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Radiative effects to the adiabatic dynamics of envelope-wave solitons

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Abstract

A general asymptotic method for analysis of radiative effects to the adiabatic dynamics of envelope-wave solitons is presented in the form of a modified soliton perturbation technique involving three asymptotic scales. This method is applied to a generalized NLS equation for description of both the instability-induced soliton dynamics near the instability threshold and exponentially weak radiative effects. The results are obtained for two particular problems: (i) a new (revised) derivation of a double-logarithmic scaling law of singularity formation at the critical soliton collapse and (ii) calculation of an inverse squared logarithmic decay rate of an amplitude of internal low-frequency oscillations excited at the background of a stable soliton near the instability threshold. © 1998 Published by Elsevier Science B.V.

1. Introduction

Soliton theory has been successfully applied to different physical problems associated with generation, motion, evolution, and interaction of localized nonlinear wave perturbations called solitons. It has been revealed that soliton may evolve *adiabatically* under the action of external or internal perturbation, i.e. their shapes remains the same during evolution while the underlying parameters are varying in time. This adiabatic dynamics is described by a regular soliton perturbation theory which enables us to simplify the governing equations and consider problems of soliton dynamics within the framework of finite-dimensional systems (see [1] for a review of the soliton perturbation theory and its different applications).

The adiabatic soliton evolution usually leads to generation of radiative waves escaping a soliton and taking away a part of its energy. This effect is generally very important because it results in modifications of dynamical laws and long-term predictions for soliton evolution. However, there is no general theory for emission of radiation and, besides, different types of solitary waves are known to generate different types of radiation during their adiabatic dynamics. For instance, solitary waves at the surface of a shallow-water fluid (see [2] and references therein) generate a strong shelf-shaped radiation. If the adiabatic dynamics is driven by a small (adiabaticity) parameter ϵ, then the radiation field appears at the first order of ϵ and it affects the soliton dynamics in the order of ϵ^2. In this case, an

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asymptotic *two-scaled* expansion technique provides an adequate description of both the soliton dynamics and the radiation emission (see [3,4]).

In contrast with the above example, there exists another type of solitary waves, e.g. bright optical solitons in dielectric waveguides with intensity-dependent refractive index (see [5]), which does not generate a strong radiation field at the first order of the adiabaticity parameter. Moreover, because of an effective gap in the linear spectrum (see [4]) the radiation does not appear at *any* order of ϵ and, in fact, is *exponentially small* with respect to ϵ. [2] In spite of this smallness, the radiation-induced effects change the long-term adiabatic evolution drastically, e.g. the scaling laws of blow-up of localized perturbation described by a critical NLS equation become modified due to the radiative effects by a double-logarithmic factor (see [7–33] for review of the contradictory literature devoted to this problem). Such exponentially small effects have been predicted in a number of different problems, e.g. for a low-frequency motion of a sin-Gordon kink oscillation in an external field or for emission of a NLS soliton scattered by local inhomogeneities (see Section VI in [1] for numerous examples). However, a regular asymptotic method for problems of this class has not been elaborated yet.

In this paper we develop results of our previous work [34] and present a regular method to analyse the long-term instability-induced dynamics of envelope-wave solitons. This problem is described by the generalized NLS equation written in the following form:

$$\mathrm{i}\Psi_t + \Psi_{xx} + f(|\Psi|^2)\Psi = 0. \tag{1}$$

Here $\Psi(x,t)$ is a complex envelope of a carrier wave, x and t stand for spatial and temporal variables, respectively, and a real function $f(|\Psi|^2)$ corresponds to a merely nonlinear correction to a wave frequency such that $f(0) = 0$. This equation describes modulations of a carrier small-amplitude high-frequency wave in a number of different physical problems and it can exhibit localized solutions known as envelope-wave solitons. For instance, in nonlinear optics (see [5]) these solitons are referred to as the bright optical solitons and they represent self-guided stationary beams of an electric field propagating in dielectrical waveguides. Within this context, x and t stand for a transverse coordinate and a propagation distance, respectively, while $f(|\Psi|^2)$ characterizes nonlinear correction to the refractive index of the optical material.

In the previous paper [34] we studied adiabatic evolution of weakly unstable solitons in the generalized NLS equation (1). This evolution was induced by development of internal perturbations at the soliton background which grew slowly in time near the instability threshold. Then, a straightforward expansion method involving *only one asymptotic scale* allowed us to find reduced equations describing different scenarios of the adiabatic soliton evolution such as long-term oscillations, decay, and collapse. The radiative waves were not taken into account in our previous paper because they appear beyond all orders of the straightforward asymptotic procedure. In particular, for the critical collapse problems our method reproduced the same results as those described by self-similar solutions to a critical NLS equation [8]. These results are only appropriate for description of an initial stage of blow-up (see [32]) while they are known to be invalid at a longer time-scale of the soliton dynamics [15].

In this paper we evaluate exponentially small radiative effects at the longer-term instability-induced dynamics of envelope-wave solitons within the generalized NLS equation (1). In Section 2 governing asymptotic equations modified due to the radiative effects are derived with the help of a regular expansion method involving *three asymptotic scales*. Then, we apply the derived equations to two particular problems of envelope-wave soliton dynamics. In Section 3 we obtain the scaling laws of the critical collapse and compare our results with numerous

[2] It is assumed here that the continuous-wave excitations of an envelope-wave soliton follow adiabatically for the soliton evolution. In a number of physical problems associated with envelope-wave solitons (see, e.g., [6]) the continuous-wave perturbations are introduced to satisfy a certain initial condition which may not be consistent with the adiabatic evolution of a soliton. This inconsistence in an initial condition results in an intermediate soliton emission already at the first order of the perturbation theory [6].

approaches presented in literature [7–33]. In Section 4 we evaluate the decay rate of internal low-frequency soliton oscillations which occur to stable solitons near the instability threshold.

