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JUSTIFICATION OF THE LOG–KdV EQUATION IN GRANULAR CHAINS: THE CASE OF PRECOMPRESSION*

ERIC DUMAS† AND DMITRY PELINOVSKY‡

Abstract. For traveling waves with nonzero boundary conditions, we justify the logarithmic Korteweg–de Vries equation as the leading approximation of the Fermi–Pasta–Ulam lattice with Hertzian nonlinear potential in the limit of small anharmonicity. We prove control of the approximation error for the traveling wave solutions satisfying differential advance-delay equations, as well as control of the approximation error for time-dependent solutions to the lattice equations on long but finite time intervals. We also show nonlinear metastability of the traveling waves on long but finite time intervals.

Key words. Fermi–Pasta–Ulam lattice, Korteweg–de Vries equation, existence and stability of nonlinear waves, justification of amplitude equations

AMS subject classifications. 34D20, 35Q53, 37K40, 37K45, 37K60

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1. Introduction.

Solitary waves in anharmonic granular chains with Hertzian interaction forces are modeled by the Fermi–Pasta–Ulam (FPU) lattices with non-smooth nonlinear potentials [21]. We write the FPU lattice in the form

$$\ddot{u}_n = V'_\alpha(u_{n+1}) - 2V'_\alpha(u_n) + V'_\alpha(u_{n-1}), \quad n \in \mathbb{Z}, \tag{1.1}$$

where $(u_n)_{n\in\mathbb{Z}}$ is a function of the time $t \in \mathbb{R}$, with values in $\mathbb{R}^{\mathbb{Z}}$, and the dot denotes the time derivative. The coordinate u_n corresponds to the relative displacement between locations of two adjacent particles. The Hertzian nonlinear potential $V_\alpha \in C^2(\mathbb{R})$ is given by

$$V_\alpha(u) = \frac{1}{1+\alpha}|u|^{1+\alpha} H(-u), \quad \alpha > 1, \tag{1.2}$$

where $H(u)$ is the standard Heaviside function. Recently, the FPU lattice (1.1)–(1.2) was formally reduced to the logarithmic Korteweg–de Vries (log–KdV) equation in the limit of small anharmonicity of the Hertzian interaction forces (that is, for $\alpha = 1+\epsilon^2$ with $\epsilon \to 0$) [5, 13]. Using the asymptotic correspondence $u_n(t) \approx -v(x,\tau)$, $x = \epsilon(n-t)$, and $\tau = \epsilon^3 t$, we obtain the log–KdV equation in the form

$$2v_\tau + \frac{1}{12}v_{xxx} + (v \log|v|)_x = 0, \quad (x,\tau) \in \mathbb{R}\times\mathbb{R}. \tag{1.3}$$

Here and in what follows, we denote partial derivatives by subscripts. Note that in the derivation of (1.3), v is assumed to be positive (otherwise, the Heaviside function should appear in the nonlinear term). Experimental evidence for validity of the limit $\alpha \to 1$ in the context of granular chains with hollow particles can be found in [22].

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†Institut Fourier, Université Grenoble 1, 38402 Saint Martin d'Hères cedex, France (Eric.Dumas@ujf-grenoble.fr).

‡Department of Mathematics, McMaster University, Hamilton, Ontario, L8S 4K1, Canada (dmpeli@math.mcmaster.ca). The research of this author was supported by the Chaire d'excellence ENSL/UJF.

The log–KdV equation (1.3) has a two-parameter family of Gaussian traveling waves

$$v(x,\tau) = e^{2b} v_G(x - b\tau - a), \quad a, b \in \mathbb{R}, \tag{1.4}$$

where v_G is a symmetric standing wave given by

$$v_G(x) := \sqrt{e} e^{-3x^2}, \quad x \in \mathbb{R}. \tag{1.5}$$

Global solutions to the log–KdV equation (1.3) in the energy space

$$X := \left\{ v \in H^1(\mathbb{R}) : \quad v^2 \log|v| \in L^1(\mathbb{R}) \right\} \tag{1.6}$$

were constructed in [4]. In addition, spectral and linearized stability of Gaussian traveling waves were proved in [4] with analysis of the linearized evolution problem. Unfortunately, technical difficulties exist to proving nonlinear orbital stability of Gaussian traveling waves, as well as to constructing solutions in spaces of higher regularity [4]. The technical difficulties are caused by the necessity to control the logarithmic nonlinearity near $v = 0$, where it is not differentiable.

This paper addresses a different problem, namely, the rigorous justification of the log–KdV equation (1.3) in the context of the FPU lattice (1.1)–(1.2). Numerical approximations of time-dependent solutions to the FPU lattice in [13] suggest that the Gaussian traveling waves represent well the stable solitary waves in granular chains, which are known to propagate robustly in physical experiments [28]. Therefore, it becomes relevant to control the approximation error between the corresponding solutions to the FPU lattice (1.1)–(1.2) and the log–KdV equation (1.3).

In a similar context to FPU lattices with sufficiently smooth nonlinear potential V, small-amplitude solutions are described by the celebrated KdV equation. In a series of papers, Friesecke and Pego [7, 8, 9, 10] justified the KdV approximation for traveling waves and proved the nonlinear stability of small-amplitude solitary waves in generic FPU chains from analysis of the orbital and asymptotic stability of KdV solitons. Later these results were extended to the proof of asymptotic stability of several solitary waves in the FPU lattices by Mizumachi [19, 20], Hoffmann and Wayne [14], and Benes, Hoffman, and Wayne [3]. Independently, validity of the KdV equation for time-dependent solutions on the time scale of $\mathcal{O}(\epsilon^{-3})$ was obtained by Schneider and Wayne [27] and Bambusi and Ponno [2]. Recently, these results were generalized for polyatomic FPU lattices in [12]. Because of the lack of smoothness of the potential V_α in (1.1)–(1.2), and therefore also of the nonlinearity in the log–KdV equation (1.3) near the origin, none of the previous results can be applied to the FPU lattice (1.1)–(1.2).

As a first step towards the ultimate goal of justification of the log–KdV equation (1.3), we shall here consider solutions with nonzero (positive) boundary conditions at infinity. In other words, we shall consider solutions bounded from below by some positive constant and satisfying the boundary conditions $v(x,t) \to v_0 > 0$ as $|x| \to \infty$ for all $t \in \mathbb{R}$. In the context of the anharmonic granular chains (1.1)–(1.2), these boundary conditions correspond to the constant precompression force applied to the granular chains.

The precompression technique is well known both numerically and experimentally for regularization of responses of granular chains [21]. Typically, small-amplitude perturbations of the constant precompressed state are handled through Taylor expansion of the nonlinearity, thanks to the smoothness of the nonlinear potential $V(u)$ or the logarithmic nonlinearity near any point $v_0 > 0$. In comparison to this standard technique, we avoid Taylor series expansion and consider large-amplitude solutions to the FPU lattice (1.1)–(1.2) with $\alpha = 1 + \epsilon^2$ in the limit $\epsilon \to 0$. In this way, we confirm the validity of the log–KdV equation (1.3) with nonzero boundary conditions for the existence and stability of large–amplitude traveling waves.

