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Asymptotic theory of plane soliton self-focusing in two-dimensional wave media

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Abstract

An asymptotic method is developed to describe a long-term evolution of unstable quasi-plane solitary waves in the Kadomtsev-Petviashvili model for two-dimensional wave media with positive dispersion. An approximate equation is derived for the parameters of soliton transversal modulation and a general solution of this equation is found in an explicit form. It is shown that the development of periodic soliton modulation, in an unstable region, leads to saturation and formation of a two-dimensional stationary wave. This process is accompanied by the radiation of a small-amplitude plane soliton. In a stable region, an amplitude of the modulation is permanently decreasing due to radiation of quasi-harmonic wave packets. The multiperiodic regime of plane soliton self-focusing is also investigated.

1. Introduction

One-dimensional nonlinear solitary waves are known to be unstable with respect to transversal perturbations in various physical media (for example, see reviews [1-3]). Starting from the classical papers by Zakharov [4] and Kadomtsev and Petviashvili [5], the phenomenon of transversal soliton instability was re-discovered, for instance, for surface and internal waves in a fluid [6,7], waves in shear flows [8], magneto-sonic and ion-acoustic waves in a magnetized plasma [9-11], as well as for envelope waves in homogeneous nonlinear dispersive (diffractive) media [12-15]. Analysis of this phenomenon was usually limited by the framework of a linear theory which allowed one to get the important and rather universal characteristic features of the initial stage of plane soliton transformation [1,2]. However, the problem of long-term development of unstable transversal modulation on a soliton background remains attractive and has no universal solution even for a two-dimensional geometry [16]. Only recently different groups of researchers have succeeded in considering this problem for two, the most important, models of Kadomtsev and Petviashvili (KP) and Zakharov and Kuznetsov (ZK) [17-21].

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The success of analytical investigation of wave processes described by the KP equation [17,18] is based on the integrability of this equation and on the existence of broad classes of exact solutions [22]. The most general conclusion of the paper [18] is that the instability leads to decay of a plane soliton into two and more stationary, transversely modulated nonlinear waves; their amount and parameters (period and modulation depth) are determined by the structure of an initial perturbation. The same scenario of the instability was observed in numerical simulations of the ZK and KP equations [20,21].

The detailed study of the obtained results enabled us to understand the reasons of failures of the asymptotic approaches which were used to describe the nonlinear stage of transversal soliton instability [16]. The approximate equations for a coordinate (phase) of a plane soliton were usually derived in a long-wave limit of transversal modulation. However, it follows from the found analytical and numerical solutions [17–21] that, in this limit, the self-focusing of the initially small perturbation leads to great growth of its amplitude so that a chain of two-dimensional solitons appearing from the unstable plane solitary wave has an amplitude exceeding essentially that of the original soliton. It is obvious that such a cardinal transformation of a wave field profile cannot be taken into account within the framework of a bounded asymptotic expansion in the vicinity of the one- dimensional soliton. As a result, singularities and ambiguities of the soliton phase versus the transversal coordinate appear within the derived approximate equations.

In the present paper we develop another approach to the problem of long-term evolution of an unstable quasi-plane soliton and apply it to the classical KP equation. It is well-known that instability of a one-dimensional soliton occurs only for long enough transversal perturbations with the wave number k bounded by a critical value k_c: $0 < k < k_c$. The main idea that allows us to describe the development of the transversal soliton instability from initial to final stages is to consider a short-wave limit in the instability region $k \to k_c$. In this limit, the amplitude of transversal modulation remains a small but finite value and hence the two-dimensional structure which is formed on the basis of a self-focusing soliton looks like a quasi-plane solitary wave with transversely modulated front [18,19]. Besides, at the nonlinear stage of the instability, there appears a radiation in the form of a one-dimensional wave with such a small amplitude that it can be taken into account within the bounded asymptotic expansion.

It should be noted that the idea of an asymptotic approach near the critical value k_c has been used in studying similar phenomena such as an appearance of a soliton collapse [13] or a bifurcation of two-dimensional stationary soliton solutions [9] accompanying the phenomenon of plane soliton transversal instability. However, neither the one-dimensional radiation nor its "solitary" features have been described yet. In our paper we shall not only construct a complete, self-consistent scheme of the asymptotic expansion but also compare our results with the known exact solutions to the KP equation. This comparison manifests an excellent correspondence between approximate and exact solutions making us expect that this method will be successful for other models of various origins.

This paper is constructed as follows. In Section 2 we discuss assumptions and a general scheme of the used asymptotic approach. Applying it up to the several first terms we find in Section 3 approximate equations for parameters of the quasi-plane soliton evolving under the action of transversal perturbations and for a radiation field distributed behind the soliton. Analysis of these equations is given in Section 4 for a particular case of a periodic soliton perturbation. In Section 5 we expand the boundaries of this analysis and consider the instability of a two-dimensional stationary solitary wave with transversely modulated front. A general solution of the initial value problem to the asymptotic equations is found in Section 6 by means of the linearization technique. Such a general solution is very useful for a description of a composite multiperiodic regime of plane soliton instability and self-focusing that is discussed in Section 7. Finally, Section 8 concludes the paper.

2. Scheme of asymptotic approach

It is convenient for further analysis to take the KP equation in the following form:

$$(4u_t + 12uu_x + u_{xxx})_x = 3u_{yy}. \tag{2.1}$$

Eq. (2.1) describes dynamics of smoothly two-dimensional, weakly nonlinear acoustic-type waves in media with weak positive dispersion in the reference frame moving with a "sound" speed. The coordinate system is chosen so that nonlinear solitary waves propagate along the x-axis, while linear quasi-harmonic waves on a zero mean background propagate in the opposite direction. The simplest solitary wave is a one-dimensional soliton expressed by the function

$$u_0(\xi; q) = \frac{q^2}{\cosh^2(q\xi)}, \qquad \xi = x - vt - s, \quad v = q^2, \tag{2.2}$$

where q is a parameter of soliton velocity, characteristic scale and amplitude and s is an arbitrary phase constant.

It should be mentioned that Eq. (2.1) was historically derived specifically for stability analysis of the solitons (2.2) [5]. As was found, one-dimensional solitons in positive-dispersion media are unstable with respect to transversal modulations of their fronts. Moreover, Zakharov succeeded in constructing an explicit form of unstable mode for Eq. (2.1) linearized on the solution (2.2) [23]:

$$\delta u_0(\xi; y, t; q, p) = \frac{\partial^2}{\partial \xi^2}\left(\frac{\exp(p\xi)}{\cosh(q\xi)}\right)\exp(\lambda t + iky). \tag{2.3}$$

This mode is bounded in space for $|p| \le q$ and has parameters λ, k and p related by the algebraic equations

$$\lambda = p(q^2 - p^2), \quad k = q^2 - p^2. \tag{2.4}$$

Eliminating p from (2.4), we find a direct relationship between the growth rate of soliton instability and transverse wave number

$$\lambda^2 = k^2(q^2 - k), \tag{2.5}$$

which is shown in Fig. 1 by a solid line. As it follows from (2.5), growth of the linear mode slows down in a long-wave limit $k \to 0$ and near a critical point $k \to k_c = q^2$ which gives us a possibility to describe evolution of a transversal soliton perturbation by means of an asymptotic technique. It is important to note that the relationship between parameters λ^2 and k differs in both the limits being quadratic for long-wave perturbations and linear near the critical point. The asymptotes of the dependence $\lambda^2(k)$ are presented in Fig. 1 by dotted lines.