2. Derivation of asymptotic equations

2.1. Equations of the soliton scale for adiabatic dynamics

Steady-state envelope-wave soliton is prescribed for the generalized NLS equation (1) by the following substitution [34]:

$$\Psi = \Phi(x : \omega)\mathrm{e}^{\mathrm{i}\omega t},$$

where ω is the soliton parameter determined by nonlinear properties of a wave system. Beside this degree of freedom, the envelope-wave solitons can also propagate along the system with a constant velocity, but this drift motion can be removed with the help of a Lorentz-transformation (see, e.g., [4]). The real function $\Phi(x;\omega)$ is *even* and *nodeless* in x and satisfies the following differential equation:

$$\Phi_{xx} - \omega\Phi + f(\Phi^2)\Phi = 0, \tag{2}$$

subject to the zero boundary conditions at infinity. We assume that these soliton solutions exist and are exponentially localized, i.e.

$$\Phi(x;\omega) \to A(\omega)\mathrm{e}^{-\sqrt{\omega}|x|} + \mathrm{o}(\mathrm{e}^{-\sqrt{\omega}|x|}) \quad \text{as } |x| \to \infty, \tag{3}$$

where $A(\omega)$ is a constant amplitude. It is clear that the parameter ω must be *positive* for the soliton solutions to exist.

Now we consider the adiabatic dynamics of envelope-wave solitons under the action of small internal perturbations supported due to a near-threshold instability. To the end, we introduce a slow time $T = \epsilon t$ through a formal small parameter ϵ and suppose that the nonlinear wave field $\Psi = \Psi(x, T)$ evolves adiabatically at the slow scale. This implies that the field remains locally close to the steady-state profile $\Phi(x;\omega)$ but the parameter ω changes slowly in time according to a dependence $\omega = \omega(T)$ to be found. Under this (adiabaticity) assumption, we expand solutions to (1) in a regular asymptotic series,

$$\Psi = \phi(x, T)\mathrm{e}^{\mathrm{i}\theta(T)}, \tag{4}$$

where $\theta = \epsilon^{-1}\int_0^T \omega(T')\,\mathrm{d}T'$ and

$$\phi = \Phi(x;\omega) + \sum_{k=1}^{\infty}\epsilon^{2k-1}\mathrm{i}\phi_{2k-1}(x;\omega;T) + \sum_{k=1}^{\infty}\epsilon^{2k}\phi_{2k}(x;\omega;T). \tag{5}$$

The leading-order term $\Phi(x;\omega)$ of this regular series is the steady-state soliton but with a varying parameter ω while the next-order correction terms $\phi_n(x;\omega;T)$ represent deviations of the field profile from the self-similar soliton shape. These correction terms are to be found from linear inhomogeneous equations associated with the main equation (2). Indeed, substitution of (4) and (5) into (1) shows that the functions ϕ_n satisfy the following linear equations:

$$\mathcal{L}_0\phi_{2k-1} = H_{2k-1}, \qquad \mathcal{L}_1\phi_{2k} = H_{2k}. \tag{6}$$

Here $\mathcal{L}_0$ and $\mathcal{L}_1$ are linear associated operators given by

$$\mathcal{L}_0 = -\frac{\partial^2}{\partial x^2} + \omega - f(\Phi^2), \qquad \mathcal{L}_1 = -\frac{\partial^2}{\partial x^2} + \omega - f(\Phi^2) - 2\Phi^2 f'(\Phi^2),$$

while the right-hand side functions H_{2k-1} and H_{2k} include corrections of lower orders. To find asymptotic equations at the leading order, we need explicit formulas only for a few first right-hand side functions,

$$H_1 = \frac{\mathrm{d}\Phi}{\mathrm{d}T}, \qquad H_2 = -\frac{\mathrm{d}\phi_1}{\mathrm{d}T} + \Phi f'(\Phi^2)\phi_1^2, \qquad H_3 = \frac{\mathrm{d}\phi_2}{\mathrm{d}T} + \phi_1 f'(\Phi^2)(2\Phi\phi_2 + \phi_1^2),$$

where we have used the following notation for a full time-derivative:

$$\frac{\mathrm{d}}{\mathrm{d}T} = \frac{\partial}{\partial T} + \frac{\mathrm{d}\omega}{\mathrm{d}T}\frac{\partial}{\partial\omega}.$$

The first terms of the regular asymptotic series (5) were found and analysed in our previous paper [34] together with an asymptotic equation imposed to $\omega = \omega(T)$. This equation governs the adiabatic soliton dynamics. Here we reproduce the main results of this analysis in order to proceed to radiation-induced modifications of the asymptotic technique. The first-order correction term ϕ_1 can be explicitly presented in the form

$$\phi_1 = -\Phi(x;\omega)\int_0^x \frac{\mathrm{d}x'}{\Phi^2(x';\omega)}\int_0^{x'} \Phi(x'';\omega)\frac{\mathrm{d}\Phi(x'';\omega)}{\mathrm{d}T}\,\mathrm{d}x'', \tag{7}$$

where the integrand is not singular because Φ is nodeless in x. It follows from this equation that the function ϕ_1 contains two types of diverging terms as $|x| \to \infty$,

$$\phi_1 \to -\frac{1}{4\sqrt{\omega}A(\omega)}\frac{\mathrm{d}N_\mathrm{s}(\omega)}{\mathrm{d}T}[\mathrm{e}^{\sqrt{\omega}|x|} + o(\mathrm{e}^{\sqrt{\omega}|x|})] - \frac{1}{8\omega}\frac{\mathrm{d}\omega}{\mathrm{d}T}A(\omega)[x^2\mathrm{e}^{-\sqrt{\omega|x|}} + \mathrm{O}(|x|\mathrm{e}^{-\sqrt{\omega}|x|})], \tag{8}$$

where $N_\mathrm{s}(\omega)$ is the soliton power given by

$$N_\mathrm{s} = \frac{1}{2}\int_{-\infty}^{\infty} \Phi^2(x;\omega)\,\mathrm{d}x. \tag{9}$$