Note that the nonlinear function $p(v) = v \log |v|$ in the log–KdV equation (1.3) satisfies for any $v \geqslant v_0 > 0$ the general assumption $p''(v) > 0$ and $p'''(v) \leqslant 0$ required by the orbital stability theory for large-amplitude traveling waves of the generalized KdV equation (see Theorem 1 in [15]). Consequently, traveling waves of arbitrary amplitudes with the boundary conditions $v \to v_0$ as $x \to \pm\infty$ are orbitally stable with respect to the time evolution of the log–KdV equation (1.3) in the classical sense [1].

We report three main results in our work. First, we study traveling wave solutions to (1.1)–(1.2) with $\alpha = 1 + \epsilon^2$, under the form $u_n(t) = -v_0(1 + w(n - ct))$ with speed $c = (v_0^{\epsilon^2}(1 + \lambda\epsilon^2))^{1/2}$ for any $\lambda > 1$. We provide a rigorous approximation of such traveling waves, in the limit $\epsilon \to 0$, by means of traveling solutions to the log–KdV equation (1.3).

Next, we show that a simple energy argument gives nonlinear metastability of the previously constructed (large-amplitude) traveling wave solutions to the FPU lattice equations (1.1)–(1.2) with $\alpha = 1 + \epsilon^2$ on the time scale $\mathcal{O}(\epsilon^{-3})$, where the approximation of the log–KdV equation (1.3) is formally applicable. The energy argument we develop here does not use the spectral information on the linearized log–KdV equation and holds for time-dependent perturbations, which may violate the scaling of space and time variables resulting in the log–KdV equation (1.3). It only uses the precise justification result for the traveling waves of the FPU lattice.

Finally, we control the error in the approximation of time-dependent solutions to the FPU lattice by solutions to the log–KdV equation up to the time scale $\mathcal{O}(\epsilon^{-3})$ by extending the same energy argument used for control of the nonlinear metastability of traveling waves.

Although our results are analogous to the outcomes of the corresponding works [7] and [27], a different analytical technique is adopted to obtain the justification and stability results. The technique is thought to be applicable to a much larger class of FPU models which result in the generalized KdV equation with possibly large-amplitude traveling waves. We also point out that the methods of neither [7] nor [27] cannot be immediately applied to the justification of the log–KdV equation (1.3) because they require the smallness of the traveling wave amplitude.

In more detail, we use the method of decomposition of solutions in the Fourier space, which was originally developed in [24] and used in [6, 16, 25] (see also Chapter 2 in [23]) for the justification of asymptotic reductions of solitary waves in the nonlinear Schrödinger equation with a periodic potential. This technique is an alternative to the method of Friesecke and Pego [7] that relies on approximations of roots of the dispersion relations and on an appropriate version of a fixed-point theorem. We also use fixed-point arguments but in a more classical way.

While the strategy adopted in [8, 9, 10] gives nonlinear stability results for FPU traveling waves globally in time, it applies only to the small-amplitude traveling waves. It also relies on the spectral information of the linearized KdV equation, modulation equations along the two-dimensional manifold of the traveling waves, and careful analysis of linearized advance-delay equations, all of which may not be available when dealing with the log–KdV equation (1.3).

The plan of the paper is as follows. Section 2 presents the main results. Section 3 is devoted to the justification of the log–KdV approximation for the traveling waves of the FPU lattice. Section 4 is devoted to the nonlinear metastability of the traveling waves in the FPU lattice on the time scale $\mathcal{O}(\epsilon^{-3})$. Section 5 describes the justification of the log–KdV equation for the time-dependent solutions to the FPU lattice. Section 6 discusses these results in the context of general FPU lattices.

2. Main results. Substituting the traveling wave ansatz $u_n(t) = u(z)$ with $z = n - ct$ for a positive speed $c > 0$ into the FPU lattice (1.1)–(1.2), we obtain the differential advance-delay equation

$$c^2 u''(z) = -\Delta |u|^\alpha H(-u)(z), \quad z \in \mathbb{R}, \tag{2.1}$$

where $\alpha > 1$, and Δ is the discrete Laplacian operator on the infinite line,

$$\Delta f(z) := f(z+1) - 2f(z) + f(z-1).$$

Since the limit $\alpha \to 1$ is considered, we set $\alpha := 1 + \epsilon^2$ for a small positive ϵ. Here and in the sequel, we shall drop the dependence of the functions (such as u) upon ϵ for simplicity, and only mention this dependence in the main statements. With a precompression level $v_0 > 0$, we set

$$u(z) = -v_0(1 + w(z)) \quad \text{and} \quad c^2 = v_0^{\epsilon^2}(1+\mu), \tag{2.2}$$

where $\mu > -1$ is an arbitrary parameter and $w(z)$ is assumed to decay to zero at infinity and to be bounded in the interval

$$-1 < C_- \leqslant w(z) \leqslant C_+ < \infty \quad \text{for every } z \in \mathbb{R}, \tag{2.3}$$

where $C_\pm$ are ϵ-independent and C_+ does not have to be smaller than one (that is, $\|w\|_{L^\infty}$ may exceed one). Under the a priori bound (2.3), we rewrite the existence problem in the form

$$(1+\mu) w''(z) = \Delta \tilde{V}_\epsilon'(w)(z), \quad z \in \mathbb{R}. \tag{2.4}$$

Here the potential

$$\tilde{V}_\epsilon(w) := \frac{1}{2+\epsilon^2}\left[(1+w)^{2+\epsilon^2} - 1\right] - w, \quad w > -1,$$

is $C^2(-1,\infty)$, positive near $w = 0$, and $\tilde{V}_\epsilon(w)/w^2$ increases strictly with w for all $w \in (0,\infty)$. For such potentials, Theorem 1 of Friesecke and Wattice [11] applies (as it was also noted in [18]). By this theorem, which is proved by a variational method based on the concentration compactness principle, there exists a nontrivial positive solution $w \in H^1(\mathbb{R})$ of the differential advance-delay equation (2.4) for some values of the parameter μ satisfying the constraint $1 + \mu > \tilde{V}_\epsilon''(0) = 1 + \epsilon^2$ (that is, for $\mu > \epsilon^2$). Moreover, recent work [29] suggests that these traveling waves are smooth and exponentially localized.

To obtain the formal limit to the stationary log–KdV equation, we set the variables $x = \epsilon z$ and $W(x) = w(z)$, use the Taylor expansions

$$\Delta w(z) = \epsilon^2 W''(x) + \frac{1}{12}\epsilon^4 W''''(x) + \mathcal{O}(\epsilon^6 W^{(6)}(x)), \tag{2.5}$$

and

$$\begin{aligned} \tilde{V}_\epsilon'(w) &= (1+w)^{1+\epsilon^2} - 1 \\ &= w + \epsilon^2(1+w)\log(1+w) + \mathcal{O}(\epsilon^4(1+w)\log^2(1+w)), \end{aligned} \tag{2.6}$$

and finally integrate (2.4) with $\mu = \epsilon^2\lambda$ twice in x subject to the zero boundary conditions for W and its derivatives. Truncating at the leading order $\mathcal{O}(\epsilon^4)$, we obtain the stationary log–KdV equation

$$\lambda W(x) = \frac{1}{12}W''(x) + (1+W)\log(1+W), \quad x \in \mathbb{R}. \tag{2.7}$$

By Proposition 3.2 below, there exists a unique positive and even solution $W_{\rm stat} \in H^\infty(\mathbb{R})$ to the stationary log–KdV equation (2.7) with $\lambda > 1$. We are now ready to formulate the main result on the rigorous justification of this formal approximation.

