $\mathrm{sn}^2(\nu x;\kappa)$ and performing computations with Jacobian elliptic functions shows that the expressions for K_1 and K_2 can be factorized as follows, which can be further simplified to be

$$K_1 = Kk_0\big\{\sqrt{1+k_0}\mathrm{sn}^2(t;k_0)\mathrm{dn}(t;k_0)\big[1-2\kappa_0^2\mathrm{sn}^2(\nu_0x;\kappa_0)+\kappa_0^2\mathrm{sn}^4(\nu_0x;\kappa_0)\big] + \mathrm{cn}^2(t;k_0)\big[1-2\mathrm{sn}^2(\nu_0x;\kappa_0)+\kappa_0^2\mathrm{sn}^4(\nu_0x;\kappa_0)\big]\big\}$$

and

$$K_2 = K\big\{\sqrt{1+k_0}\big[1-2\kappa_0^2\mathrm{sn}^2(\nu_0x;\kappa_0)+\kappa_0^2\mathrm{sn}^4(\nu_0x;\kappa_0)\big] - \mathrm{dn}(t;k_0)\big[1-2\mathrm{sn}^2(\nu_0x;\kappa_0)+\kappa_0^2\mathrm{sn}^4(\nu_0x;\kappa_0)\big]\big\},$$

where

$$K = \frac{2(\mathrm{dn}(t;k_0)+\sqrt{1-k_0})}{(\sqrt{1+k_0}-\sqrt{1-k_0})[1+\kappa_0\mathrm{sn}^2(\nu_0x;\kappa_0)]^2}.$$

We shall now use the half-argument formula (see 16.18 in Ref. [42]),

$$\mathrm{sn}^2(\nu_0x;\kappa_0) = \frac{1-\mathrm{cn}(2\nu_0x;\kappa_0)}{1+\mathrm{dn}(2\nu_0x;\kappa_0)},$$

which yields

$$1-2\kappa_0^2\mathrm{sn}^2(\nu_0x;\kappa_0)+\kappa_0^2\mathrm{sn}^4(\nu_0x;\kappa_0) = \frac{2\mathrm{dn}(2\nu_0x;\kappa_0)\big[1-\kappa_0^2+\mathrm{dn}(2\nu_0x;\kappa_0)+\kappa_0^2\mathrm{cn}(2\nu_0x;\kappa_0)\big]}{[1+\mathrm{dn}(2\nu_0x;\kappa_0)]^2}$$

and

$$1-2\mathrm{sn}^2(\nu_0x;\kappa_0)+\kappa_0^2\mathrm{sn}^4(\nu_0x;\kappa_0) = \frac{2\mathrm{cn}(2\nu_0x;\kappa_0)\big[1-\kappa_0^2+\mathrm{dn}(2\nu_0x;\kappa_0)+\kappa_0^2\mathrm{cn}(2\nu_0x;\kappa_0)\big]}{[1+\mathrm{dn}(2\nu_0x;\kappa_0)]^2}.$$

As a result, the expression (A9) with K_1 and K_2 given above simplifies to the explicit expressions:

$$Q(x,t) = \frac{k_0}{\sqrt{1-k_0^2\mathrm{sn}^4(t;k_0)}}\frac{\sqrt{1+k_0}\mathrm{sn}^2(t;k_0)\mathrm{dn}(t;k_0)\mathrm{dn}(2\nu_0x;\kappa_0)+\mathrm{cn}^2(t;k_0)\mathrm{cn}(2\nu_0x;\kappa_0)}{\sqrt{1+k_0}\mathrm{dn}(2\nu_0x;\kappa_0)-\mathrm{dn}(t;k_0)\mathrm{cn}(2\nu_0x;\kappa_0)}. \tag{A10}$$

Computing $\dot{\theta}$ from $\dot{\theta} = z_1+z_2+z_3-2z$ yields

$$\dot{\theta} = 2[1-z(t)] = \frac{2\mathrm{dn}^2(t;k_0)}{1-k_0^2\mathrm{sn}^4(t;k_0)}. \tag{A11}$$

We claim that

$$\theta(t) = t + \arctan\varphi(t), \quad \varphi(t) := \frac{\mathrm{sn}(t;k_0)\mathrm{dn}(t;k_0)}{\mathrm{cn(t;k_0)}}. \tag{A12}$$