The first term in (8) is strongly (exponentially) diverging in x while the second localized term indicates an appearence of a quadratically growing complex phase in the exponential representation (4). The latter terms are usually referred to as *virtual* secular divergences.

If one neglects the second type secular divergence, then an asymptotic procedure of removing the exponentially growing terms produces a certain differential equation for the dependence $\omega = \omega(T)$. In the absence of any external perturbations to (1) this differential equation is non-trivial only in the vicinity of the instability threshold, where the derivative $N_\mathrm{s}'(\omega) = \mathrm{d}N_\mathrm{s}/\mathrm{d}\omega$ vanishes (see [34]). Assuming a balance $N_\mathrm{s}'(\omega) \sim \mathrm{O}(\epsilon^2)$, we remove the exponentially growing term of the function ϕ_1 (8) to the order of $\mathrm{O}(\epsilon^3)$, where this term is in balance with an exponentially diverging component of the function ϕ_3. The latter component is responsible for inertial effects to the adiabatic dynamics of envelope-wave solitons. Taking this into account, we find the following asymptotic representation for the regular asymptotic series (5) at infinity:

$$\phi = u(x,T)\exp\left(-\frac{\mathrm{i}\epsilon}{8\omega}\frac{\mathrm{d}\omega}{\mathrm{d}T}x^2\right), \tag{10}$$

where $u(x, T)$ approaches the following limiting expressions as $|x| \to \infty$:

$$u \to A(\omega)\mathrm{e}^{-\sqrt{\omega}|x|} - \frac{\mathrm{i}\epsilon}{4\sqrt{\omega}A(\omega)}\frac{\mathrm{d}N_0}{\mathrm{d}T}\mathrm{e}^{\sqrt{\omega}|x|}. \tag{11}$$

In this representation we have kept only the leading-order terms for the quadratic complex phase as well as for the exponentially localized and exponentially diverging terms. Here N_0 stands for a localized wave field power to exhibit the following asymptotic expansion found in [34],

$$N_0 = N_\mathrm{s}(\omega) + \epsilon^2\left[M_\mathrm{s}(\omega)\frac{\mathrm{d}^2\omega}{\mathrm{d}T^2} + \frac{1}{2}\frac{\mathrm{d}M_\mathrm{s}(\omega)}{\mathrm{d}\omega}\left(\frac{\mathrm{d}\omega}{\mathrm{d}T}\right)^2\right] + \mathrm{O}(\epsilon^4), \tag{12}$$

where the coefficient $M_\mathrm{s}(\omega)$ is given by

$$M_\mathrm{s} = \int_{-\infty}^{\infty}\left[\frac{1}{\Phi(x;\omega)}\int_0^x \Phi(x';\omega)\frac{\partial\Phi(x';\omega)}{\partial\omega}\,\mathrm{d}x'\right]^2\mathrm{d}x. \tag{13}$$

In the previous paper [34] we analyse the asymptotic Eq. (12) in the approximation, $N_0 = \text{const}$, when there is no exponentially growing term in (11). However, this approximation corresponds to neglection of radiation generated due to the adiabatic soliton dynamics. The radiative effects are displayed in the representation (10) in the form of a quadratically growing complex phase of the wave function ψ (see (4) and (10)). This quadratic phase is usually referred to as a “chirp” which is responsible for the radiation field “frozen” to an adiabatically varying soliton. The quadratic phase appears from virtual secular divergences of each correction term ϕ_n after summation of the regular asymptotic series (5). Therefore, this actual secular divergence is beyond all orders of the regular expansion (5) and represents exponentially weak radiative effects. In the present analysis we extend (5) in the region outside the soliton core and find a modified expansion according to the limiting representation (10) and (11). This modification allows us to find an asymptotic equation describing variations of the integral power N_0 induced due to radiative effects.

2.2. *Equations of the quadratic phase scale for radiative effects*

The nonlinear term in the generalized NLS Eq. (1) is negligible outside the soliton core and, hence, the radiation-induced problems are solely linear. Furthermore, the substitution (10) reduces (1) to a simple equation,

$$u_{xx} - \omega u + \frac{1}{4}\epsilon^2 Bx^2u + \mathrm{i}\epsilon\left[u_\mathrm{T} - \frac{1}{4\omega}\frac{\mathrm{d}\omega}{\mathrm{d}T}(u + 2xu_x)\right] = 0, \tag{14}$$

where $B(T)$ is defined through $\omega(T)$ according to the formula,

$$B = \frac{1}{2\omega}\frac{\mathrm{d}^2\omega}{\mathrm{d}T^2} - \frac{3}{4\omega^2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}T}\right)^2. \tag{15}$$