In order to develop our asymptotic method in the short-wave region of the instability domain we introduce the "fast" coordinates along the longitudinal (ξ) and transversal (y) directions with respect to the stationary soliton (2.2) as well as the "slow" variables of time $T = \epsilon t$ and of transversal coordinate $Y = \epsilon^2 y$, where ϵ is supposed to be a small parameter. The "slow" variables describe an appearance of the linear soliton instability when the transverse wave number deviates from the critical value k_c to a long-wave region.

It follows from (2.3) that the linear mode near k_c ($p \sim 0$) is expressed by an even function δu_0 with respect to the variable ξ and, therefore, it represents an amplitude modulation of the original soliton. Nevertheless, nonlinear effects might also excite a phase modulation which is expressed by an odd function δu_0 at $p = q$. The phase modulation can be taken into account if the parameter s is supposed to be a function of slow variables $s = s(T, Y)$. The discussed difference between the dependence λ^2 on k in both the asymptotic limits leads to

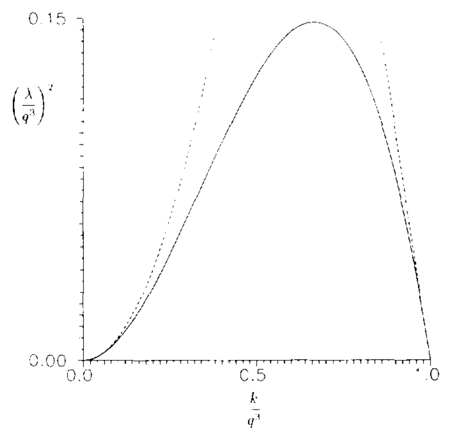

Fig. 1. Dependence of square growth rate on transverse wave number for the exact solution (2.5) (solid line) and its asymptotes at $k \to 0$ and $k \to q^2$ (dotted lines).

the soliton instability with respect to the transversal perturbations of the soliton phase in the higher-order terms compared to the instability caused by the growth of amplitude modulation.

Generally speaking, soliton acceleration which is proportional to s_{TT} gives rise to algebraically secular terms of an asymptotic series [11,14,19]. There exist several methods to localize the expansion; for example, regrouping secular terms and proving their convergence [11], or matching fields outside the soliton and in its neighborhood [14]. In our paper this problem is solved by means of introducing a secular component of radiation field which is proportional to a function $C(T, X, Y, \tau)$ depending, excepting the discussed above variables, on a smooth longitudinal coordinate $X = \epsilon x$ ("outer" with respect to exponentially localized soliton) and still slower time $\tau = \epsilon^3 t$ along which the radiation field evolves.

Combining all assumptions, we rewrite Eq. (2.1) in the reference frame ξ of a perturbed plane soliton (2.2) and introduce new smooth independent variables:

$$\begin{aligned}\left(-4vu + 6u^2 + u_{\xi\xi}\right)_{\xi\xi} - 3u_{yy} &= \epsilon\left(4s_T u_{\xi\xi} - 4u_{\xi T} + 4vu_{\xi X} - 12(u^2)_{\xi X} - 4u_{\xi\xi\xi X}\right)\\ &+\epsilon^2\left(4s_T u_{\xi X} - 4u_{XT} - 6(u^2)_{XX} - 6u_{\xi\xi XX} + 6u_{yY} - 6s_Y u_{\xi y}\right)\\ &+\epsilon^3\left(-4u_{\xi\tau} - 4u_{\xi XXX}\right)\\ &-\epsilon^4\left(-4u_{X\tau} - u_{XXXX} + 3u_{YY} - 6s_Y u_{\xi Y} + 3(s_Y)^2 u_{\xi\xi} - 3s_{YY}u_\xi\right). \end{aligned} \tag{2.6}$$

Let us expand a solution to Eq. (2.6) in series with respect to a small parameter ϵ^ν where the power ν must be chosen so that the nonlinear effects, accompanying the growth of soliton perturbations, be indicated in the same order of smallness as linear instability effects. The detailed analysis reveals that this condition is fulfilled if $\nu = 1/2$. Thus, the formal expansion of a solution to Eq. (2.6) has the form:

$$u = u_0(\xi; q) + \sum_{n=1}^{\infty} \epsilon^{n/2} u_{n/2}(\xi, y; T, X, Y, \tau; q), \tag{2.7}$$

where the original plane soliton u_0 is expressed by the formula (2.2), the first correction $u_{1/2}$ is a solution of a linearized problem

$$\boldsymbol{L}u_{1/2} = 0, \quad \boldsymbol{L} = -4v\frac{\partial^2}{\partial\xi^2} + 12\frac{\partial^2}{\partial\xi^2}(u_0) + \frac{\partial^4}{\partial\xi^4} - 3\frac{\partial^2}{\partial y^2}, \tag{2.8}$$

and the higher-order corrections $u_{n/2}$ are calculated consecutively from the linear inhomogeneous equations

$$Lu_{n/2} = H_{n/2}(u_0, u_{1/2}, \ldots, u_{n/2-1}), \quad n \geq 2. \tag{2.9}$$

The right-hand sides of Eqs. (2.9) are easily expressed by the lower-order terms as

$$H_1 = 4s_T u_{0\xi\xi} - 6\left(u_{1/2}^2\right)_{\xi\xi},$$
$$H_{3/2} = 4s_T u_{1/2\xi\xi} - 4u_{1/2\xi T} - 12\left(u_{1/2}u_1\right)_{\xi\xi},$$
$$H_2 = 4s_T u_{1\xi\xi} - 4u_{1\xi T} - 6\left(u_1^2 + 2u_{1/2}u_{3/2}\right)_{\xi\xi},$$

and so on.

An arbitrary solution to the linearized equation (2.8) with amplitude $b(T,Y)$ specifies all higher-order terms of the asymptotic series (2.7). For a uniform convergence of the series we demand that each correction $u_{n/2}$ should satisfy the boundary conditions

$$u_{n/2}(\xi, y; T, X, Y, \tau; q) \to 0 \quad \text{when} \begin{cases} \xi \to \pm\infty \\ X \to \pm\infty \end{cases}. \tag{2.10}$$

The conditions (2.10) imply that all higher-order terms are localized functions throughout the region where the field of the original soliton is negligible. Since solutions to Eqs. (2.9) contain both exponentially and algebraically growing terms [11], we remove them restricting the functions $b(T,Y)$, $s(T,Y)$ and $C(T,X,Y,\tau)$ by certain partial differential equations.

3. Approximate equations for soliton parameters and for soliton radiation field

The linearized equation (2.8) has only two solutions localized at $\xi \to \pm\infty$ which are expressed by the function (2.3) at $p = q$, $k = 0$ and $p = 0$, $k = k_c$. In the first case, the function δu_0 gives a simple correction to the soliton phase s and is taken into account in the zero order of an asymptotic series (2.7). In the second case, this function describes a perturbation which grows slowly in time and oscillates intensively along the transversal coordinate. The evolution of such a perturbation is to be considered in our paper. Therefore, we choose a solution $u_{1/2}$ in the following form:

$$u_{1/2} = \left[b(T,Y)\exp(ik_c y) + \text{c.c.}\right] \frac{\partial^2}{\partial \xi^2}\left(\frac{1}{\cosh(q\xi)}\right). \tag{3.1}$$

Other solutions to Eq. (2.8) have nonzero asymptotes at $\xi \to \pm\infty$ and do not satisfy the boundary conditions (2.10). Hence, they should be eliminated from the perturbation $u_{1/2}$.

