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Universal power law for the energy spectrum of breaking Riemann waves

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The universal power law for the spectrum of one-dimensional breaking Riemann waves is justified for the simple wave equation. The spectrum of spatial amplitudes at the breaking time $t = t_b$ has an asymptotic decay of $k^{-4/3}$, with corresponding energy spectrum decaying as $k^{-8/3}$. This spectrum is formed by the singularity of the form $(x - x_b)^{1/3}$ in the wave shape at the breaking time. This result remains valid for arbitrary nonlinear wave speed. In addition, we demonstrate numerically that the universal power law is observed for long time in the range of small wave numbers if small dissipation or dispersion is accounted in the viscous Burgers or Korteweg–de Vries equations.

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I. Introduction. One-dimensional traveling nonlinear waves in dispersiveless systems are called Riemann or simple waves; their dynamics is well studied in various physical media, e.g. acoustic waves in the compressible fluids and gases [1, 2], surface and internal waves in oceans [3–7], tidal and tsunami waves in rivers [8, 9], ion- and magneto-sound waves in plasmas [10], electromagnetic waves in transmission lines [11], and optical tsunami in fiber optics [12]. In homogeneous and stationary media, the Riemann waves continuously deform and transform to the shock waves yielding breaking in a finite time.

In nonlinear acoustics where the wave intensity is not very high, the nonlinear deformation of the Riemann wave has been studied in many details since this stage occurs during many wavelengths [13, 14, 15, 16]. Corresponding nonlinear evolution equation is a so-called simple wave equation

$$u_t + V(u)u_x = 0, \tag{1}$$

where u is a wave function, and $V(u)$ is a characteristic local velocity of the various points of the wave shape. If $V'(u) > 0$ for all u (that is, if V is invertible), equation (1) can be written for the local velocity:

$$V_t + VV_x = 0, \tag{2}$$

which is equivalent to the inviscid Burgers equation

$$v_t + vv_x = 0, \quad v := V(u). \tag{3}$$

The Cauchy problem for Eq. (3) starts with the initial condition given by a smooth function that decays to zero at infinity:

$$v(x,0) = F(x), \qquad \lim_{|x|\to\infty} F(x) = 0, \tag{4}$$

and results in the implicit solution called a simple or Riemann wave:

$$v(x,t) = F\left[x - tv(x,t)\right]. \tag{5}$$

We are interested in the asymptotic behavior of these solutions near the breaking time $t = t_b$. The appearance of the singularity in the wave shape yields a power law in the Fourier spectrum of wave turbulence [17]. It was argued earlier in [18] that for simple (Riemann) waves this singularity is of the form $v - v_b \sim (x_b - x)^{1/2}$; this corresponds to the spectrum of spatial amplitudes decaying at the breaking time as $k^{-3/2}$, with corresponding energy spectrum decaying as k^{-3}.

However, we will show that this assumption is in fact incorrect and the spectrum of spatial amplitudes decays at the breaking time as $k^{-4/3}$, with corresponding energy spectrum decaying as $k^{-8/3}$, because the simple waves develop singularities of the form $v - v_b \sim (x - x_b)^{1/3}$. This analytical result confirms earlier numerical observations in Ref. [19, 20].

We note that the wave field in the vicinity of the breaking time was also reviewed in a number of earlier works [21–24]. Although the result $v - v_b \sim (x - x_b)^{1/3}$ has appeared in these works in different physical contexts, the authors of these works did not study the Fourier spectrum of the breaking Riemann wave, which was studied only recently in numerical works [19, 20].