It is clear that solution to (14) are still quasi-adiabatic, i.e. they depend on the evolution time T only through the two varying parameters ω and B, while the time-derivative term u_T is a small perturbation. In this case, an asymptotic representation for quasi-adiabatic solutions of (14) can be thought in the form,

$$u = U(x;\omega,\epsilon^2B) + \sum_{n=1}^{\infty}\epsilon^n u_n(x;\omega,\epsilon^2B;T), \tag{16}$$

where the leading-order term U satisfies the parabolic cylinder equation,

$$U_{xx} - \omega U + \tfrac{1}{4}\epsilon^2Bx^2U = 0, \tag{17}$$

subject to the boundary conditions (11) taken in the asymptotic limit $|x| \to 0$. Solutions to the parabolic cylinder equation essentially depend on a sign of B. If $B \le 0$, then the function U has only exponentially decaying or growing solutions with respect to x. In this case, the exponentially growing term in (11) produces a global divergence for solutions of (17) in the limit $|x| \to \infty$. Therefore, this term is to be removed by the equation imposed on evolution of N_0, i.e.

$$\frac{\mathrm{d}N_0}{\mathrm{d}T} = 0 \quad \text{if } B(T) \le 0. \tag{18}$$

This means that the radiation is not excited during the time intervals when $B(T) \le 0$. On the other hand, for $B(T) > 0$ the parabolic cylinder equation (17) contains the turning points located at

$$|x| = X_{\mathrm{p}} = \frac{2\sqrt{\omega}}{\epsilon\sqrt{B}}.$$

Outside the turning points, where $|x| > X_{\mathrm{p}}$, solutions to (17) have oscillatory behaviour which indicates excitation of radiative waves escaping a soliton core. Thus, the regular asymptotic expansion (5) is applicable only within an *inner* soliton interval, where $|x| \ll X_{\mathrm{p}}$. Otherwise, for $|x| \ge X_{\mathrm{p}}$ small-amplitude radiative waves are generated through a quadratic potential and they should be described separately at an *outer* radiation interval. An *intermediate* quadratic phase interval for $|x| \sim X_{\mathrm{p}}$ serves as a “buffer” between the inner and outer scales and is responsible both for generation of radiative waves and for derivation of governing equations of soliton dynamics. The main problem arising at this intermediate interval is to relate the two exponentially growing and decaying fields from inner side of the turning points (see (11)) to the incoming and outcoming waves from outside (see (25) below).

The aforementioned problem is typical in quantum mechanics and can be easily solved in a quasi-classic approximation valid when potentials are slowly varying in space (see, e.g., [35]). As an alternative way, we simplify here intermediate calculations and use exact solutions to the parabolic cylinder equation. As is well-known (see, e.g., [36]) a general solution to (17) can be expressed through the Weber functions,

$$U = \alpha_1 W(\gamma, \nu) + \alpha_2 W(\gamma, -\nu), \tag{19}$$

where

$$\gamma = \frac{\omega}{\epsilon\sqrt{B}}, \quad \nu = \sqrt{\epsilon} B^{1/4} x,$$

and α_1, α_2 are some coefficients. We find these coefficients by matching the asymptotic values of (19) in the limit $|\nu| \ll 2\sqrt{\gamma}$ with the boundary values for the regular series (11). To do this, we use the asymptotic formulas [36]

$$W(\gamma, \pm\nu) \to \frac{1}{(4\gamma - \nu^2)^{1/4}} \exp\left[\mp\frac{1}{2}\int_0^{\nu} \sqrt{4\gamma - \nu^2}\,\mathrm{d}\nu\right] \quad \text{as } |\nu| \ll 2\sqrt{\gamma}. \tag{20}$$

The matching conditions with the boundary values (11) lead to the following expressions for α_1 and α_2:

$$\frac{\alpha_1}{(4\gamma)^{1/4}} = A(\omega), \qquad \frac{\alpha_2}{(4\gamma)^{1/4}} = -\frac{\mathrm{i}\epsilon}{4\sqrt{\omega}A(\omega)}\frac{\mathrm{d}N_0}{\mathrm{d}T}. \tag{21}$$

Next, we use the asymptotic values for the Weber functions in the limit $|\nu| \gg 2\sqrt{\gamma}$ [36] and find the following expression for the function U extending far from the turning points as $|\nu| \gg 2\sqrt{\gamma}$:

$$U \to \frac{\alpha_1\sqrt{2k}}{(\nu^2 - 4\gamma)^{1/4}} \cos\left[\frac{\pi}{4} + \frac{1}{2}\int_{2\sqrt{\gamma}}^{\nu} \sqrt{\nu^2 - 4\gamma}\,\mathrm{d}\nu\right] + \frac{\alpha_2\sqrt{2k^{-1}}}{(\nu^2 - 4\gamma)^{1/4}} \sin\left[\frac{\pi}{4} + \frac{1}{2}\int_{2\sqrt{\gamma}}^{\nu} \sqrt{\nu^2 - 4\gamma}\,\mathrm{d}\nu\right], \tag{22}$$

where

$$\kappa = \sqrt{1+e^{2\pi\gamma}} - e^{\pi\gamma} \approx \tfrac{1}{2}e^{-\pi\gamma},$$

the last simplification is applied in the asymptotic limit $\epsilon \to 0$ and $\gamma \to \infty$. Expression (22) indicates generation of a radiation field outside the quadratic phase interval. The radiation field is presented by a superposition of two waves propagating outside the turning points with the complex phase factor, $\pm\int p\,d\nu$, where the quasi-momentum p is given by $p = \frac{1}{2}\sqrt{\nu^2 - 4\gamma}$. The positive sign in the phase factor corresponds to an outcoming wave escaping the quadratic potential to infinity while the negative sign corresponds to an incoming wave running from infinity to the quadratic potential (see [35]). It is clear from physical motivations that the latter wave should be eliminated by a *radiation condition* so that soliton dynamics might generate only waves escaping the soliton core. To eliminate the incoming wave, we define the coefficients α_1 and α_2 to be related as follows,