THEOREM 1. *Set $\mu := \epsilon^2\lambda$ with fixed ϵ-independent parameter $\lambda > 1$. There exist positive constants ϵ_0 and C_0 such that for every $\epsilon \in (0, \epsilon_0)$, there exists a unique even solution $w_{{\rm stat},\epsilon}$ to the differential advance-delay equation* (2.4) *in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that*

$$\sup_{z\in\mathbb{R}} |w_{{\rm stat},\epsilon}(z) - W_{\rm stat}(\epsilon z)| \leqslant C_0\epsilon^{1/6}, \tag{2.8}$$

where $W_{\rm stat}$ is the unique positive and even solution to the stationary log–KdV equation (2.7)*. Moreover, $w_{{\rm stat},\epsilon} \in H^\infty(\mathbb{R})$ and for every $k \in \mathbb{N}$, there is a positive ϵ-independent constant C_k such that*

$$\sup_{z\in\mathbb{R}} |\partial_z^k w_{{\rm stat},\epsilon}(z) - \epsilon^k\partial_x^k W_{\rm stat}(\epsilon z)| \leqslant C_k\epsilon^{k+1/6}. \tag{2.9}$$

Remark 2.1. It follows from analysis of the roots of the dispersion relation associated with the differential advance-delay equation (2.4) that $w_{{\rm stat},\epsilon}$ decays to zero exponentially at infinity (see section 5 in [7]).

Remark 2.2. While the result of Theorem 1 does not exclude the sign-indefinite solution $w_{{\rm stat},\epsilon}$, the negative parts are as small as $\mathcal{O}(\epsilon^{1/6})$ in the L^∞ norm, because $W_{\rm stat}$ is positive. Nevertheless, based on the results of the variational theory in [11], we anticipate that $w_{{\rm stat},\epsilon}$ is also positive.

Using the scaling transformation

$$u_n(t) = -v_0\left(1 + w_n(t')\right), \quad t' = v_0^{\epsilon^2/2}t,$$

we can write the FPU lattice in the (formally) equivalent form of the first-order system

$$\begin{cases} \dot{w}_n = p_{n+1} - p_n, \\ \dot{p}_n = \tilde{V}_\epsilon'(w_n) - \tilde{V}_\epsilon'(w_{n-1}), \end{cases} \quad n \in \mathbb{Z}. \tag{2.10}$$

Any $(w,p) \in C^1(\mathbb{R}, l^2(\mathbb{Z}))$ solution to the first-order system (2.10), with $w_n > -1$ for all $n \in \mathbb{Z}$, provides a $C^2(\mathbb{R}, l^2(\mathbb{Z}))$ solution u to the scalar second-order equation (1.1). The FPU lattice equations (2.10) admit the conserved energy

$$H := \frac{1}{2}\sum_{n\in\mathbb{Z}} p_n^2 + \sum_{n\in\mathbb{Z}} \tilde{V}_\epsilon(w_n). \tag{2.11}$$

Note that the dot in (2.10) applies with respect to the new variable t'. In what follows, we will use the same notation t for the independent time variable of the FPU system (2.10) for convenience.

Since shift operators are bounded in $l^2(\mathbb{Z})$, it is easy to show the local (in time) well-posedness of the Cauchy problem associated with the FPU system (2.10) in $l^2(\mathbb{Z})$.

Furthermore, the energy conservation (2.11) and the embedding of $l^2(\mathbb{Z})$ in $l^\infty(\mathbb{Z})$ ensure global existence of the solutions, at least for small initial data. For large initial data, any solution to (2.10) provides a solution to (1.1) as long as all components of u remain strictly negative, that is, as long as

$$-1 < C_- \leqslant w_n(t) \leqslant C_+ < \infty \quad \text{for every } n \in \mathbb{N}, \tag{2.12}$$

where $C_\pm$ are ϵ and t-independent constants. It may be hard to control this condition during evolution for general initial data, but our study addresses time-dependent solutions near the traveling wave of Theorem 1, which definitely satisfies the bounds (2.12); see Remark 2.2. Let us emphasize once again that the traveling waves and the solutions we consider are not small-amplitude solutions to the FPU lattice (2.10).

We define a reference traveling wave $(w_{\text{trav}}, p_{\text{trav}}) \in C^1(\mathbb{R}, l^2(\mathbb{Z}))$ solution to the FPU lattice (2.10) by

$$(w_{\text{trav}})_n(t) = w_{\text{stat}}(n - ct), \quad (p_{\text{trav}})_n(t) = p_{\text{stat}}(n - ct), \tag{2.13}$$

where $c^2 = 1 + \epsilon^2\lambda$ is the squared wave speed, w_{stat} is given by Theorem 1, and p_{stat} is found from the advance equation $-cw'_{\text{stat}}(z) = p_{\text{stat}}(z+1) - p_{\text{stat}}(z)$. We now ask if the traveling wave given by (2.13) is stable in the time evolution of the FPU lattice (2.10) with small ϵ at least on the time scale of $\mathcal{O}(\epsilon^{-3})$, when the approximation of the log–KdV equation is applicable.

The following theorem gives the affirmative answer to the question of the nonlinear metastability of the FPU traveling waves and specifies the precise conditions, in which the nonlinear metastability of the traveling wave is understood. In particular, this result ensures existence of the time-dependent solution (w, p) to the FPU lattice (2.10) up to $\mathcal{O}(\epsilon^{-3})$ times.

THEOREM 2. *As in Theorem* 1, *set* $\mu := \epsilon^2\lambda$ *with fixed* ϵ*-independent parameter* $\lambda > 1$. *For every* $\tau_0 > 0$, *there exist positive constants* ϵ_0, δ_0, *and* C_0 *such that, for all* $\epsilon \in (0, \epsilon_0)$, *when initial data* $(w_{\text{ini},\epsilon}, p_{\text{ini},\epsilon}) \in l^2(\mathbb{R})$ *satisfy*

$$\delta := \|w_{\text{ini},\epsilon} - w_{\text{trav},\epsilon}(0)\|_{l^2} + \|p_{\text{ini},\epsilon} - p_{\text{trav},\epsilon}(0)\|_{l^2} \leqslant \delta_0, \tag{2.14}$$

then the unique solution (w_ϵ, p_ϵ) *to the FPU lattice equations* (2.10) *with initial data* $(w_{\text{ini},\epsilon}, p_{\text{ini},\epsilon})$ *belongs to* $C^1([-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}], l^2(\mathbb{Z}))$ *and satisfies*

$$\|w_\epsilon(t) - w_{\text{trav},\epsilon}(t)\|_{l^2} + \|p_\epsilon(t) - p_{\text{trav},\epsilon}(t)\|_{l^2} \leqslant C_0\delta, \quad t \in \left[-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}\right]. \tag{2.15}$$

Remark 2.3. According to [15], the solitary wave W of the stationary log–KdV equation (2.7) is orbitally stable in the time evolution of the log–KdV equation

$$2W_\tau + \frac{1}{12}W_{\xi\xi\xi} + (g(W))_\xi = 0, \quad g(W) := (1+W)\log(1+W), \tag{2.16}$$

where $\tau = \epsilon^3 t$ and $\xi = \epsilon(n - t)$ are scaled variables of the FPU lattice (2.10). From Theorem 2 and this orbital stability result, one can expect that the time-dependent version of the log–KdV equation (2.16) is a valid approximation of the time-dependent solutions to the FPU lattice (2.10) modulated on the spatial scale $\mathcal{O}(\epsilon^{-1})$ up to the time scale of $\mathcal{O}(\epsilon^{-3})$.