Let us now illustrate the wave breaking and the main result with examples. First, it is easy to see that the wave steepness v_x increases on the wave front where F' is negative:

$$v_x = \frac{F'(\zeta)}{1 + tF'(\zeta)}, \quad \zeta = x - tv(x,t). \tag{6}$$

The breaking time is computed explicitly as

$$t_b = \frac{1}{\max_x[-F'(x)]} = \frac{-1}{\min_x[F'(x)]}. \tag{7}$$

This process is illustrated at Fig. 1 for an initial pulse

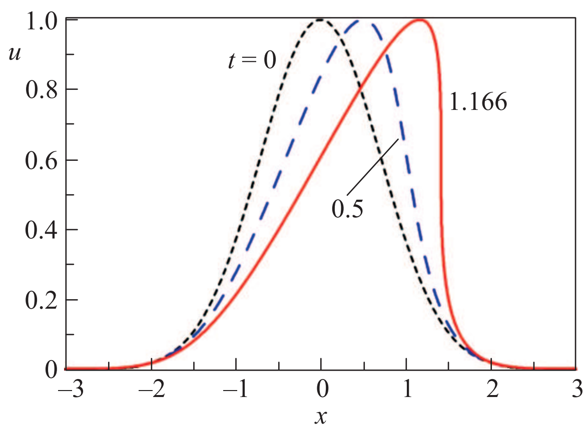

Fig. 1. Deformation of the Riemann wave for the case of quadratic nonlinearity and initial pulse of Gaussian shape $F(x) = \exp(-x^2)$. The wave shape at the moments of time $t = 0, 0.5, 1.166$ is shown as small dashed, long dashed and bold lines correspondingly

of the Gaussian shape $F(x) = \exp(-x^2)$. The shock is formed at the point $x_b = 2^{-1/2}$ and $v_b = \exp(-1/2)$ at the moment of time $t_b = 2^{-1/2}\exp(1/2) \approx 1.166$. At the breaking time the wave shape contains singularity on its front (i.e. its steepness becomes infinite) and the solution (5) is not valid anymore.

If $f(x) = \sin x$, an analytical solution for the Fourier spectrum of the simple waves (5) is known. Corresponding spectrum is known as the Bessel–Fubini spectrum and is given in Ref. [14]:

$$v(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{nt} J_n(nt)\sin(nx), \tag{8}$$

where J_n is Bessel function of the first order with integer n and the breaking time is $t_b = 1$. The amplitude of the Fourier spectrum at the breaking time reads

$$|A_n| = \frac{2}{n} J_n(n) \tag{9}$$

and is distributed close to the rate of $n^{-4/3}$ (see Fig. 2).

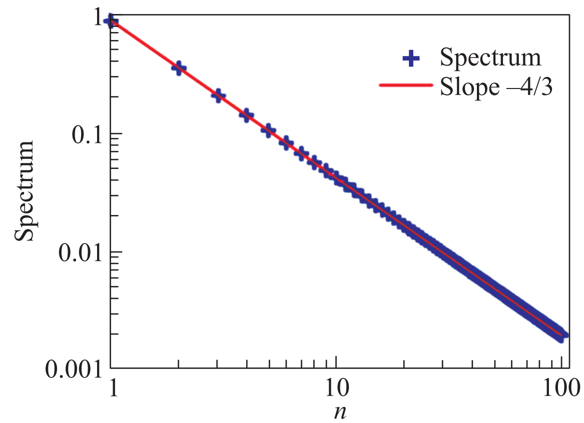

Fig. 2. (Color online) Amplitude spectrum of Riemann waves for the solution (8) at the breaking time $t_b = 1$ is shown by blue crosses. The power rate $n^{-4/3}$ is shown as the red solid line. Wave number n (horizontal axes) and the amplitude spectrum $|A_n|$ (vertical axes) are shown in logarithmical coordinates

We shall now prove that the power rate of the Fourier amplitude spectrum at the time of wave breaking is $k^{-4/3}$ for a simple (Riemann) wave supported by an arbitrary smooth initial pulse and an arbitrary local velocity $V(u)$. Moreover, the same rate remains valid in the range of small wave numbers if small dissipation or dispersion is added in the framework of the viscous Burgers or Korteweg–de Vries equations.