Indeed, by chain rule, we obtain

$$\dot{\theta} = 1 + \frac{\mathrm{dn}^2(t;k_0) - k_0^2\mathrm{sn}^2(t;k_0)\mathrm{cn}^2(t;k_0)}{\mathrm{cn}^2(t;k_0) + \mathrm{sn}^2(t;k_0)\mathrm{dn}^2(t;k_0)},$$

which recovers (A11). By using the following elementary formulas:

$$\cos(\arctan\varphi) = \frac{1}{\sqrt{1+\varphi^2}}, \quad \sin(\arctan\varphi) = \frac{\varphi}{\sqrt{1+\varphi^2}},$$

we substitute (A12) into (45) and obtain the explicit expression:

$$\psi(x,t) = \frac{Q(x,t) - \delta(t)\varphi(t) + i\delta(t) + iQ(x,t)\varphi(t)}{\sqrt{1+\varphi^2(t)}} e^{it}. \tag{A13}$$

Substituting (A2), (A10), and (A12) into (A13) results in the explicit solution (2) with k_0 and κ_0 being written again as k and κ in (2) for the sake of notations.

APPENDIX B: DERIVATION OF THE EXPLICIT SOLUTION (3) FROM (69) AND (75)

Setting $z_1 = 2\xi$ in the solution (69) yields $\zeta = 1$. By using the scaling invariance, we can also normalize the solution by $\xi^2 + \eta^2 = \frac{1}{4}$ which yields $\mu = 2$. Furthermore, k is related to ξ by $k = 2\xi$. This simplifies the expression (69) to the form:

$$z(t) = k\frac{1 - \mathrm{cn}(2t;k)}{2}, \tag{B1}$$

where k is a free parameter. With the help of the double argument formula (see 16.18.4 in Ref. [42]), the exact solution (B1) can be rewritten in the form:

$$z(t) = \frac{k\mathrm{sn}^2(t;k)\mathrm{dn}^2(t;k)}{1 - k^2\mathrm{sn}^4(t;k)}, \tag{B2}$$

so that we extract smoothly the square root and obtain

$$\delta(t) = \frac{\sqrt{k}\mathrm{sn}(t;k)\mathrm{dn}(t;k)}{\sqrt{1 - k^2\mathrm{sn}^4(t;k)}}. \tag{B3}$$

In order to express $Q_{1,2}$ and $Q_{3,4}$ from $z(t)$ and $\dot{\delta}(t)$ by (74), we obtain directly:

$$z_1 - z = \frac{k\mathrm{cn}^2(t;k)}{1 - k^2\mathrm{sn}^4(t;k)}$$

and

$$\xi - z = \frac{k[1 - 2\mathrm{sn}^2(t;k) + k^2\mathrm{sn}^4(t;k)]}{2[1 - k^2\mathrm{sn}^4(t;k)]}.$$

We claim the following remarkable formula:

$$\sqrt{(\xi - z)^2 + \eta^2} \pm (\xi - z) = \frac{(1 \pm k)[1 \mp k\mathrm{sn}^2(t;k)]}{2[1 \pm k\mathrm{sn}^2(t;k)]}. \tag{B4}$$

Indeed, we have

$$\begin{aligned}(\xi - z)^2 + \eta^2 &= \frac{k^2[1 - 2\mathrm{sn}^2(t;k) + k^2\mathrm{sn}^4(t;k)]^2 + (1 - k^2)[1 - k^2\mathrm{sn}^4(t;k)]^2}{4[(1 - k^2\mathrm{sn}^4(t;k)]^2} = \frac{[1 - 2k^2\mathrm{sn}^2(t;k) + k^2\mathrm{sn}^4(t;k)]^2}{4[(1 - k^2\mathrm{sn}^4(t;k)]^2}\\ &= \left\{\frac{(1 \pm k)[1 \mp k\mathrm{sn}^2(t;k)]}{2[1 \pm k\mathrm{sn}^2(t;k)]} \mp \frac{k[1 - 2\mathrm{sn}^2(t;k) + k^2\mathrm{sn}^4(t;k)]}{2[1 - k^2\mathrm{sn}^4(t;k)]}\right\}^2,\end{aligned}$$

which proves (B4). As a result, we obtain from (74) that

$$Q_{1,2} = \frac{\sqrt{k}\mathrm{cn}(t;k) \pm \sqrt{k+1}[1 - k\mathrm{sn}^2(t;k)]}{\sqrt{1 - k^2\mathrm{sn}^4(t;k)}}$$

and

$$\alpha = -\frac{\sqrt{k}\mathrm{cn}(t;k)}{\sqrt{1 - k^2\mathrm{sn}^4(t;k)}}, \quad \beta = \frac{\sqrt{1-k}[1 + k\mathrm{sn}^2(t;k)]}{\sqrt{1 - k^2\mathrm{sn}^4(t;k)}}.$$