Remark 2.4. Compared to the log–KdV equation (2.16), Theorem 2 also gives metastability of the FPU traveling waves with respect to modulations on any other spatial scale, nevertheless, up to the time scale of $\mathcal{O}(\epsilon^{-3})$.

Finally, we justify the approximation of time-dependent solutions to the FPU lattice (2.10) by the log–KdV equation (2.16). Technically, when a solution W to (2.16) is given, we define

$$P_\epsilon := -W + \frac{\epsilon}{2} W_\xi - \frac{\epsilon^2}{8} W_{\xi\xi} - \frac{\epsilon^2}{2} g(W) + \frac{\epsilon^3}{48} W_{\xi\xi\xi} + \frac{\epsilon^3}{4} (g(W))_\xi, \tag{2.17}$$

so that (W, P_ϵ) solves the first equation in (2.10) up to $\mathcal{O}(\epsilon^4)$ terms. The following theorem controls the approximation error up to $\mathcal{O}(\epsilon^{-3})$ times.

THEOREM 3. *Let $W \in C([-\tau_0, \tau_1], H^s(\mathbb{R}))$ be a solution to the log–KdV equation* (2.16) *for some integer $s \geqslant 6$ and some $\tau_0, \tau_1 \geqslant 0$. Assume that there exists $r_W > -1$ such that $W \geqslant r_W$. Then there exist positive constants ϵ_0 and C_0 such that, for all $\epsilon \in (0, \epsilon_0)$, when initial data $(w_{\text{ini},\epsilon}, p_{\text{ini},\epsilon}) \in l^2(\mathbb{R})$ are given such that*

$$\|w_{\text{ini},\epsilon} - W(\epsilon\cdot, 0)\|_{l^2} + \|p_{\text{ini},\epsilon} - P_\epsilon(\epsilon\cdot, 0)\|_{l^2} \leqslant \epsilon^{3/2} \tag{2.18}$$

with P_ϵ given by (2.17)*, the unique solution (w_ϵ, p_ϵ) to the FPU lattice equations* (2.10) *with initial data $(w_{\text{ini},\epsilon}, p_{\text{ini},\epsilon})$ belongs to $C^1([-\tau_0\epsilon^{-3}, \tau_1\epsilon^{-3}], l^2(\mathbb{Z}))$ and satisfies*

$$\begin{aligned} &\|w_\epsilon(t) - W(\epsilon(\cdot - t), \epsilon^3 t)\|_{l^2} + \|p_\epsilon(t) - P_\epsilon(\epsilon(\cdot - t), \epsilon^3 t)\|_{l^2} \leqslant C_0 \epsilon^{3/2}, \\ &t \in \left[-\tau_0\epsilon^{-3}, \tau_1\epsilon^{-3}\right]. \end{aligned} \tag{2.19}$$

Remark 2.5. The Cauchy problem associated with the log–KdV equation (1.3) is not understood in full generality: global solutions in some subspace of H^1 are constructed in [4], but the question of propagation of regularity remains open. However, the classical approach (see, for example, Kato [17]) allows us to construct short-time solutions with H^s regularity, $s > 3/2$, given initial data satisfying a lower bound as in the assumption of Theorem 3, namely, $W \geqslant r_W > -1$ (in the neighborhood of which the nonlinearity g is smooth).

Remark 2.6. Using higher-order asymptotic expansions and $\epsilon^{K+3/2}$-close initial data, the approximation in (2.19) could be improved to be $\mathcal{O}(\epsilon^{K+3/2})$ for any $K \in \mathbb{N}$; see Remark 5.1 below.

Remark 2.7. Even if the traveling wave solution $W = W_{\text{stat}}(\xi - \lambda\tau/2)$ to the log–KdV equation (2.16) is used in bounds (2.18) and (2.19), where W_{stat} is a solution to the stationary log–KdV equation (2.7), the results of Theorems 1 and 3 do not recover the result of Theorem 2, because the small parameter δ in Theorem 2 does not depend on the small parameter ϵ.

3. Justification analysis for traveling waves. Adopting the Fourier transform on $L^2(\mathbb{R})$ functions

$$\hat{w}(k) = \mathcal{F}(w)(k) := \int_{-\infty}^{\infty} w(z) e^{-ikz} dz,$$

with the inverse Fourier transform

$$w(z) = \mathcal{F}^{-1}(\hat{w})(z) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{w}(k) e^{ikz} dk,$$

we can rewrite the existence problem (2.4) as the fixed-point equation

$$w(z) = \frac{1}{1+\mu} \int_{-1}^{1} \Lambda(y) \tilde{V}_\epsilon'(w(z-y)) dy, \quad z \in \mathbb{R}, \tag{3.1}$$

where $\Lambda(z) = (1-|z|)_+$ is the hat function, or in the equivalent Fourier form

$$\hat{w}(k) = \frac{1}{1+\mu}\hat{\Lambda}(k)\mathcal{F}(\tilde{V}'_\epsilon(w))(k), \quad k \in \mathbb{R}, \tag{3.2}$$

where $\hat{\Lambda}(k) := \frac{4}{k^2}\sin^2\left(\frac{k}{2}\right)$. This section presents the proof of Theorem 1, after several auxiliary results have been obtained.

3.1. Nonzero solutions to the fixed-point equation (3.1). We shall first investigate if nonzero solutions to the fixed-point equation (3.1) exist for $\mu = \mathcal{O}(\epsilon^2)$. Therefore, we set $\mu := \epsilon^2\lambda$ with an ϵ-independent parameter λ. The following proposition shows that, when $\lambda > 1$ is fixed and $R > 0$ is small enough, there is no solution to the fixed-point equation (3.1) with the $L^2 \cap L^\infty$ norm less than R other than the trivial (zero) solution.

Proposition 3.1. *Set $\mu := \epsilon^2\lambda$. For every $R > 0$, there exists $\lambda_R > 1$ such that for all $\lambda > \lambda_R$ and all $\epsilon \in (0,1)$ the only solution to the fixed-point equation* (3.1) *in*

$$B_R := \{w \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) : \quad \|w\|_{L^2\cap L^\infty} \leqslant R, \quad w \geqslant 0\} \tag{3.3}$$

is the trivial zero solution. Furthermore, λ_R may be chosen so that $\lambda_R \underset{R\to 0}{\longrightarrow} 1$.