$$i\alpha_1 = 2\alpha_2 e^{\pi\gamma}. \tag{23}$$

Together with (21), this relation leads to the differential equation imposed to the dependence $\omega = \omega(T)$,

$$\frac{dN_0}{dT} = -\frac{1}{\epsilon}2\sqrt{\omega}A^2(\omega)\exp\left[-\frac{\pi\omega}{\epsilon\sqrt{B}}\right] \quad \text{if } B(T) > 0, \tag{24}$$

where N_0 and B are given by (12) and (15) respectively. The asymptotic equations (18) and (24) describe long-term instability-induced dynamics of an envelope-wave soliton within the multi-scale reduction of the generalized NLS equation (1). It follows from (22) that generation of the radiation field $u(x,T)$ is provided with the following boundary condition as $|x| \to \infty$:

$$u \to A\left(\frac{4\omega}{\epsilon\sqrt{B}}\right)^{1/4} \frac{e^{-\pi\omega/2\epsilon\sqrt{B}}}{\nu^{1/2+i\omega/\epsilon\sqrt{B}}} \exp\left(\frac{i}{4}\epsilon\sqrt{B}x^2 + i\varphi\right), \tag{25}$$

where

$$\varphi = \frac{\pi}{4} - \frac{\omega}{2\epsilon\sqrt{B}} + \frac{\omega}{2\epsilon\sqrt{B}}\log\left(\frac{\omega}{\epsilon\sqrt{B}}\right).$$

The radiation field evolves at the outer radiation interval non-adiabatically according to a linear NLS equation, $i\Psi_{\infty t} + \Psi_{\infty xx} = 0$, where Ψ_∞ is a profile of the radiation field. The boundary conditions for the radiation field Ψ_∞ at $x \to 0$ follow from the limiting representations (4), (10), and (25). We would like to point out that the amplitude of generated waves is exponentially small in terms of the adiabaticity parameter ϵ (see formula (25))). Here we do not consider evolution of radiative waves dealing with a description of envelope-wave soliton dynamics. Some explicit solutions for evolution of radiative waves were recently constructed in paper [23–25].

3. Application to critical collapse

3.1. Results

Here we apply the general asymptotic equations (18) and (24) to the problem of singularity formation in a critical NLS equation which has the following dimensionless form:

$$i\Psi_t + \Psi_{xx} + 3|\Psi|^4\Psi = 0. \tag{26}$$

The critical NLS equation follows from (1) for $f(|\Psi|^2) = 3|\Psi|^4$. The soliton solution $\Phi(x;\omega)$ (see (2)) is given by the explicit formula

$$\Phi = \omega^{1/4}\mathrm{sech}^{1/2}[2\sqrt{\omega}x]. \tag{27}$$

Using (27) and the asymptotic representation (3) we evaluate the constant $A(\omega)$ in the form $A = \sqrt{2}\omega^{1/4}$. The soliton solutions (27) are well known to be critically unstable (see [34] and references therein) because the soliton power $N_s(\omega)$ [see (9)] does not depend on ω and has the form $N_s(\omega) = N_{cr} = \frac{1}{4}\pi$. Moreover, small perturbations to the steady-state solitons (27) blow up in finite time if an initial total power N_0 exceeds the critical value N_{cr} and decay to small-amplitude dispersive waves if N_0 is less than N_{cr}. Evolution of the soliton perturbations occurs to be adiabatic if the difference between N_0 and N_{cr} is small and, in this case, the asymptotic method and equations used in Section 2 are applicable. Thus, it follows from (13) and (27) that the coefficient $M_s(\omega)$ has the form $M_s = m\omega^{-3}$, where

$$m = \frac{1}{16}\int_{-\infty}^{\infty} x^2\Phi^2(x;1)\,\mathrm{d}x = \frac{\pi^3}{512}. \tag{28}$$

To simplify analysis of the asymptotic equations we introduce a conventional transformation of the time T into τ and the soliton parameter $\omega(T)$ into $a(\tau)$ and $b(\tau)$ according to the following formulas:

$$\tau = \frac{1}{\varepsilon}\int_0^T \omega(s)\,\mathrm{d}s, \qquad a(\tau) = \frac{\varepsilon}{2\omega^2}\frac{\mathrm{d}\omega}{\mathrm{d}T}, \qquad b(\tau) = a_\tau + a^2. \tag{29}$$

Then, the new variable $b(\tau)$ defines a deviation of the integral power N_0 from the critical value N_{cr} as well as the parameter B,

$$b = \frac{1}{2m}(N_0 - N_{cr}), \qquad B = \varepsilon^{-2}\omega^2 b. \tag{30}$$

These formulas follow from (12), (15) and (29). The blow-up occurs for $b > 0$, i.e. for $N > N_{cr}$, and, in this case, the asymptotic Eq. (24) reduces with the help of (30) to the single equation,

$$\frac{\mathrm{d}b}{\mathrm{d}\tau} = -\frac{2}{m}\exp\left(-\frac{\pi}{\sqrt{b}}\right). \tag{31}$$