It is obvious that the operator L admits for separating the variables ξ, y. As a consequence, solutions $u_{n/2}$ of the linear inhomogeneous equations (2.9) can be presented by a sum $\sum_{m=0}^{n} b_{n/2}^{(m)}(\xi)\exp(imk_c y) + \text{c.c.}$ where the amplitudes $b_{n/2}^{(m)}$ are found from solutions of ordinary differential equations for each harmonic of the transverse wave number. Thus solving Eq. (2.9) for $n = 2$ we get a correction u_1 which satisfies the boundary conditions (2.10)

$$u_1 = \left[s_T - 3q^2|b|^2\right]\frac{\partial u_0}{\partial v} - 2|b|^2\cos^2(\Psi)\frac{\partial^2}{\partial \xi^2}\left(\frac{1}{\cosh^2(q\xi)}\right), \tag{3.2}$$

where $b = |b|\exp(i\phi)$ and $\Psi = k_c y + \phi$. The function u_1 describes a velocity shift of the perturbed soliton as well as the zero and double harmonics of a transversal perturbation appearing due to nonlinear effects.

However, the higher-order corrections $u_{3/2}$ and u_2 include exponentially growing terms and do not satisfy the boundary conditions (2.10). So, solutions to Eqs. (2.9) for $n = 3, 4$ are as follows:

$$u_{3/2} = \left[\frac{b_T}{v}\exp(ik_c y) + \text{c.c.}\right]\frac{\partial^2}{\partial\xi^2}\left(\frac{\xi}{\cosh(q\xi)}\right) + q^4\left[s_T - q^2|b|^2\right]|b|\cos(\Psi)\tilde{u}_{3/2}(\xi)$$
$$+2\left[s_T - 3q^2|b|^2\right]|b|\cos(\Psi)\frac{\partial}{\partial v}\frac{\partial^2}{\partial\xi^2}\left(\frac{1}{\cosh(q\xi)}\right) + \tfrac{8}{3}|b|^3\cos^3(\Psi)\frac{\partial^2}{\partial\xi^2}\left(\frac{1}{\cosh^3(q\xi)}\right), \tag{3.3}$$

$$u_2 = -\frac{1}{8q^3}\left[s_{TT} + 3q^2\left(|b|^2\right)_T\right]\frac{\partial^2}{\partial\xi^2}\left(\xi^2\tanh(q\xi)\right) + \left[s_{TT} - q^2\left(|b|^2\right)_T\right]\tilde{u}_2(\xi)$$
$$-\frac{1}{q^2}\left[\left(|b|^2\right)_T + \tfrac{1}{2}\left((b^2)_T\exp(2ik_c y) + \text{c.c.}\right)\right]\frac{\partial^2}{\partial\xi^2}\left(\frac{\xi}{\cosh^2(q\xi)}\right) + \tfrac{1}{2}\left[s_T - 3q^2|b|^2\right]^2\frac{\partial^2 u_0}{\partial v^2}$$
$$-3|b|^2\left[s_T + q^2|b|^2\right]\frac{\partial u_0}{\partial v} - 2\left[s_T - 3q^2|b|^2\right]|b|^2\cos^2(\Psi)\frac{\partial}{\partial v}\frac{\partial^2}{\partial\xi^2}\left(\frac{1}{\cosh^2(q\xi)}\right)$$
$$-4|b|^4\cos^4(\Psi)\frac{\partial^2}{\partial\xi^2}\left(\frac{1}{\cosh^4(q\xi)}\right), \tag{3.4}$$

where the functions $\tilde{u}_{3/2}, \tilde{u}_2$ grow exponentially along the ξ-axis. Appearance of exponential divergences in solutions of (2.9) is caused by the existence of localized eigenfunctions of the adjunctive operator L^* at the zero and the first harmonics of k_c. In order to remove both the divergences, two restrictions must be specified for variations of parameters b and s.

However, as follows from (3.3), (3.4), only one equation is a sufficient condition for removing both the exponentially growing terms:

$$s_T - q^2|b|^2 = O(\epsilon). \tag{3.5}$$

This equation gives a simple relationship between an amplitude of the transversal perturbation and a correction to the velocity of a quasi-plane soliton: the larger the transversal perturbation, the faster the solitary wave. Such a relationship is valid both for a non-stationary, evolving soliton and for a stationary weakly modulated solitary wave. In order to investigate the effects associated with the development of soliton instability we need to consider higher-order terms of the expansion (2.7).

It is essential for our analysis that the first term of the correction u_2 (3.4) includes a non-localized component and does not satisfy the boundary conditions (2.10) along the ξ-coordinate. Unlike the exponentially growing terms, such algebraically secular terms cannot be removed from the asymptotic series (2.7). However, we can make the expansion converging along the longitudinal coordinate X which is "outer" with respect to the soliton rather than along the "inner" coordinate ξ. For this purpose, we introduce a new component of the correction u_2 which is a non-localized eigenfunction of the operator L:

$$\delta u_2 = C(T, X, Y, \tau)\left(1 - 3\frac{\partial u_0}{\partial v}\right). \tag{3.6}$$

The superposition field $u_2 + \delta u_2$ at $\xi \to \pm\infty$ tends to the following values:

$$u^{\pm}(T, X, Y, \tau) = C \mp D, \quad D(T, Y) = \frac{s_{TT} + 3q^2\left(|b|^2\right)_T}{4q^3}. \tag{3.7}$$

The components $u^{\pm}$ are those of the radiation field propagating in front of ($\xi \to +\infty$) and behind ($\xi \to -\infty$) the moving soliton and varying along the smooth coordinate X. Under the condition that the functions $u^{\pm}$ fall to zero at $X \to \pm\infty$ faster than X^{-m}, where m is an arbitrary natural number, the second-order and the higher-order terms $u_{n/2}$ satisfy the conditions (2.10) outside the soliton location.