Proof. We write

$$\tilde{V}'_\epsilon(w) = (1+w)^{1+\epsilon^2} - 1^{1+\epsilon^2} = (1+\epsilon^2)\int_0^w (1+x)^{\epsilon^2}\,dx.$$

Let $A_{\lambda,\epsilon}(w)$ denote the right-hand side of the fixed-point equation (3.1) with $\mu = \epsilon^2\lambda$. Since $\|\Lambda\|_{L^1} = 1$ and $\|\Lambda\|_{L^2} = \frac{\sqrt{2}}{\sqrt{3}} < 1$, we apply Young's inequality and obtain

$$\begin{aligned} \|A_{\lambda,\epsilon}(w)\|_{L^2\cap L^\infty} &\leqslant \frac{1}{1+\epsilon^2\lambda}\|\Lambda\|_{L^1\cap L^2}\|\tilde{V}'_\epsilon(w)\|_{L^2} \\ &\leqslant \frac{1+\epsilon^2}{1+\epsilon^2\lambda}(1+\|w\|_{L^\infty})^{\epsilon^2}\|w\|_{L^2}. \end{aligned}$$

Consider the ball of positive functions in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ centered at zero with the radius $R > 0$, denoted by B_R, and defined by (3.3). If R is fixed, there exists an ϵ-independent constant C_R such that

$$(1+\|w\|_{L^\infty})^{\epsilon^2} \leqslant 1 + C_R\epsilon^2\log(1+R) \quad \text{for every } \epsilon \in (0,1).$$

Furthermore, C_R may be chosen so that $C_R \underset{R\to 0}{\longrightarrow} 1$.

For $\lambda \geqslant \lambda_R := 1 + 2C_R\log(1+R)$ and $\epsilon \in (0,1)$, we have $A_{\lambda,\epsilon} : B_R \to B_R$. Moreover, using similar bounds

$$\begin{aligned} \|A_{\lambda,\epsilon}(w_1) - A_{\lambda,\epsilon}(w_2)\|_{L^2} &\leqslant \frac{1}{1+\epsilon^2\lambda}\|\Lambda\|_{L^1}\|\tilde{V}'_\epsilon(w_1) - \tilde{V}'_\epsilon(w_2)\|_{L^2} \\ &\leqslant \frac{1+\epsilon^2}{1+\epsilon^2\lambda}(1+\max\{\|w_1\|_{L^\infty}, \|w_2\|_{L^\infty}\})^{\epsilon^2}\|w_1 - w_2\|_{L^2} \\ &\leqslant \frac{1+\epsilon^2}{1+\epsilon^2\lambda}(1 + C_R\epsilon^2\log(1+R))\|w_1 - w_2\|_{L^2}, \end{aligned}$$

we have the desired contraction property for the operator $A_{\lambda,\epsilon} : B_R \to B_R$ if $\lambda > \lambda_R$. Since $A_{\lambda,\epsilon}(0) = 0$, the contraction principle guarantees that the trivial solution $w = 0$ is the only fixed point of $A_{\lambda,\epsilon}$ in the set B_R. ∎

Next we set $\mu := \epsilon^2\lambda$ with $\lambda \in (1,\infty)$ being fixed and ϵ-independent. Proposition 3.1 does not rule out the existence of nonzero solutions in B_R to the fixed-point equation (3.1) for sufficiently large R. In what follows, we will consider the nonzero solutions to the fixed-point equation (3.1), which are close to large-amplitude traveling waves given by the stationary log–KdV equation (2.7).

Let us now recapture the formal limit to the stationary log–KdV equation (2.7). Using the Taylor series expansion as $k \to 0$,

$$\hat{\Lambda}(k) = \frac{4}{k^2}\sin^2\left(\frac{k}{2}\right) = 1 - \frac{1}{12}k^2 + \mathcal{O}(k^4), \tag{3.4}$$

and the power series (2.6) for $\tilde{V}'_\epsilon(w)$, we truncate the fixed-point equation (3.2) with $\mu = \epsilon^2\lambda$ at the leading-order terms as follows:

$$\epsilon^2\lambda\widehat{w_{\text{lead}}}(k) = -\frac{1}{12}k^2\widehat{w_{\text{lead}}}(k) + \epsilon^2\mathcal{F}((1+w_{\text{lead}})\log(1+w_{\text{lead}}))(k). \tag{3.5}$$

Using the inverse Fourier transform and setting the variables $x = \epsilon z$ and $W(x) = w_{\text{lead}}(z)$, we then recover the stationary log–KdV equation (2.7).

3.2. Solitary waves for the stationary log–KdV equation. A standard construction of solutions to the stationary log–KdV equation (2.7) is based on a dynamical system approach and gives the following result.

PROPOSITION 3.2. *For any $\lambda > 1$, there exists a unique (up to the spatial translation) solution W_{stat} to the stationary log–KdV equation* (2.7) *in $H^1(\mathbb{R})$, such that $W_{\text{stat}}(x) > 0$ for all $x \in \mathbb{R}$. Moreover, $W_{\text{stat}} \in H^\infty(\mathbb{R})$, W'_{stat} vanishes only at one point on $\mathbb{R}$, and*

$$W_{\text{stat}}(x) \leqslant C_\lambda e^{-\kappa_\lambda|x|}, \quad x \in \mathbb{R}, \tag{3.6}$$

for some λ-dependent positive constants C_λ and κ_λ.

Proof. Integrating the second-order differential equation (2.7), we obtain the energy

$$E(W) := \frac{1}{24}\left(\frac{dW}{dx}\right)^2 + \frac{1}{2}(1+W)^2\log(1+W) - \frac{1}{4}(1+W)^2 - \frac{1}{2}\lambda W^2 = E_0,$$

which is constant in x. Since any solution in $H^1(\mathbb{R})$ should decay to zero at infinity, we set $E_0 = -\frac{1}{4}$. Because $E(W) \to \infty$ as $W \to \infty$, the turning point $W_0 > 0$ such that $E(W_0) = E_0$ exists if $E(W)$ is concave near $W = 0$. This is ensured by the condition $\lambda > 1$.

Further analysis of the nonlinear potential shows that if $\lambda > 1$, there is a unique turning point W_0 and a unique homoclinic orbit in the right-half of the phase plane (W, W') that connects the saddle point $(0,0)$ for $E_0 = -\frac{1}{4}$. For this homoclinic orbit, W' vanishes at exactly one point x_0, where $W(x_0) = W_0$. By the theory of stable and unstable manifolds, the homoclinic orbit for the nondegenerate saddle point decays exponentially fast at infinity with the precise decay rate $\kappa_\lambda := \sqrt{12(\lambda-1)}$. Furthermore, bootstrapping arguments for the differential equation (2.7) yield $W_{\text{stat}} \in H^\infty(\mathbb{R})$ because $W \mapsto \log(1+W)$ is C^∞ on $(0,\infty)$. ∎

Remark 3.1. By the translational symmetry, we can always shift W_{stat} so that $x_0 = 0$, in which case, $W'_{\text{stat}}(0) = 0$ and W_{stat} is even.