II. Power law of wave breaking. Using the method of characteristics, we write the inviscid Burgers equation (3) as the system of two ordinary differential equations

$$\frac{dx}{dt} = v(x,t), \qquad \frac{dv}{dt} = 0, \tag{10}$$

therefore, each point on the wave shape moves with velocity proportional to the magnitude of v. The solution is now written in the parametric form:

$$u(t) = F(\zeta), \qquad x(t) = \zeta + tF(\zeta). \tag{11}$$

Let ζ_b be the global minimum of F' (which always exists since F is smooth and decays to zero at infinity). We assume that the minimum is not degenerate, hence, $F''(\zeta_b) = 0$ and $F'''(\zeta_b) > 0$. Let t_b be the time of breaking defined by Eq. (7) such that $1 + t_b F'(\zeta_b) = 0$. The wave breaks at the point $x_b = \zeta_b + t_b F(\zeta_b)$ and $v_b = F(\zeta_b)$. Using the decomposition $\zeta = \zeta_b + \eta$ and expanding the exact solution (11) into Taylor series, we obtain at the time of breaking:

$$x = \zeta_b + \eta + t_b F(\zeta_b + \eta) = \\ = x_b + \frac{1}{6} t_b F'''(\zeta_b)\eta^3 + \mathcal{O}(\eta^4) \tag{12}$$

and

$$v(x, t_b) = F(\zeta_b + \eta) = \\ = v_b + F'(\zeta_b)\eta + \mathcal{O}(\eta^2). \tag{13}$$

Solving (12) for a unique small real root of η, we obtain an explicit relation between u and x at the time of breaking:

$$\eta \sim (x - x_b)^{1/3}, \quad v - v_b \sim (x - x_b)^{1/3}. \tag{14}$$

Therefore, the wave profile changes near the breaking point at the breaking time as $(x - x_b)^{1/3}$, contrary to the behavior $(x - x_b)^{1/2}$ suggested in [18]. The behavior (14) leads to the power spectrum with slope $-4/3$ as it was established numerically in [19, 20]. The energy spectrum in this case has the slope $-8/3$.

If $F'''(\zeta_b) = 0$, then $F^{(4)}(\zeta_b) = 0$ since F is smooth and ζ_b is the point of minimum of F'. If $F^{(5)}(\zeta_b) \neq 0$ (in which case $F^{(5)}(\zeta_b) > 0$), then the modification of the previous analysis shows that the wave profile changes near the breaking point at the breaking time as $(x - x_b)^{1/5}$, leading to the power spectrum with slope $-6/5$. We can continue this analysis if ζ_b is a degenerate minimum of a higher order.

Note that the above analysis holds for a general nonlinear evolution Eq. (1) under the assumption that $V'(u) > 0$ (that is, when V is invertible). In this case, if $F'''(\zeta_b) \neq 0$, the wave field $u(x, t_b)$ of the nonlinear evolution Eq. (1) changes according to the behavior (14), or explicitly, as

$$u(x, t_b) \approx V^{-1}\left[v_b + \alpha(x - x_b)^{1/3}\right] \approx \\ \approx u_b + \frac{\alpha}{V'(u_b)}(x - x_b)^{1/3}, \tag{15}$$

where V^{-1} is the inverse function to V, $u_b = V^{-1}(v_b)$, and $\alpha \neq 0$ is a numerical coefficient. Thus, we conclude that the above universal behavior extends to a general nonlinear evolution Eq. (1) and a general initial data (4) under some restrictive assumptions that are physically relevant.

III. Small dissipation and dispersion effects. The simple wave Eq. (1) is only valid before the moment of breaking; the study of the wave field evolution at the later times is usually conducted by including effects of dissipation or dispersion. Corresponding terms added to the right hand side of (1) produce different types of equations such as the viscous Burgers equation

$$u_t + 6uu_x = \nu u_{xx}, \quad \nu > 0 \tag{16}$$

and the Korteweg–de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{17}$$

In both cases, we consider the initial-value problem starting with initial data $u(x, 0) = S_0 \sin x$.