In original variables, T and $\omega(T)$, the blow-up implies formation of singularities of the soliton parameter ω in finite time $T = T_0$, i.e. $\omega(T) \to \infty$ as $T \to T_0$. The transformation (29) leads, however, that singularities appear in the limit $\tau \to \infty$ when the new amplitudes a and b vanish, i.e. $b(\tau) \to 0$ as $\tau \to \infty$. Analysing the main equation (31) in this asymptotic limit (see [17] for details), we find the leading-order behaviour of the amplitude $b(\tau)$ as $\tau \to \infty$,

$$b \to \frac{\pi^2}{\log^2\tau}\left[1 - 6\frac{\log\log\tau}{\log\tau}\right]. \tag{32}$$

In the original variables, this dependence corresponds to the well-known double-log scaling law for the critical blow-up of envelope-wave solitons,

$$\omega \to \frac{\log|\log(T_0 - T)|}{(T_0 - T)} \quad \text{as } T \to T_0. \tag{33}$$

Finally, we mention that for $b < 0$, i.e. for $N_0 < N_{cr}$, a soliton spreads out and decays into small-amplitude dispersive waves. In this case, the exponentially small radiation is not tunneled through a quadratic-phase potential according to the asymptotic equations (18) and (30) and a value of b remains constant. Therefore, in contrast with the collapsing soliton, the decaying soliton remains self-similar everywhere and the soliton parameter ω has the following limiting behaviour, $\omega \to T^{-1}$ as $T \to \infty$ (see [14,34]). The case $b = 0$, i.e. $N_0 = N_{\rm cr}$, is very special because the quadratic phase potential vanishes in this case (see (17) for $B = 0$). In this special case the asymptotic approach described above reproduces an exact self-similar solution to (26) which describes the completely radiationless dynamics of envelope-wave solitons (see [34] for discussions).

3.2. *Historical remarks*

Here we recall a contradictory history of searches of scaling laws for the critical collapse of envelope-wave solitons which finally lead to discovery of the double-log dependence (33). This brief essay help us compare the approaches used in literature with the asymptotic method presented in this paper.

In 1970 Talanov [7] was the first who discovered a remarkable transformation of the NLS equation referred to as the lens transformation. Here we reproduce this transformation only for Eq. (26) in one dimension though it exists for the critical NLS equation in multi dimensions. The lens transformations for (26) has the form

$$\Psi = \omega^{1/4} u(\xi, \tau) \exp\left[\mathrm{i}\tau - \frac{\mathrm{i}}{4} a(\tau)\xi^2 \right], \tag{34}$$

where $\xi = \sqrt{\omega} x$, $\omega = \omega(T)$, while τ and a are the same as in (29). According to (34) the critical NLS equation (26) transforms to a simple equation with a quadratic phase term,

$$\mathrm{i}u_\tau + u_{\xi\xi} + 3|u|^4 u - u + \tfrac{1}{4} b(\tau)\xi^2 u = 0, \tag{35}$$

where $b(\tau)$ is given by (29). This transformation immediately leads to the exact self-similar solutions of the critical NLS equation, $u(\xi, \tau) = u(\xi)$, subject to the condition $b(\tau) = \text{const}$ [8]. It is clear that the self-similar solutions can be reproduced by the above described asymptotic method by neglecting the exponentially small radiation (in this case, the right-hand side of (31) is negligible). Omitting here a special case $b = 0$ which is structurally unstable (see [8]), the self-similar solutions produce the following scaling law of the singularity formation, $\omega \to (t_0 - t)^{-1}$ as $t \to t_0$. Emission of radiation being not included into this approach, the further numerous papers devoted to this problem were aimed at modifying the adiabatic scaling law by a radiation-induced factor.

Earlier arguments proposed by Zakharov and Synakh [9] (see also [10,11]) and alternatively by different authors [12–14] were based either on representation of the radiation field by a constant ("frozen") flat plateau or by "naive" versions of asymptotic expansions. The scaling laws predicted by those approaches were subsequently "checked" by numerical experiments carried out with a lack of numerical accuracy. Two different scaling laws for the critical collapse were thus predicted, the first in one dimension is given by $\omega \to (t_0 - t)^{-4/7}$ [9–11] while the second is $\omega \to |\log(t_0 - t)|(t_0 - t)^{-1}$ [12–14]. However, a more accurate numerical analysis of this phenomenon [15] refuted the validity of these predictions.

It was Fraiman [16] (see also [17]) who first obtained in 1985 reduced asymptotic equations describing both the evolution of the soliton spikes and radiation field and who was able to predict the correct scaling law (33) by analysing his equations. The asymptotic method he used (see [17]) was based on a rather tricky separation of soliton and radiation fields into two independent components and decomposition of the quadratic phase potential in (35) inside and outside turning points.

At the same time, an independent approach was proposed by Landman et al. [18] and LeMesurier et al. [19]. They considered the family of self-similar solutions with proper boundary conditions existing in a weakly supercritical case, i.e. when the dimension of the generalized NLS equation is slightly bigger than the critical value realized at the same power of nonlinearity. [3] Using a direct asymptotic approach, the authors of [18,19] related the derivativve of $a(\tau)$ with respect to τ with the deviation of the dimension from the critical value. Then, analysis revealed that this deviation is exponentially small in terms of a and the equation similar to (31) but with undetermined coefficients was derived. Although the further development of this idea by means of a rigorous technique of matching asymptotic expansions (see [20,21]) allowed one to find the constants of proportionality in (31) the method used seems to be rather limited and physically inconsistent. Indeed, the problem of evolution of a localized self-focusing pulse implies that variations of the parameter $a(\tau)$ occurs due to development of internal soliton perturbations at a *fixed* value of the spatial dimension rather than due to the varying dimension of the governing model.