Using these assumptions, we are able to find solutions to Eqs. (2.9) for $n = 5, 6$ which are bounded along the coordinates ξ and/or X. Let us omit the bulky expressions for these corrections and write only equations for functions b and s which are the conditions of removing exponentially growing terms at the first and zero harmonics of the series (2.7) up to the fifth- and sixth-order terms:

$$\epsilon^{3/2}\left[q^4\left(s_T - q^2|b|^2\right)b\right] + \epsilon^{5/2}\left[iq^4 b_Y - b_{TT} - 4q^6 b|b|^4 - 3q^4 bC_0\right] + O(\epsilon^{7/2}) = 0, \tag{3.8a}$$

$$\epsilon^2\left[s_{TT} - q^2\left(|b|^2\right)_T\right] + \epsilon^3\left[-3q^2\left(|b|^4\right)_T - C_{0T} - vC_{0X}\right] + O(\epsilon^4) = 0, \tag{3.8b}$$

where $C_0(T, Y) = C|_{X=X_s(T),\tau=0}$ is a component of the radiation field at the point of location of the moving soliton and X_s is its coordinate along the X-axis so that in the leading order $X_s(T) = X_0 + v(T - T_0)$, where T_0 is an initial time moment of soliton evolution, X_0 is its initial coordinate, and $v = q^2$. Since the operator

$$\left(\frac{\partial}{\partial T} + v\frac{\partial}{\partial X}\right)\Bigg|_{X=X_s(T)} = \frac{d}{dT}\Bigg|_{X=X_s(T)}$$

is an operator of the complete derivative in the reference frame propagating together with the soliton, Eq. (3.8b) can be integrated and parameter s_T can be eliminated from all subsequent expressions. Then, we get a unique equation describing a leading order of amplitude variations:

$$b_{TT} - iq^4 b_Y + q^6 b|b|^4 + 2q^4 bC_0 = 0. \tag{3.9}$$

In order to close this equation, we need to relate the component of radiation field C_0 at the moving soliton to the amplitude b. For this, we find equations describing the evolution of the radiation field $u^{\pm}$ far from the soliton. Let us consider terms in r.h.s. of Eq. (2.6) which do not contain derivatives with respect to the coordinate ξ. On integration these terms give rise to algebraic secularities of the solutions to Eqs. (2.6) ($\sim \xi^2$) if we do not put restrictions on the $u^{\pm}$ variations. As follows from (2.6), the first such secularities are expected in the order $n = 8$ and are removed if the equations

$$4\left(u^{\pm}\right)_{XT} = 0 \tag{3.10}$$

are met. These equations imply that the soliton radiation does not move in the laboratory reference frame on the time scales along which the instability of a perturbed soliton develops. It is obvious that if there are no non-localized perturbations in front of the soliton at the initial time moment, they cannot appear there during the instability development. Hence $u^+ = 0$, and Eq. (3.7) determines the component C_0,

$$C_0 = D = \frac{\left(|b|^2\right)_T}{q} + O(\epsilon). \tag{3.11}$$

This relation completes Eq. (3.9) which takes the form

$$b_{TT} - iq^4 b_Y + q^6 b|b|^4 + 2q^3 b\left(|b|^2\right)_T = 0. \tag{3.12}$$

Eq. (3.12) looks like the Eckhaus equation which is known to be completely integrable [24,25]. However, we note that the initial-value problem for the Eckhaus equation was usually considered as a first-order problem along variable Y. In our case soliton perturbations evolve along variable T and we have to consider the second-order initial-value problem for Eq. (3.12).

On the other hand, evolution of the radiation field u^- escaping behind the soliton occurs on the slower time scale described by the variable τ. The corresponding equation for u^- is found by integrating Eqs. (2.9) for $n = 12$,

$$\left(4u^-_\tau + 12u^-u^-_X + u^-_{XXX}\right)_X = 3u^-_{YY}, \tag{3.13}$$

and is nothing but the original KP equation which is rewritten for smooth variables X, Y, τ and for the small function u^-. It is important to emphasize that the nonlinear properties of the wave medium are no less essential for evolution of the radiation than the dispersive ones.

The initial value problem for the evolution equation (3.13) is as follows. Since variations of the perturbations localized near the soliton and of the radiation field escaping from the soliton are separated in time, we may consider these two processes independently. We may suppose that, at first, the original soliton goes to infinity leaving behind the immobile distribution

$$u^-(X,Y,\tau=0) = \begin{cases} 2q^{-1}\left(|b|^2\right)_T|_{T=T_s(X)} & \text{for } X \geq X_0 \\ 0 \quad \text{for } X < X_0 \end{cases}, \tag{3.14}$$

and, then, the radiation field evolves along the variable τ. Here $T_s(X)$ is the inverse function of $X_s(T)$ and, in the leading order, it has the simple form, $T_s(X) = T_0 + (X - X_0)/q^2$. Therefore, the distribution (3.14) depending on the variables X and Y is an initial condition for the initial-value problem to Eq. (3.13).

Eqs. (3.12)–(3.14) conclude construction of the leading order of an asymptotic expansion near the perturbed quasi-plane soliton. As we shall show in the next section, these equations describe successfully the dynamics of unstable transversal perturbations from initial growth to final saturation. The higher-order terms of the series (2.7), which are negligible in the limit $k \to k_c$, do not change the principal picture of the soliton dynamics but define more accurately some quantitative results.

4. Soliton dynamics under a single-periodic perturbation

Eq. (3.12) governs evolution of a transversal soliton perturbation depending on the type of linear stability and the nonlinear effects preventing its growth against the background of the original, strongly nonlinear wave. The last term in this equation describes nonlinear dissipative effects which are caused by the energy flow from the soliton to the radiation field. Here we analyze a combined action of these three effects (linear instability, nonlinear stabilization and radiation-induced dissipation) for a periodic transversal perturbation. In this case, the amplitude has the form $b(T,Y) = B(T)\exp(i\Delta kY)$ and Eq. (3.12) reduces to the second-order ordinary differential equation

$$\ddot{B} + q^4\Delta kB + q^6B^5 + 4q^3B^2\dot{B} = 0. \tag{4.1}$$

First, we consider a perturbation with transverse wave number lying in the left vicinity of the critical value k_c ($\Delta k = -p^2 < 0$) where a non-modulated state $B = 0$ is unstable. In this region, besides the zero equilibrium state of a saddle type with the growth rate $\lambda = pq^2$, there exists a symmetrical pair of nontrivial equilibrium states $B = \pm B_0$, where $B_0 = (p/q)^{1/2}$. They appear from the non-modulated solution at $\Delta k = 0$ as a result of a "triple equilibrium" bifurcation. New equilibrium states are images of a stationary solitary wave with a front weakly modulated in a transversal direction. As is well-known, far from the point k_c, this branch of stationary solutions describes a periodic chain of two-dimensional KP solitons and transforms in the limit $k \to 0$ to the individual KP soliton [18,19].

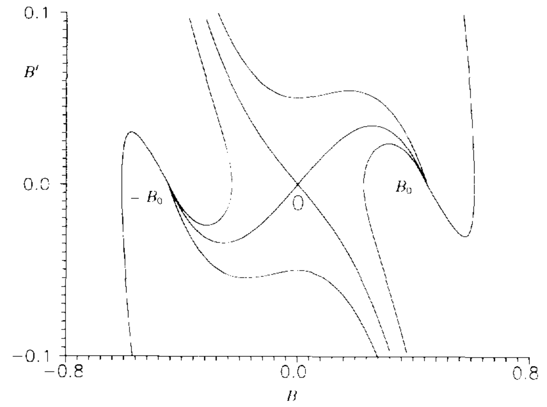

Fig. 2.

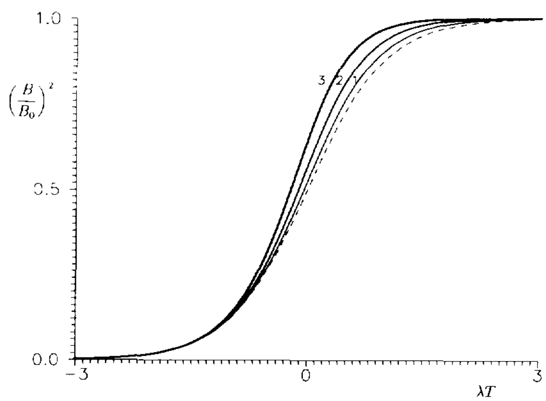

Fig. 3.