Taking into account dissipative and dispersive effects will inevitably change the wave spectrum for large wave numbers. For instance, it is well-known that shock waves in the viscous Burgers Eq. (16) have spectral density that decays as k^{-2} for large wave numbers [2].

Nevertheless, our numerical simulations of the viscous Burgers Eq. (16) demonstrate that universal power $k^{-8/3}$ of the energy spectrum of breaking Riemann waves is clearly visible in the range of small wave numbers at least for the evolution times $t \sim t_b \approx 25.5$, where t_b is the breaking time in the inviscid Burgers equation. For longer times, dissipative or dispersive effects become fully developed and drift the energy spectrum away from the power $k^{-8/3}$.

Figures 3 and 4 illustrate solutions of the viscous Burgers Eq. (16) in physical and Fourier space consequently. The boundary between the spectral asymptotics of $k^{-8/3}$ and k^{-2} can be roughly found from the balance between the terms $6uu_x$ and νu_{xx} in the viscous Burgers equation. If a is the wave amplitude, then the boundary is found at $k \sim 6a/\nu$. If a decreases, then the boundary moves towards the low frequency part of

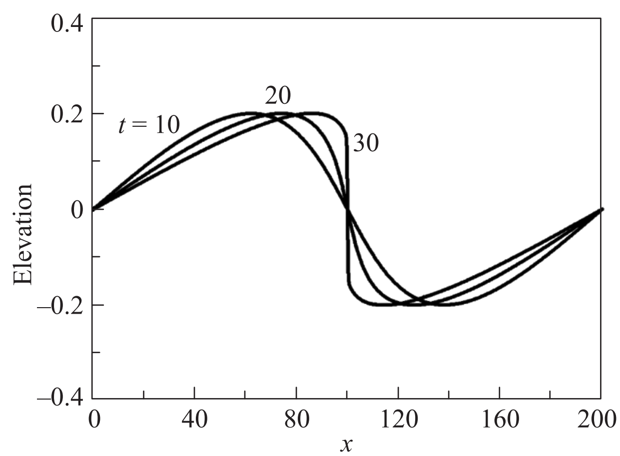

Fig. 3. Solutions of the Burgers Eq. (16) with $\nu = 0.1$ and initial data $u(x, 0) = S_0 \sin x$ in physical space, for different values of time $t = 10, 20, 30$

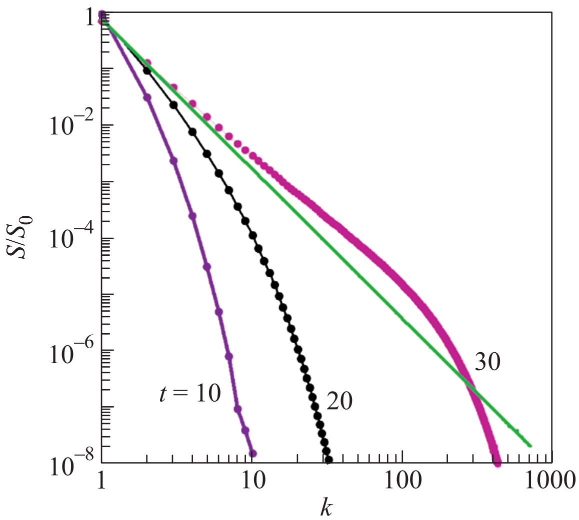

Fig. 4. Solutions of the Burgers Eq. (16) as in Fig. 3 but in Fourier space: the normalized energy spectrum S/S_0 is shown versus wave numbers k, with axes in logarithmic coordinates. The universal power law $k^{-8/3}$ is shown by green solid line

the Fourier spectrum. Note that the fast decay of the energy spectrum for large wave numbers on Fig. 4 is due to numerical effects.