Computing parameters of the expression (75) yields $\nu = \sqrt{2}$ and $2\kappa^2 = 1-k$. In order to compute γ, we obtain

$$\gamma^2 = \frac{(Q_2-\alpha)^2+\beta^2}{(Q_1-\alpha)^2+\beta^2} = \frac{1-2k^2\mathrm{sn}^2(t;k)+k^2\mathrm{sn}^4(t;k)+2k\mathrm{cn}^2(t;k)-2\sqrt{k(1+k)}\mathrm{cn}(t;k)[1-k\mathrm{sn}^2(t;k)]}{1-2k^2\mathrm{sn}^2(t;k)+k^2\mathrm{sn}^4(t;k)+2k\mathrm{cn}^2(t;k)+2\sqrt{k(1+k)}\mathrm{cn}(t;k)[1-k\mathrm{sn}^2(t;k)]},$$

which simplifies to

$$\gamma = \frac{1-2k^2\mathrm{sn}^2(t;k)+k^2\mathrm{sn}^4(t;k)+2k\mathrm{cn}^2(t;k)-2\sqrt{k(1+k)}\mathrm{cn}(t;k)[1-k\mathrm{sn}^2(t;k)]}{1-k^2\mathrm{sn}^4(t;k)}$$

and leads to

$$\gamma+1 = \frac{2[1+k-\sqrt{k(1+k)}\mathrm{cn}(t;k)]}{1+k\mathrm{sn}^2(t;k)}, \quad \gamma-1 = \frac{2[k\mathrm{cn}^2(t;k)-\sqrt{k(1+k)}\mathrm{cn}(t;k)]}{1+k\mathrm{sn}^2(t;k)}. \tag{B5}$$

Substituting $Q_{1,2}$ and (B5) into (75) yields after some lengthy but direct computations:

$$Q(x,t) = \frac{k\sqrt{k(1+k)}\mathrm{cn}(t;k)\mathrm{sn}^2(t;k)+\mathrm{dn}^2(t;k)\mathrm{cn}(\sqrt{2}x;\kappa)}{\sqrt{1-k^2\mathrm{sn}^4(t;k)}[\sqrt{1+k}-\sqrt{k}\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa)]}. \tag{B6}$$

Recall the representation (45) with $Q(x,t)$ in (B6) and $\delta(t)$ in (B3). Computing $\dot{\theta}$ from $\dot{\theta} = z_1+z_2+z_3-2z$ yields

$$\dot{\theta} = 2[k-z(t)] = \frac{2k\mathrm{cn}^2(t;k)}{1-k^2\mathrm{sn}^4(t;k)}. \tag{B7}$$

We claim that

$$\theta(t) = kt + \arctan\varphi(t), \quad \varphi(t) := \frac{k\mathrm{sn}(t;k)\mathrm{cn}(t;k)}{\mathrm{dn}(t;k)}. \tag{B8}$$

Indeed, by chain rule we obtain

$$\dot{\theta} = k + \frac{k\{\mathrm{dn}^2(t;k)[\mathrm{cn}^2(t;k)-\mathrm{sn}^2(t;k)]+k^2\mathrm{cn}^2(t;k)\mathrm{sn}^2(t;k)\}}{\mathrm{dn}^2(t;k)+k^2\mathrm{cn}^2(t;k)\mathrm{sn}^2(t;k))},$$

which recovers (B7). Substituting (B2), (B6), and (B8) into

$$\psi(x,t) = \frac{Q(x,t)-\delta(t)\varphi(t)+i\delta(t)+iQ(x,t)\varphi(t)}{\sqrt{1+\varphi^2(t)}}e^{ikt}.$$

yields the explicit solution (3).

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