Fig. 2. Phase plane of Eq. (4.1) in an unstable case $q = 1$, $p = 0.2$.

Fig. 3. Evolution of the square amplitude of a single-periodic soliton perturbation for the exact solution (4.5) at $q = 1$, $p = 0.1$ (curve 1), $p = 0.2$ (2), $p = 0.3$ (3) and for the approximate solution (4.2) (dashed line).

The equilibrium states $B = \pm B_0$ are stable "degenerate nodes" with a double decrement -2λ which attract almost all the trajectories of the dynamical system (4.1). The phase plane $(B, \dot{B})$ is illustrated in Fig. 2. The separatrix curves connecting the unstable and stable equilibrium states correspond to the solutions bounded everywhere which describe the instability development of a soliton non-modulated at $T \to -\infty$. We found an explicit form of the dependence $B(T)$ for this process

$$B = \pm B_0 \sqrt{\tfrac{1}{2}(1 + \tanh(\lambda T))}. \tag{4.2}$$

The function (4.2), in dimensionless variables $(B/B_0)^2$ and λT, is shown in Fig. 3 by a dashed line. Thus, the initial growth of transversal modulation on the soliton background, which occurs in accordance with the relations (2.4) in the limit $k \to k_c$, slows down under the action of nonlinear effects and results in saturation of amplitude B near a new, modulated equilibrium state. The similar picture of the development of soliton instability in the same asymptotic limit was revealed earlier for the ZK equation [9].

Such an irreversible scenario of plane soliton instability is accounted for energy flow to the radiation field which is distributed behind the soliton according to (3.14). Since the solution (4.2) is bounded everywhere, the problem of soliton instability along the separatrix curves may be considered along the infinite time interval $-\infty < T < \infty$, i.e. we may choose the initial time moment T_0 to be $T_0 = -\infty$. Then, the radiation field is determined throughout the X-axis and has the form

$$u^-(X, \tau = 0) = \frac{p^2}{\cosh^2(pX)}. \tag{4.3}$$

Time evolution of the radiation field escaping from the soliton with a single-periodic transversal perturbation obeys a one-dimensional analog of Eq. (3.13) which is referred to as the Korteweg–de Vries (KdV) equation (see, i.e. [22]). As was found a long time ago, all the initial energy is distributed in the evolution process between the nonlinear discrete part of the wave field (solitons) and the linear continuous part (dispersive waves). As the function (4.3) completely coincides with the profile of the one-dimensional soliton $u_0(X - \delta v\tau; p)$ at the time moment $\tau = 0$, the evolution of the initial distribution (4.3) along the variable τ is presented by a simple stationary drift of the soliton with velocity $\delta v = p^2$. Thus, in the leading order of our asymptotic

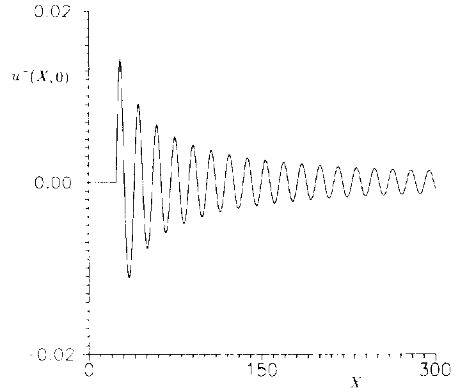

Fig. 4. Profile of the radiation field behind quasi-plane soliton in a stable region for $q = 1$, $r = 0.2$.

theory all the energy emitted by the unstable soliton is condensed in the nonlinear soliton structure (4.3) without excitation of quasi-harmonic wave packets.

Since all the trajectories which are different from the separatrix one go to infinity at $T \to -\infty$, we should consider the corresponding soliton dynamics only for a finite time interval $T \geq T_0$, where T_0 is finite, and $B(T_0)$ is supposed to have a sufficiently small initial value. Then, as follows from Fig. 2, an arbitrary single-periodic soliton perturbation lying outside the input separatrix curves leads to formation of a weakly modulated wave with amplitude $\pm B_0$. However, the component u^- emitted by the soliton has, in a general case, a more complicated form than (4.3) and is determined only for $X \geq X_0$. Obviously, the evolution of such an initial distribution leads to energy losses in a non-soliton, dispersive component of the wave field.

These facts strengthens our confidence that the processes of pure soliton instability discovered in [18] for a field profile presented at $t \to -\infty$ by a one-dimensional soliton (2.2) with a small but growing perturbation (2.3) are not exceptional mathematical artifacts and could be observed in a broad range of real soliton perturbations. In particular, such processes were easily revealed in numerical simulations carried out for the KP equation [21].

Let us consider the opposite case $\Delta k > 0$ when a non-modulated soliton is neutrally stable in a linear approximation. For this case, the state $B = 0$ is a center and there are no other equilibrium states. The nonlinear dissipation results in the decrease of the amplitude of oscillating transversal perturbations and the non-modulated soliton becomes asymptotically stable at $T \to +\infty$. A general solution of Eq. (4.1) for $\Delta k = r^2$ has the form

$$B = \frac{C\cos(\omega T + \Phi)}{q^{3/2}\left[1 + C^2\left(T + \sin(2(\omega T + \Phi))/2\omega\right)\right]^{1/2}}, \tag{4.4}$$

where $\omega = rq^2$, and C and Φ are arbitrary constants of the amplitude and phase of oscillating perturbation.

The formula (4.4) gives a slow, power law of soliton modulation damping $B \sim T^{-1/2}$ at $T \to +\infty$. The energy loss of an initial soliton perturbation is accounted for excitation of a one-dimensional wave packet behind the unstable soliton which looks like a quasi-harmonic wave with a slowly varying amplitude (Fig. 4). As a result of the evolution process, such a wave packet disperses and becomes uniformly vanished, while small-amplitude solitons do not appear in this process. So, the excitation of linear dispersive waves rather than nonlinear solitary waves leads to asymptotic stability of the original non-modulated soliton in the right vicinity of the critical point k_c. It should be noted that the same mechanism of soliton stability was discovered in a whole range of k for negative-dispersion media described by the KP equation [26–29]. However, in such media the radiation field was excited in a linear approximation, while in our case the dissipative effects have a

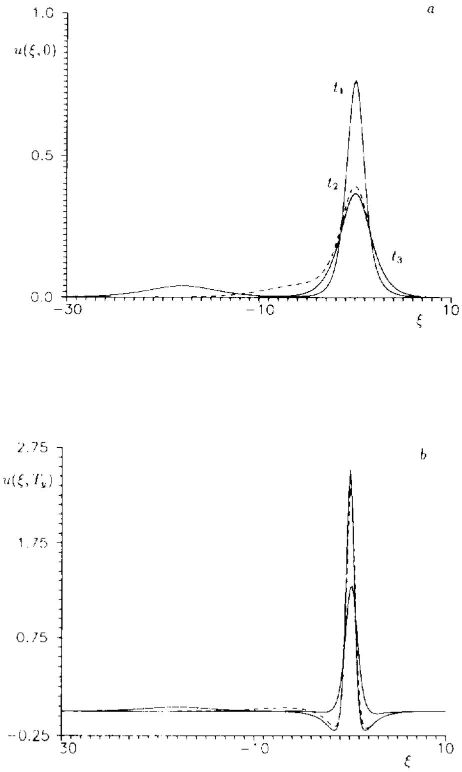

Fig. 5. Successive stages of plane soliton instability for the exact solution (4.5) at $q = 1, p = 0.2$ ($t_3 > t_2 > t_1$). (a) corresponds to the projection $u(x, y, t)$ at the minimum of the transversal perturbation and (b) corresponds to that at the maximum. The reference frame moves with velocity of the soliton maximum.

nonlinear origin.