Similar results of numerical simulations for the Korteweg–de Vries Eq. (17) are shown in Fig. 5 and 6.

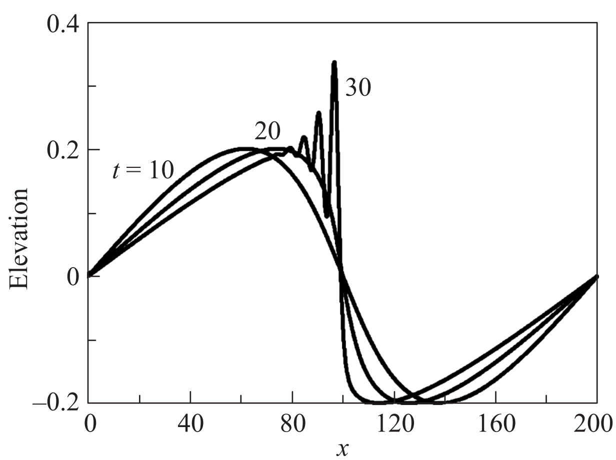

Fig. 5. Solutions of the KdV Eq. (17) with initial data $u(x,0) = S_0 \sin x$ in physical space, for different values of time $t = 10, 20, 30$

It is clearly visible that although wave breaking is absent in the KdV Eq. (17), the universal power $k^{-8/3}$ appear in the energy spectrum for small wave numbers for $t \sim t_b \approx 25.5$.

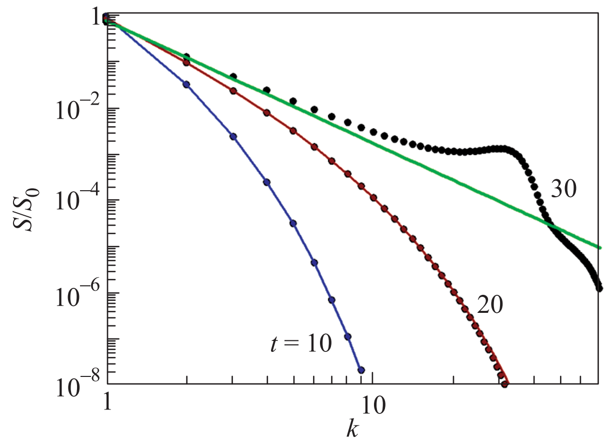

Fig. 6. Solutions of the KdV Eq. (17) as in Fig. 5 but in Fourier space: the normalized energy spectrum S/S_0 is shown versus wave numbers k, with axes in logarithmic coordinates. The universal power law $k^{-8/3}$ is shown by green solid line

We also mention results of the numerical simulations of the reduced Ostrovsky equation [25]

$$(\eta_t + \eta\eta_x)_x = \gamma\eta, \quad \gamma > 0, \tag{18}$$

starting with the same initial data $u(x,0) = S_0 \sin x$. A similar effect is observed, namely, the universal power $k^{-8/3}$ appear in the energy spectrum for large wave numbers regardless of the rotation parameter γ and the initial wave amplitude S_0. The universal behavior is now observed in the range of large wave numbers because the dispersion term in the reduced Ostrovsky Eq. (18) affects the wave dispersion for small wave numbers.

IV. Summary. We have justified the universal power law $k^{-8/3}$ in the energy spectrum of one-dimensional breaking Riemann waves in the context of the simple wave Eq. (1) with smooth initial data (4). This result remains valid for arbitrary nonlinear wave speed provided that the wave speed is an invertible function of the wave amplitude. In addition, we have demonstrated that the same power law is observed for long times in the range of small wave numbers in the context of the viscous Burgers (16) and Korteweg–de Vries (17) equations. These universal power law also occurs in other nonlinear evolution equations that reduce to the simple wave equation in the dissipationless and dispersionless limit.

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