Linearizing the nonlinear differential equation (2.7) at the solitary wave $W_{\rm stat}$, we obtain the Schrödinger operator with a bounded and decaying potential

$$L_\lambda := -\frac{1}{12}\frac{\partial^2}{\partial x^2} + \lambda - 1 - \log(1 + W_{\rm stat}) : H^2(\mathbb{R}) \to L^2(\mathbb{R}). \tag{3.7}$$

Although the exact location of the spectrum of L_λ is unknown, several facts follow from the Sturm theory (see Chapter 5.5 in [30] for a review of the Sturm theory).

PROPOSITION 3.3. *For any $\lambda > 1$, the spectrum of L_λ in $L^2(\mathbb{R})$ includes one negative eigenvalue λ_{-1} and the zero eigenvalue $\lambda_0 = 0$ with the eigenfunction $W_0 = W'_{\rm stat}$. The rest of the spectrum of L_λ lies in $(0,\infty)$ and is bounded away from zero by a positive number. Consequently, the linear operator L_λ is invertible with bounded inverse on the subspace of functions in $L^2(\mathbb{R})$ orthogonal to W_0.*

Proof. Since L_λ is self-adjoint, it has a real spectrum. The zero eigenvalue is due to the possible translation of the solitary wave $W_{\rm stat}$ in space: $L_\lambda W'_{\rm stat} = 0$. Since $W'_{\rm stat}$ has exactly one zero, there exists exactly one negative eigenvalue λ_{-1}. The continuous spectrum of L_λ is bounded from below by the positive number $\lambda - 1$, thanks to the fact that the potential $\log(1 + W_{\rm stat})$ of the Schrödinger operator L_λ is bounded and exponentially decaying at infinity. By Sturm's theory, there may exist a finite number of positive eigenvalues between 0 and $\lambda - 1$. □

For iterations of the fixed-point equation (3.1), it is more convenient to work with the operator

$$S_\lambda := \left(-\frac{1}{12}\frac{\partial^2}{\partial x^2} + \lambda - 1\right)^{-1} \log(1 + W_{\rm stat}) : L^2(\mathbb{R}) \to H^2(\mathbb{R}). \tag{3.8}$$

The following result is an equivalent reformulation of Proposition 3.3.

PROPOSITION 3.4. *For any $\lambda > 1$, the spectrum of S_λ in $L^2(\mathbb{R})$ lies in $(0,\infty)$ and includes one simple eigenvalue μ_{-1} bigger than 1, a simple eigenvalue $\mu_0 = 1$ with the eigenfunction $W_0 = W'_{\rm stat}$, and the rest of the spectrum of S_λ is located in the interval $(0,1)$ bounded away from $\mu_0 = 1$. Consequently, the linear operator $I - S_\lambda$ is invertible with bounded inverse on the subspace of functions in $L^2(\mathbb{R})$ orthogonal to W_0.*

Proof. The operator S_λ is conjugated via the positive operator $\left(-\frac{1}{12}\partial_x^2 + \lambda - 1\right)^{1/2}$ to a self-adjoint operator in $L^2(\mathbb{R})$. Hence the spectrum of S_λ is real. Moreover, since $\log(1 + W_{\rm stat}(x)) > 0$ for all $x \in \mathbb{R}$, the spectrum of S_λ is positive.

By Sylvester's inertia law (see Chapter 4.1.2 in [23]), operators L_λ and $I - S_\lambda$ have the same number of negative eigenvalues and the same multiplicity of the zero eigenvalue. By Proposition 3.3, L_λ has one simple negative eigenvalue and a simple zero eigenvalue. Equivalently, S_λ has one simple eigenvalue $\mu_{-1} > 1$ with an eigenfunction W_{-1} and a simple eigenvalue $\mu_0 = 1$ with an eigenfunction W_0. Moreover, the eigenfunction of S_λ for $\mu_0 = 1$ is the same as that of L_λ for $\lambda_0 = 0$, that is, $W_0 = W'_{\rm stat}$.

Because the positive spectrum of L_λ is bounded away from zero, the rest of the spectrum of S_λ is located in the interval $(0,1)$ and bounded away from $\mu_0 = 1$. Consequently, $\|S_\lambda\|_{X_0^\perp \to L^2} < 1$, where $X_0 := \operatorname{span}\{W_{-1}, W_0\}$, and, by von Neumann's theorem, $I - S_\lambda$ is invertible with bounded inverse on the subspace $X_0^\perp \subset L^2(\mathbb{R})$. Furthermore, since $\mu_{-1} > 1$, it is also invertible on the subspace of functions in $L^2(\mathbb{R})$ orthogonal to W_0. □

Remark 3.2. It follows from the criterion given by Pego [26] that S_λ is actually a compact operator in $L^2(\mathbb{R})$. However, we do not need to use this fact here nor to construct the spectrum of S_λ explicitly.

3.3. Strategy to prove Theorem 1. Let us divide the infinite line for the Fourier variable k into two sets $\mathcal{I}_p := [-\epsilon^p, \epsilon^p]$ and $\mathcal{J}_p := \mathbb{R}\backslash\mathcal{I}_p$, where a positive ϵ-independent parameter p is to be defined later. Let χ_S be the characteristic function of the set $S \subset \mathbb{R}$. Then, we decompose the solution in the Fourier form into two parts:

$$\hat{w}(k) = \hat{u}(k) + \hat{v}(k), \quad \text{where } \hat{u}(k) := \chi_{\mathcal{I}_p}(k)\hat{w}(k), \quad \hat{v}(k) := \chi_{\mathcal{J}_p}(k)\hat{w}(k). \tag{3.9}$$

The original problem (3.2) with $\mu = \epsilon^2\lambda$ is now written as a system of two equations

$$\hat{v}(k) = \frac{1}{1+\epsilon^2\lambda}\chi_{\mathcal{J}_p}(k)\hat{\Lambda}(k)\mathcal{F}(\tilde{V}_\epsilon'(u+v))(k), \quad k \in \mathcal{J}_p \tag{3.10}$$

and

$$\hat{u}(k) = \frac{1}{1+\epsilon^2\lambda}\chi_{\mathcal{I}_p}(k)\hat{\Lambda}(k)\mathcal{F}(\tilde{V}_\epsilon'(u+v))(k), \quad k \in \mathcal{I}_p. \tag{3.11}$$

Here we set $\lambda > 1$ to be ϵ-independent. For $R > 0$ and $r \in (-1, 0)$, we define

$$B_{R,r} := \left\{ u \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) : \quad r \leqslant \inf_{\mathbb{R}} u, \quad \sup_{\mathbb{R}} u \leqslant R \right\} \tag{3.12}$$

to consider functions which may have small negative and large positive values.

First, we show that for any $u \in B_{R,r}$ and for any small ϵ, there exists a unique solution v to the first equation (3.10) in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that

$$\|v\|_{L^2\cap L^\infty} \leqslant C_{R,r}\epsilon^{2-2p}\|u\|_{L^2}, \tag{3.13}$$

where the positive constant $C_{R,r}$ is independent of ϵ and $\|u\|_{L^2}$. We use the contraction principle for (3.10) which holds if $p < 1$. Let us denote the solution by $v^*(u)$.