In order to estimate an accuracy of our asymptotic approach, we compare the instability development within the framework of the approximate solutions (4.2), (4.3) and of the corresponding exact solution found in [18]. The exact formula for the original variable $u(x, y, t)$ has the following form:

$$u = \frac{\partial^2}{\partial x^2} \ln \left[1 + \exp(2q\xi) + \exp(2p\xi + 2\lambda t) + \frac{4\sqrt{pq}}{p+q} \exp((p+q)\xi + \lambda t)\cos(ky) \right], \tag{4.5}$$

where the parameters λ, k are related to p, q by (2.4). According to [18], the solution (4.5) describes the decay of the original soliton (2.2) under the action of a small but growing perturbation (2.3) into a new plane soliton $u_0(x - p^2 t; p)$ and a quasi-plane solitary wave with transversely modulated front

$$u_p(\eta, y; q, p) = (q-p)^2 \frac{1 + 2B_0 \cos(ky)\cosh((q-p)\eta)}{[\cosh((q-p)\eta) + 2B_0\cos(ky)]^2}, \tag{4.6}$$

where $B_0 = \sqrt{pq}/(p+q)$, $\eta = x - wt$, and $w = p^2 + q^2 + pq$.

Figs. 5a,b show successive stages of this process near the point of neutral stability ($q = 1$, $p = 0.2$) for two cross-sections of the function $u(x, y, t)$ at the minimum and maximum of the transversal modulation. The saturation of the amplitude of the growing perturbation and the formation of a "tail" behind the perturbed soliton are well manifested in these figures. Note that the approximate solution (4.3) describes the profile of radiated soliton exactly, while the first terms of the asymptotic series (3.1)–(3.4) at $b = (p/q)^{1/2}\exp(-ip^2Y)$ correspond to the expansion of the solution (4.6) in the limit $p \ll q$.

The quantity $(u_{\max} - u_{\min})/2$ can easily be calculated from the distribution of a wave field at fixed time moments and may be taken roughly as an amplitude of the transversal modulation B. Our asymptotic theory predicts an universal form of the dependence $B(T)$ in the dimensionless variables $(B/B_0)^2$, λT (Fig. 3, dashed line). Comparison of this dependence with the ones calculated from (4.5) for the parameter k moving away from the critical value k_c (Fig. 3, solid lines) reveals a good correspondence.

5. Instability and decay of a weakly modulated wave

Although a weakly modulated wave appears as a result of self-focusing of a plane soliton, there is a possibility of its own instability with respect to perturbations with other transverse wave numbers. Indeed, one can readily find a pair of such perturbations B_+, B_- which grow in a linear approximation against the background of a weakly modulated wave with amplitude B_0

$$b = \big(B_0 + B_+ \exp[\gamma T + i(\delta k Y + \psi)] + B_- \exp[\gamma T - i(\delta k Y + \psi)]\big)\exp(i\Delta k Y), \tag{5.1}$$

where $B_0 = (-\Delta k)^{1/4}/q^{1/2}$, $(B_+ - B_-)/(B_+ + B_-) = -(\delta\lambda/\gamma)^2$, $\lambda = (-\Delta k)^{1/2}q^2$, $\delta\lambda = (\delta k)^{1/2}q^2$, ψ is an arbitrary phase displacement with respect to the phase of the original wave, and the growth rate γ is found from an algebraic fourth-order equation

$$\gamma^2(\gamma + 2\lambda)^2 = (\delta\lambda)^4. \tag{5.2}$$

It follows from (5.2) that an arbitrary small deviation of the length of a transversal perturbation from the period of the original wave with the wave number $k_0 = k_c + \Delta k < k_c$ generates its instability and growth of two symmetrical wave-satellites B_+, B_- in the right and left vicinities of the value k_0 : $k_\pm = k_c + \Delta k \pm \delta k$. We may expect a competition between the growing perturbations due to nonlinear and dissipative effects and a selection only of one transverse wave number in the limit $T \to +\infty$.

As the various multiplied and combined harmonics of the original wave number are excited due to the instability development, the amplitude of the transversal perturbation $b(T,Y)$ is no longer described by a finite-dimensional dynamical system. Nevertheless, we succeed in seeking a partial solution to Eq. (3.12) which corresponds to the development of the perturbation (5.1):

$$b = \left[\frac{1 + \exp(\gamma T - i\delta k Y)}{\big(1 + 4p\exp(\gamma T)\cos(\delta k Y)/(p+s) + p\exp(2\gamma T)/s\big)^{1/2}}\right] B_0 \exp(i\Delta k Y), \tag{5.3}$$

where we introduce the parametrization $\Delta k = -p^2$, $\delta k = s^2 - p^2$, $B_0 = (p/q)^{1/2}$, $B_+/B_- = -p/s$, and $\gamma = (s-p)q^2$ and set the phase constants to be zero.

The profile of the function $|b|^2$ vs. Y at successive time instants is shown in Fig. 6 for $q = 1$, $p = 0.2$, and $s = 0.4$. It demonstrates that the competition between the wave-satellites (5.1) results in damping short-wave perturbation with the wave number k_+ and formation, at $T \to +\infty$, of a long-wave structure with the wave number $k_- = q^2 - s^2 < k_0$ and amplitude $B_\infty = (s/q)^{1/2}$. Thus, the deviation of the transverse wave number of a quasi-plane soliton from the critical value k_c increases due to such a secondary instability of weakly modulated wave.

The instability development described by Eq. (5.3) is accompanied by emission of a radiation field which exactly coincides with the solitary wave with transversely modulated front $u_p(X - \delta w\tau, Y; s, p)$ (see formula

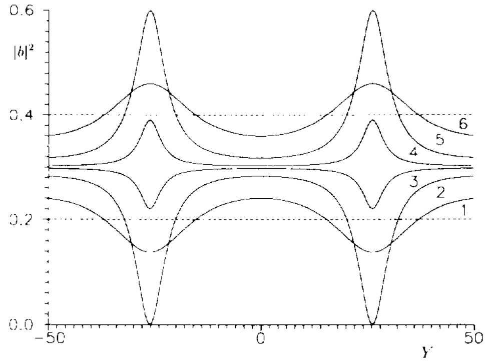

Fig. 6. The instability development of modulated wave for the approximate solution (5.3) at $q = 1, p = 0.2, s = 0.4$. The figures correspond to successive time moments counted in equal intervals. Dashed lines designate the initial ($|b|^2 = B_0^2 = 0.2$) and the final ($|b|^2 = B_\infty^2 = 0.4$) states.