The aforementioned difficulty did not take place in the approach of Dyachenko et al. [22] which originated from problems of nuclear physics. These authors analysed the same equation (35) but modified by an imaginary correction to an "eigenvalue" which was exponentially small in terms of b. This correction was introduced to find a correct asymptotic behaviour of the radiation field at infinity. On the other hand, the introduced correction has also modified the equation for evolution of the parameter b according to (31). Although the authors of [22] missed the correct coefficient in (31) a more careful analysis (see [21]) reveals that the method used was correct [4].

Nowadays the modified asymptotic law (33) and the main asymptotic equation for the amplitude parameter b are regarded as to be valid and numerically verified. Nevertheless, some delicate mathematical problems related to generation of radiation outside the soliton core were considered by Malkin [23] and Berge and Pesme [24,25] who constructed explicit non-self-similar solutions. Furthermore, extentions of the previous approaches were recently used to analyse effects of the critical collapse arrest [26–29] and appearence of long-term soliton oscillations [30–32]. We would like to point out that all the methods mentioned in this historical essay were actually based on the lens transformation (34) which simplified the critical NLS equation to the form (35) (see also [33]).

In the contrast with those approaches, our method uses a *direct* asymptotic multi-scale expansion of the generalized NLS Eq. (1) based on physical motivations coming from consideration of envelope-wave soliton dynamics and radiative waves. Therefore, our approach is not solely related to the remarkable properties of the critical collapse but can be extended to many other problems including the long-term internal oscillations of an envelope-wave soliton (see [34]).

4. Application to internal oscillations

Here we consider an oscillatory dynamics of a stable envelope-wave soliton within the asymptotic method developed. It has become clear (see [34]) that a stable soliton of the generalized NLS equation (1) may have a non-trivial localized mode in the linearized spectral problem arising after the substitution, $\Psi = [\Phi(x;\omega) + \chi(x;\omega,\Omega)e^{i\Omega t}]e^{i\omega t}$. The mode $\chi(x;\omega,\Omega)$ described internal oscillations of the soliton shape with the frequency Ω. The localized mode exists only if the frequency Ω fits into a gap of the continuous-wave spectrum, i.e. $|\Omega| < \omega$. However, due to nonlinearity, the internal mode generates multiple frequencies which may lead to emission of radiative waves into the continuous spectrum. This mechanism results in radiation-induced damping of the oscillatory soliton dynamics. For the case when Ω is comparable with ω, rates for the decay of an oscillation amplitude can be evaluated by a standard asymptotic multi-scale technique. However, when the frequency Ω becomes small, $\Omega \ll \omega$, the multiple frequencies are still arrested in the gap of the continuous-wave spectrum and only exponentially amall "tunnelling" effects may lead to generation of an effective radiation. This limiting case for the mode frequency fits

[3] As a matter of fact, the authors of [18,19] considered only the blow-up in the cubic NLS equation in two dimensions but their results can be generalized for the critical NLS equation in any dimensions.

[4] Instead of the coefficient $2/m$ in (31) the authors of [22] derived the coefficient N_{cr}/m.

into the asymptotic analysis presented above. Therefore, we study this limiting case by applying the asymptotic equations (18) and (24).

For simplification of the analysis the internal soliton oscillations are supposed to have very small amplitudes. In this linear limit, we introduce a small perturbation $\omega_1(T)$ to a mean value ω_0 of the soliton parameter $\omega(T)$ according to the expansion, $\omega(T) = \omega_0 + \mu\omega_1(T)$, where μ is a small amplitude of the perturbation. Then, subject to the condition $\mu \ll \varepsilon^2$ the nonlinear terms of the asymptotic expansion (12) are removed into higher orders and the asymptotic equations (12) and (15) take a simplified form

$$N_0 = N_{\rm s}(\omega_0) + m\left(\frac{{\rm d}^2\omega_1}{{\rm d}T^2} + \Omega^2\omega_1\right), \tag{36}$$

$$B = \frac{1}{2\omega_0}\frac{{\rm d}^2\omega_1}{{\rm d}T^2}. \tag{37}$$

Here $m = M_{\rm s}(\omega_0), \Omega$ is the internal mode frequency which is given asymptotically by the equation, $\Omega^2 = m^{-1}N_{\rm s}'(\omega_0)$ valid in the limit $\Omega \ll \omega$, and the derivative $N_{\rm s}'(\omega)$ is supposed to be small and positive for a stable soliton near the instability threshold [34]. Besides, we have renormalized the formal parameters ε and μ to be equal to unity, Using these formulas, we rewrite the dynamical equations (18) and (24) as follows:

$$\frac{{\rm d}N_0}{{\rm d}T} = 0 \quad \text{for } \omega_1'' \le 0, \tag{38}$$

$$\frac{{\rm d}N_0}{{\rm d}T} = -2\sqrt{\omega_0}A^2(\omega_0)\exp\left[-\frac{\pi\sqrt{2\omega_0^3}}{\sqrt{\omega_1''}}\right] \text{for } \omega_1'' > 0, \tag{39}$$

where $\omega_1'' = {\rm d}^2\omega_1/{\rm d}T^2$. If one neglects the radiative effects, then the adiabatic equation (36) described linear small-amplitude oscillations of a soliton perturbation according to a simple solution

$$N_0 = N_{\rm s}(\omega_0), \qquad \omega_1(T) = c\sin[\Omega T + \varphi], \tag{40}$$

where c and φ are oscillation amplitude and phase, respectively. These parameters are arbitrary within the linear radiationless approximation. Using the solution (40) as the leading-order approximation, we notice that the radiation is excited according to (38) and (39) only for half-periods of oscillations, when $\omega_1'' > 0$, i.e. for $\pi(1+2n) - \varphi < \Omega T < 2\pi(1+n) - \varphi$, where n is integer. For the other half-periods, when $\omega_1'' \le 0$, the oscillating soliton is locked by a quadratic-phase potential and the radiative waves are not emitted.