Second, we show that, when $v^*(u)$ is substituted into (3.11), there exists a unique solution u to (3.11) in $B_{R,r}$ near the solution $w_{\text{lead}} = W_{\text{stat}}(\epsilon\cdot)$ to the stationary log–KdV equation in the Fourier form (3.5):

$$\|u - W_{\text{stat}}(\epsilon\cdot)\|_{L^2\cap L^\infty} \leqslant C_{R,r,\lambda}\max\{\epsilon^{4p-2}, \epsilon^{2-2p}\}\|W_{\text{stat}}(\epsilon\cdot)\|_{L^2}, \tag{3.14}$$

where the positive constant $C_{R,r,\lambda}$ is independent of ϵ. We use a fixed-point argument for (3.11). Note that no contraction principle can be applied directly either to the full equation (3.2) or to the reduced equation (3.11) because even if the fixed point exists, the nonlinear operator on the right-hand side is not a contraction operator in the neighborhood of the fixed point. In particular, this fact is justified by the appearance of eigenvalue $\mu_{-1} > 1$ in Proposition 3.4. Therefore, we have to regroup the left-hand and right-hand side terms of (3.11) before applying fixed-point arguments.

Note that $\|W_{\text{stat}}(\epsilon\cdot)\|_{L^2} = \mathcal{O}(\epsilon^{-1/2})$ as $\epsilon \to 0$, therefore, both corrections $u - W_{\text{stat}}$ and v are small in L^∞ norm if

$$2 - 2p - \frac{1}{2} > 0 \quad \text{and} \quad 4p - 2 - \frac{1}{2} > 0, \tag{3.15}$$

that is, for $p \in (\frac{5}{8}, \frac{6}{8})$. The optimal (smallest) bound occurs at $p = 2/3$ and corresponds to the power $1/6$ in the bound (2.8). Thanks to the positivity of W_{stat}, we have $r = \mathcal{O}(\epsilon^{1/6})$ as $\epsilon \to 0$. At the same time, $R = \mathcal{O}(1)$ depends on $\lambda > 1$ and can be as large as necessary (but ϵ-independent).

We now follow the scheme above and prove bounds (3.13) and (3.14). As explained above, these bounds yield the first part of Theorem 1.

3.4. Proof of the bound (3.13). The following lemma yields the bound (3.13).

LEMMA 3.1. *For $R > 0$ and $r \in (-1, 0)$, let u belong to the set $B_{R,r}$ defined in* (3.12)*. For any $\lambda > 1$, $p \in (0, 1)$, and sufficiently small ϵ, there exists a unique solution to* (3.10) *in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that*

$$\|v\|_{L^2 \cap L^\infty} \leqslant C_{R,r} \epsilon^{2-2p} \|u\|_{L^2}, \tag{3.16}$$

where the positive constant $C_{R,r}$ is independent of ϵ and $\|u\|_{L^2}$. Moreover, the map $B_{R,r} \ni u \mapsto v \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is C^1.

Proof. We write $\tilde{V}_\epsilon'(w) = w + N_\epsilon(w)$, where

$$N_\epsilon(w) = (1+w)^{1+\epsilon^2} - 1 - w = \log(1+w) \int_0^{\epsilon^2} (1+w)^{1+x} dx$$

and

$$N_\epsilon(w_1) - N_\epsilon(w_2) = \epsilon^2 \int_{w_2}^{w_1} (1+x)^{\epsilon^2} dx + \int_{w_2}^{w_1} \log(1+x) \left(\int_0^{\epsilon^2} (1+x)^y dy \right) dx.$$

The function $f(w) := \log(1+w)/w$ is strictly decreasing for $w > -1$ with $f(0) = 1$. As a result, for every $r \in (-1, 0)$, there is a positive constant C_r such that

$$|N_\epsilon(w)| \leqslant \epsilon^2 C_r (1 + |w|)^{1+\epsilon^2} |w|, \quad w \geqslant r$$

and

$$|N_\epsilon(w_1) - N_\epsilon(w_2)| \leqslant \epsilon^2 C_r \left(1 + \max\{|w_1|, |w_2|\}\right)^{1+\epsilon^2} |w_1 - w_2|, \quad w_1, w_2 \geqslant r.$$

Note that C_r may be chosen so that $C_r \underset{r \to 0}{\longrightarrow} 1$.

Therefore, we rewrite (3.10) in the equivalent form

$$\hat{v}(k) = \hat{\mathcal{A}}_{\lambda,\epsilon}(\hat{u}, \hat{v}) := \frac{1}{1+\epsilon^2\lambda} \hat{\Lambda}_{\mathcal{J}_p}(k) \left(\hat{v}(k) + \mathcal{F}(N_\epsilon(u+v))(k) \right), \quad k \in \mathcal{J}_p, \tag{3.17}$$

where $\hat{\Lambda}_{\mathcal{J}_p}(k) := \chi_{\mathcal{J}_p}(k) \hat{\Lambda}(k)$. Because $|k| \geqslant \epsilon^p$ for $k \in \mathcal{J}_p$, we note from (3.4) that there exists an ϵ-independent positive constant C such that

$$\|\hat{\Lambda}_{\mathcal{J}_p}\|_{L^\infty} \leqslant 1 - C\epsilon^{2p}.$$

Let $\mathcal{A}_{\lambda,\epsilon}(u, v) := \mathcal{F}^{-1}(\hat{\mathcal{A}}_{\lambda,\epsilon}(\hat{u}, \hat{v}))$ and consider $\mathcal{A}_{\lambda,\epsilon}(u, \cdot)$ for a given $u \in B_{R,r}$ as a bounded operator from $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ to itself. By Plancherel's theorem, we obtain

$$\begin{aligned}
\|\mathcal{A}_{\lambda,\epsilon}(u, v)\|_{L^2} &= \frac{1}{\sqrt{2\pi}} \|\hat{\mathcal{A}}_{\lambda,\epsilon}(\hat{u}, \hat{v})\|_{L^2} \\
&\leqslant \|\hat{\Lambda}_{\mathcal{J}_p}\|_{L^\infty} \left(\|v\|_{L^2} + \|N_\epsilon(u+v)\|_{L^2} \right) \\
&\leqslant (1 - C\epsilon^{2p}) \left(\|v\|_{L^2} + \epsilon^2 C_r (1 + \|u+v\|_{L^\infty})^{1+\epsilon^2} \|u+v\|_{L^2} \right).
\end{aligned}$$

By the Cauchy–Schwarz inequality, we also have

$$\begin{aligned}
\|\mathcal{A}_{\lambda,\epsilon}(u, v)\|_{L^\infty} &\leqslant \frac{1}{2\pi} \|\hat{\mathcal{A}}_{\lambda,\epsilon}(\hat{u}, \hat{v})\|_{L^1} \\
&\leqslant \|\Lambda\|_{L^2} \left(\|v\|_{L^2} + \|N_\epsilon(u+v)\|_{L^2} \right) \\
&\leqslant \frac{\sqrt{2}}{\sqrt{3}} \left(\|v\|_{L^2} + \epsilon^2 C_r (1 + \|u+v\|_{L^\infty})^{1+\epsilon^2} \|u+v\|_{L^2} \right).
\end{aligned}$$