(4.6)). This wave propagates with a small constant velocity $\delta w = p^2 + s^2 + ps$ behind the fast quasi-plane soliton. We would like to point out that the transverse wave number δk for the radiated wave may lie far from the critical value $\delta k_c = s^2$. Moreover, it may be infinitesimal and, in the limit $\delta k \to 0$, we may observe the formation of an individual two-dimensional soliton in the radiation field.

Summarizing these results we conclude that the instability of a weakly modulated wave with the transverse wave number $k_0 = q^2 - p^2$ leads to its decay into two modulated waves, the fast one with the wave number $k_- = q^2 - s^2$ described by the first terms (3.1)–(3.4) of the asymptotic series (2.7) and the slow one with the wave number $\delta k = s^2 - p^2$ described by the component u^- of the radiation field. The same conclusion follows from the exact solutions to the KP equation [18]. For example, the growth rate of the instability has an exact expression $\gamma = (s - p)(q - s)(q + p + s)$ and corresponds to our result in the limit $p, s \ll q$.

6. General solution to the asymptotic equation

Partial solutions to Eq. (3.12) which were found in an explicit form in Sections 4 and 5 give us a hope to reveal hidden symmetries of this nonlinear equation. It is not surprising, of course. As the original KP equation is an integrable system and has a broad class of exact solutions [22], we may expect that its asymptotic reduction, Eq. (3.12), inherits some remarkable properties. In this section we shall show that Eq. (3.12) can be linearized by replacing the dependent variables similarly to the Eckhaus equation discussed by Calogero and de Lillo [25]. Such a linearization enables us to investigate the evolution of a quasi-plane soliton and its radiation field in a general case.

Let us introduce a complex function $g(T, Y)$ and a real one $f(T, Y)$ by means of the following equations:

$$b = \frac{g}{\sqrt{2q^3 f}}, \quad |b|^2 = \frac{f_T}{2q^3 f}, \tag{6.1}$$

Substituting (6.1) into (3.12), we reduce the nonlinear equation to a system

$$f_T = gg^*, \tag{6.2}$$

$$iq^4(gf_Y - 2g_Y f) + gf_{TT} - 2g_T f_T + 2g_{TT} f = 0. \tag{6.3}$$

We can eliminate f from Eq. (6.3) by adding this equation multiplied by the function g^* to the complex conjugated one and using the relationship (6.2). Then, the final equation, which is

$$iq^4(gg_Y^* - g_Y g^*) + gg_{TT}^* + g_{TT}g^* = 0, \tag{6.4}$$

immediately leads to the linear Schrödinger equation

$$iq^4 g_Y = g_{TT} \tag{6.5}$$

if the function g is bounded along the transversal coordinate Y.

Thus, the original equation (3.12) which is nonlinear and dissipative can be reduced to the linear Schrödinger equation (6.5). Now we shall show that such a linearization gives a general solution of the initial value problem to Eq. (3.12). Let us take at an initial time instant T_0 the distributions $b(T_0, Y) \equiv b_0(Y)$, $b_T(T_0, Y) \equiv b_1(Y)$, and, additionally, $b|b|^2(T_0, Y) \equiv b_2(Y)$. Our aim is to construct the function $b(T, Y)$ for $T \geq T_0$ according to Eq. (3.12). The solution of this problem is expressed by the formulas (6.1), where the functions g and f can be found in an explicit form from solutions to Eqs. (6.5) and (6.2)

$$g(T,Y) = \int_{-\infty}^{+\infty} G_1(\kappa) \exp[\sqrt{-\kappa}q^2(T - T_0) + i\kappa Y] d\kappa$$

$$+ \int_{-\infty}^{+\infty} G_2(\kappa) \exp[-\sqrt{-\kappa}q^2(T - T_0) + i\kappa Y] d\kappa, \; f(T,Y) = 1 + \int_{T_0}^{T} |g|^2 dT. \tag{6.7}$$

The integrals in the formula (6.6) are nothing but the Fourier-transform of the function $g(T, Y)$ with respect to the transversal coordinate Y at a certain time moment T. Using Eqs. (6.1) we can relate the spectral kernels $G_1(\kappa)$, $G_2(\kappa)$ of the Fourier-transform to the corresponding kernels of the given initial functions $b_0(Y), b_1(Y)$, and $b_2(Y)$ which we designate as $\hat{b}_0(\kappa), \hat{b}_1(\kappa)$, and $\hat{b}_2(\kappa)$. It is not difficult to prove that such relations are expressed by the following formulas:

$$G_1 = \sqrt{\frac{q^3}{2}} \left(\hat{b}_0 + \frac{\hat{b}_1 + q^3 \hat{b}_2}{q^2\sqrt{-\kappa}} \right), \quad G_2 = \sqrt{\frac{q^3}{2}} \left(\hat{b}_0 - \frac{\hat{b}_1 + q^3 \hat{b}_2}{q^2\sqrt{-\kappa}} \right). \tag{6.8a,b}$$

The formulas (6.1), (6.6), (6.7), and (6.8) give a general solution of the initial value problem to the nonlinear equation (3.12). Note, that the function b (6.1) is finite and hence is meaningful for any $T \geq T_0$ because $f \geq 1$ according to (6.7). So, the evolution of the transversal soliton perturbation can be investigated in a general case within the framework of our asymptotic equation. Moreover, we can also find an explicit expression for the radiation field which the soliton generates in a non-stationary process:

$$u^-(X, Y, \tau = 0) = \begin{cases} \left. \dfrac{\partial^2}{\partial X^2} \ln(f) \right|_{T=T_s(X)}, & X \geq X_0 \\ 0, & X < X_0 \end{cases}, \tag{6.9}$$

where $T_s(X) = T_0 + (X - X_0)/q^2$. However, as the initial distribution (6.9) is bounded only for semi-interval of the X-axis ($X \geq X_0$) we cannot solve the problem of evolution of the radiation field in an explicit form. Indeed, considering the initial value problem for the radiation field we have to investigate Eq. (3.13) which is not simpler than the original KP equation (2.1). Even in the one-dimensional case, when Eq. (3.13) is the KdV equation, the evolution of an arbitrary initial distribution is considered by means of a complicated technique of the inverse scattering transform which is elaborated to generalize the Fourier-transform for nonlinear equations [22].

However, there is one lucky opportunity to consider the evolution of the radiation field in an explicit form. This is the case of "pure" soliton instability when the function f is positive for any T, even for $T < T_0$. In this case, we can set the parameters T_0 and X_0 to be equal to $-\infty$. Then, the initial distribution u^- is determined throughout X-axis and has the form

$$u^-(X, Y, \tau = 0) = \frac{\partial^2}{\partial X^2} \ln(f), \quad -\infty < X < +\infty. \tag{6.9'}$$

It is well-known [22] that the evolution of the distribution (6.9′) which depends on the functions f and g satisfying (6.7) and (6.5) at $T = X/q^2$ reduces to an additional linear equation determining the dependence of all the functions on the new variable τ:

$$g_\tau + g_{XXX} = 0. \tag{6.10}$$

Thus, for a particular class of solutions bounded along T and X everywhere, both the approximate equation (3.12) for the amplitude of soliton transversal perturbations and the KP equation (3.13) for radiation field can be linearized and integrated in an explicit form. We would like to point out that the same algebraic expression (6.9′) with the functions (6.5), (6.7), and (6.10) generates an exact solution to the KP equation describing the processes of "pure" soliton instability for a whole range of the transverse wave number variations [18]. In particular, such a correspondence is connected with the fact that secondary instabilities of wave structures in the radiation field are described again by the original KP equation. In the next section we use general formulas of the found solutions for investigation of the characteristic features of a multiperiodic regime of plane soliton instability.