In order to evaluate the exponentially small radiative effects at the long-term soliton oscillations (40) we apply the method of Van-der-Pol which is well-known for similar problems in oscillation theory (see, e.g., [37]). Thus, we consider the parameters N_0, ω_0, c, and φ to be slowly varying functions at the scale of soliton oscillations and separate the asymptotic equation (36) into two constant and oscillatory components,

$$N_{\rm s}(\omega_0) = \overline{N}_0 \equiv \frac{\Omega}{2\pi}\oint N_0(T)\,{\rm d}T, \tag{41}$$

$$\frac{{\rm d}^2\omega_1}{{\rm d}T^2} + \Omega^2\omega_1 = \frac{1}{m}[N_0(T) - \overline{N}_0]. \tag{42}$$

Using the Van-der-Pol method [37] we reduce (42) to differential equations describing radiation-induced variations of the oscillation amplitude and phase,

$$\frac{{\rm d}c}{{\rm d}T} = \frac{1}{2\pi m}\oint [N_0(T) - \overline{N}_0]\cos[\Omega T + \varphi]\,{\rm d}T, \tag{43}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}T} = -\frac{1}{2\pi mc} \oint [N_0(T) - \overline{N}_0] \sin[\Omega T + \varphi] \,\mathrm{d}T. \tag{44}$$

It can be easily shown from (44) that the leading-order approximation does not induce variations of the oscillation frequency and we can put $\varphi = 0$. On the other hand, using (38), (39), and (43) we find a single equation for the amplitude variations:

$$\frac{\mathrm{d}c}{\mathrm{d}T} = \frac{\sqrt{\omega_0} A^2(\omega_0)}{\pi m \Omega^2} \int_{\pi}^{2\pi} \sin\theta \exp\left[-\frac{\pi \sqrt{2\omega_0^3}}{\Omega \sqrt{c} \sqrt{-\sin\theta}} \right] \mathrm{d}\theta. \tag{45}$$

The leading-order term of the integral in (45) can be evaluated in the asymptotic limit $c \to 0$ by means of the Laplace method. As a result of these calculations, we obtain an explicit equation for $c(T)$,

$$\frac{\mathrm{d}c}{\mathrm{d}T} = -\beta c^{1/4} \exp\left(-\frac{\alpha}{\sqrt{c}} \right), \tag{46}$$

where

$$\alpha = \frac{\pi \sqrt{2\omega_0^3}}{\Omega}, \qquad \beta = \frac{2A^2(\omega_0)}{\pi m \Omega^{3/2} (2\omega_0)^{1/4}}.$$

This differential equation enables us to evaluate the asymptotic behaviour of the oscillation amplitude decaying due to exponentially small radiative losses. This decay is described by the following asymptotic expression as $T \to \infty$:

$$c \to \frac{\alpha^2}{\log^2 T} \left[1 - 5 \frac{\log\log T}{\log T} \right]. \tag{47}$$

Besides the predictions of the long-term oscillatory dynamics, we also study the radiation-induced changes of the soliton parameter ω_0. To do this, we average the differential equation (39) and then simplify it as above in the asymptotic limit $c \to 0$. This procedure leads to the following calculation:

$$\begin{aligned} \frac{\mathrm{d}\overline{N}_0}{\mathrm{d}T} &= -\frac{\sqrt{\omega_0} A^2(\omega_0)}{\pi} \int_{\pi}^{2\pi} \exp\left[-\frac{\pi \sqrt{2\omega_0^3}}{\Omega \sqrt{c} \sqrt{-\sin\theta}} \right] \mathrm{d}\theta \\ &\approx -m\Omega^2 \beta c^{1/4} \exp\left[-\frac{\alpha}{\sqrt{c}} \right] = m\Omega^2 \frac{\mathrm{d}c}{\mathrm{d}T}. \end{aligned} \tag{48}$$

Therefore, the total shift of the averaged integral power $\overline{N}_0$ and related shift of the mean value of the soliton parameter ω_0 (see (41)) can be estimated through an initial value for the oscillation amplitude $c(0)$ as

$$\Delta \overline{N}_0 = -m\Omega^2 c(0), \tag{49}$$

$$\Delta\omega_0 = \frac{\Delta \overline{N}_0}{N_s'(\omega)} = -c(0). \tag{50}$$

Thus, emission of radiation leads to decrease of the soliton parameter ω_0. This effect may result in a complete decay of an envelope-wave soliton if the final value of ω_0 becomes less than the critical value for an onset of soliton instability. This case is relevant for dependences $N_s(\omega)$ concaved upward at the critical point (see [4]). Thus, excitation of internal oscillatory dynamics of envelope-wave solitons may lead to their disappearence and decay into small-amplitude spreading wave packets.

Finally, we mention that it is possible to generalize the analysis described above for internal oscillations of strong amplitudes. The corresponding approaches were recently developed by Malkin [30] and Fibich [31] for the problem of collapse arrest and formation of a multi-focusing oscillating soliton. Furthermore, other scenarios of the instability-induced dynamics of envelope-wave solitons such as the non-critical collapse and decay (see [34]) can also be analysed by asymptotic methods described in the present paper.

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