7. Multiperiodic regime of plane soliton instability

Let us choose a solution of Eqs. (6.2), (6.5) in the following form:

$$g = \sum_{n=1}^{N} \exp(\Psi_n - i\Theta_n) + \sum_{m=1}^{M} C_m \exp(i\Phi_m) \tag{7.1}$$

$$f = \delta + \sum_{n=1}^{N}\sum_{n'=1}^{N} \frac{1}{p_n + p_{n'}} \exp(\Psi_n + \Psi_{n'}) \cos(\Theta_n - \Theta_{n'}) + \sum_{m=1}^{M}\sum_{m'=1}^{M} \frac{C_m C_{m'}}{r_m - r_{m'}} \sin(\Phi_m - \Phi_{m'})$$
$$+ \sum_{n=1}^{N}\sum_{m=1}^{M} \frac{2C_m}{p_n^2 + r_m^2} \exp(\Psi_n) \left(p_n \cos(\Theta_n + \Phi_m) + r_m \sin(\Theta_n + \Phi_m)\right), \tag{7.2}$$

where $\Psi_n = p_n q^2 (T + \sigma_n)$, $\Theta_n = p_n^2 Y + \psi_n$, $\Phi_m = r_m q^2 T + r_m^2 Y + \phi_m$, and δ is an arbitrary constant of integration.

For $\delta \equiv 1$, the corresponding solution describes the evolution of quasi-periodic transversal perturbations with $N + M$ transverse wave numbers, where the N wave numbers are located in an unstable region of the non-modulated soliton and the M ones are located in a stable region. For $M = 0$, $N = 2$, $p_1 = -p_2 = p$, and $\psi_1 = \psi_2$ this solution transforms to a general, 2-parametric (σ_1, σ_2) solution to Eq. (4.1) and generates trajectories on the phase plane which are shown in Fig. 2. For $M = 2$, $N = 0$, $r_1 = -r_2 = r$, and $C_1 = C_2 = C$ the solution can be rewritten in the form (4.4).

If there are no unstable perturbations in the given multiperiodic profile ($N = 0$), the transversal oscillating modulation damps as $T \to +\infty$, and the amplitudes C_m of all M perturbations give an additive contribution to damping $b \sim (T\sum_{m=1}^{M} C_m^2)^{-1/2}$ as $T \to +\infty$ due to independent emission of quasi-harmonic wave packets with M frequencies. On the other hand, if $N \neq 0$ and soliton transversal instability starts to develop, the contribution from the modes lying in a stable region is exponentially small as $T \to +\infty$. Therefore, for simplicity we suppose below $M = 0$.

If there are negative values in the set $\{p_n\}_{n=1}^{N}$, the function b is regular only starting from some finite time T^* because $f < 0$ for $T < T^*$. As we discussed in Section 4, this is connected with the stable manifolds which are inputting to the saddle equilibrium state $b = 0$ as $T \to +\infty$ and are going to infinity as $T \to -\infty$. If an initial value of b is not large enough, the influence of such modes on the motion along the unstable manifolds is negligible. Therefore, we shall consider the case, when all the parameters p_n are positive and ordered as follows:

$$0 < p_1 < p_2 < \ldots < p_N. \tag{7.3}$$

In this case, we have the non-modulated state $b = 0$ as $T \to -\infty$ and a modulated one $b = (p_N/q)^{1/2} \times \exp[-i(p_N^2 Y + \psi_N)]$ as $T \to +\infty$ with the transverse wave number deviated maximally from the critical value k_c. Concrete ways of the instability development depend on relations between the time-phase constants σ_n of each mode of the transversal perturbation. Let these constants be separated from each other far enough and

$$\sigma_1 \gg \sigma_2 \gg \ldots \gg \sigma_N. \tag{7.4}$$

Then, in a time interval $T_{n-1} \ll T \ll T_n$, where $T_n = (p_n\sigma_n - p_{n+1}\sigma_{n+1})/(p_{n+1} - p_n)$, a perturbation with the wave number $k_n = k_c - p_n^2$ dominates in the amplitude b. This perturbation appears at $T = T_{n-1}$ due to the instability of a modulated wave with the wave number $k_{n-1} > k_n$ (in the case $n = 1$ of non-modulated soliton) and is changed at $T = T_n$ by a modulated wave with the wave number $k_{n+1} < k_n$. At each successive decay, a new stationary structure is emitted by a quasi-plane soliton. For $n = 1$ such a structure is the plane soliton (4.3) with the parameter $p = p_1$; for $n > 1$ it is the modulated wave (4.6) with the transverse wave number $\kappa_n = p_n^2 - p_{n-1}^2$. So, a mechanism of a degradation of the original one-dimensional soliton to a long-wave two-dimensional structure with increasing modulation depth is well manifested in this multiperiodic process.

For relations which are inverse to (7.4), a perturbation with the wave number k_N having a maximal growth rate starts to dominate at the initial stage of soliton instability. The presence of perturbations with the other periods does not essentially influence the evolution of the basic quasi-plane soliton. However, these perturbations are revealed in the evolution of the radiation field and determine the secondary instabilities of the emitted solitary waves. As was investigated in [18], these processes can be regarded as a chain of successive decays of a plane soliton with parameter p_n, starting from p_N, into a plane soliton of smaller amplitude with parameter $p_{n-1} < p_n$ and a modulated wave with the transverse wave number κ_n. Such a process results in the formation of a plane soliton with the smallest amplitude which is determined by the parameter p_1.

Thus, our asymptotic approach is equally good to describe processes of the transversal soliton instability both concentrated on the original quasi-plane soliton and distributed in a small component of the radiation field. For arbitrary relations between the parameters σ_n, part of modes develops against the background of the original soliton and the rest develop in the radiation field behind it. In any case the fast quasi-plane soliton transforms as $T \to +\infty$ to a modulated wave with the longest period and the plane soliton with the smallest amplitude is revealed in the radiation field among the other $N - 1$ intermediate modulated waves.

For the case $\delta \equiv 0$, we have degeneration of the multiperiodic solution. In this case, the bounded solution with $M = 0$ and p_n ordered according to (7.3) describes a similar scenario of successive decays of a modulated wave starting from the wave with the transverse wave number k_1 and ending with the wave with the wave number k_N. However, the non-modulated state (a plane soliton) is not revealed in the whole interval of wave dynamics.

8. Conclusion

We believe that the asymptotic approach discussed above is a powerful tool for investigation of long-term development of plane soliton instability not only for the integrable KP model but also for a much broader class of equations of modern nonlinear wave theory. Within this approach, construction of an asymptotic expansion needs only an analytical form of a non-modulated steady-state soliton solution, of a critical value of the transverse wave number of soliton perturbation and of the corresponding linear mode. All the other terms of the asymptotic series may be calculated from solutions to linear equations analytically or numerically. We may expect that, if there is no soliton collapse in the original physical model, an approximate equation does not include singularities which appear in a long-wave region and allows us to describe the characteristic features of all stages of the transversal soliton instability, which we have done in this paper for the classical KP equation.

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