Let $u \in B_{R,r}$ be defined by (3.12), where $R > 0$ and $r \in (-1, 0)$ are fixed independently from ϵ. Recall that if $p < 1$, then $\epsilon^{2p} \gg \epsilon^2$ as $\epsilon \to 0$. Also note that

$$\|\mathcal{A}_{\lambda,\epsilon}(u, 0)\|_{L^2 \cap L^\infty} \leqslant C_{R,r}\epsilon^2 \|u\|_{L^2},$$

where the positive constant $C_{R,r}$ is independent of ϵ and $\|u\|_{L^2}$. Then, for every u in $B_{R,r}$, $\lambda > 1$ and $\delta > 0$, for sufficiently small $\epsilon > 0$ (say, $\epsilon^{2(1-p)} \leqslant C\delta$ for a small u-dependent constant C), the operator $\mathcal{A}_{\lambda,\epsilon}(u, \cdot)$ maps the ball of functions v in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ centered at zero with the radius δ to itself. Moreover, the operator $\mathcal{A}_{\lambda,\epsilon}(u, \cdot)$ is a contraction in this ball, using similar bounds,

$$\begin{aligned}
&\|\mathcal{A}_{\lambda,\epsilon}(u, v_1) - \mathcal{A}_{\lambda,\epsilon}(u, v_2)\|_{L^2 \cap L^\infty} \\
&\quad \leqslant \|\hat{\Lambda}_{\mathcal{J}_p}\|_{L^\infty \cap L^2} \left(\|v_1 - v_2\|_{L^2} + \|N_\epsilon(u + v_1) - N_\epsilon(u + v_2)\|_{L^2}\right) \\
&\quad \leqslant (1 - C\epsilon^{2p})(1 + \epsilon^2 C_r (1 + R + \delta)^{1+\epsilon^2}) \|v_1 - v_2\|_{L^2}.
\end{aligned}$$

Again, the contraction is ensured by the fact that $\epsilon^{2p} \gg \epsilon^2$ as $\epsilon \to 0$. Note that the Lipschitz constant is bounded from above by $1 - C\epsilon^{2p}$ independently from R.

By the contraction mapping principle, for every given u in $B_{R,r}$, $\lambda > 1$, $p < 1$, and sufficiently small $\epsilon > 0$, there exists a unique fixed point of the operator equation $v = \mathcal{A}_{\lambda,\epsilon}(u, v)$ in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying the bound (3.16), where ϵ^{2p} is lost because of the proximity of the Lipschitz constant to unity. Differentiability of the mapping $B_{R,r} \ni u \mapsto v \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ also follows from the contraction mapping principle, since the nonlinear operator $\mathcal{A}_{\lambda,\epsilon}(u, v)$ is differentiable with respect to both u and v. ∎

3.5. Proof of the bound (3.14). The following lemma yields the bound (3.14).

LEMMA 3.2. *For any fixed $\lambda > 1$ and $p \in \left(\frac{5}{8}, \frac{6}{8}\right)$, let $v \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be uniquely expressed in terms of $u \in B_{R,r}$ for some $R > 0$ and $r \in (-1, 0)$ by Lemma* 3.1, *where $B_{R,r}$ is defined by* (3.12). *For sufficiently small ϵ, there exists a unique solution to* (3.11) *in $B_{R,r}$ such that*

$$\|u - W_{\mathrm{stat}}(\epsilon \cdot)\|_{L^2 \cap L^\infty} \leqslant C_{R,r,\lambda} \max\{\epsilon^{4p-2}, \epsilon^{2-2p}\} \|W_{\mathrm{stat}}(\epsilon \cdot)\|_{L^2}, \tag{3.18}$$

where W_{stat} is the unique positive and even solution to the stationary log–KdV equation (2.7) *and the positive constant $C_{R,r,\lambda}$ is independent of ϵ.*

Proof. By the Taylor expansion (3.4), we can represent $\hat{\Lambda}(k)$ for any $k \in \mathcal{I}_p$ as

$$\hat{\Lambda}(k) = \frac{1 + \hat{\Lambda}_{\mathrm{Rem}}(k)}{1 + \frac{1}{12}k^2}, \quad |k| \leqslant \epsilon^p,$$

where the remainder term satisfies the bound

$$\|\chi_{\mathcal{I}_p} \hat{\Lambda}_{\mathrm{Rem}}\|_{L^\infty} \leqslant C_\Lambda \epsilon^{4p}$$

for a positive ϵ-independent constant C_Λ. We now write

$$\tilde{V}_\epsilon'(w) = w + \epsilon^2 (1 + w) \log(1 + w) + M_\epsilon(w),$$

where

$$\begin{aligned}
M_\epsilon(w) &= (1 + w)^{1+\epsilon^2} - 1 - w - \epsilon^2 (1 + w) \log(1 + w) \\
&= \log^2(1 + w) \int_0^{\epsilon^2} \left(\int_0^x (1 + w)^{1+y} dy \right) dx.
\end{aligned}$$

Recall that the function $f(w) := \log(1+w)/w$ is strictly decreasing for $w > -1$ with $f(0) = 1$. Therefore, for any $r \in (-1, 0)$, there is a positive constant C_r such that

$$|M_\epsilon(w)| \leqslant \frac{1}{2}\epsilon^4 C_r (1+|w|)^{1+\epsilon^2} w^2, \quad w \geqslant r,$$

and

$$|M_\epsilon(w_1) - M_\epsilon(w_2)| \leqslant \epsilon^4 C_r \left(1 + \max\{|w_1|, |w_2|\}\right)^{1+\epsilon^2} \max\{|w_1|, |w_2|\}|w_1 - w_2|, \\ w_1, w_2 \geqslant r.$$

Note again that C_r may be chosen so that $C_r \underset{r\to 0}{\longrightarrow} 1$.

Equation (3.11) can be rewritten in the equivalent form

$$\left(\lambda + \frac{k^2}{12\epsilon^2}\right)\hat{u}(k) - \chi_{\mathcal{I}_p}(k)\mathcal{F}((1+u+v)\log(1+u+v))(k) = \hat{H}_\epsilon(\hat{u}, \hat{v})(k), \tag{3.19}$$

where

$$\hat{H}_\epsilon(\hat{u}, \hat{v})(k) := -\frac{\lambda k^2}{12}\hat{u}(k) + \epsilon^{-2}\chi_{\mathcal{I}_p}(k)\mathcal{F}(M_\epsilon(u+v))(k) \\ + \epsilon^{-2}\chi_{\mathcal{I}_p}(k)\hat{\Lambda}_{\mathrm{Rem}}(k)\mathcal{F}(\tilde{V}_\epsilon'(u+v))(k).$$

It follows from the above estimates that for sufficiently small ϵ, the right-hand side of (3.19) satisfies the estimate

$$\frac{1}{\sqrt{2\pi}}\|\hat{H}_\epsilon(\hat{u}, \hat{v})\|_{L^2} \leqslant \frac{\lambda\epsilon^{2p}}{12}\|u\|_{L^2} + \frac{1}{2}\epsilon^2 C_r (1 + \|u+v\|_{L^\infty})^{1+\epsilon^2}\|u+v\|_{L^\infty}\|u+v\|_{L^2} \\ + C_\Lambda \epsilon^{4p-2}(1 + \epsilon^2 C_r (1+\|u+v\|_{L^\infty})^{1+\epsilon^2})\|u+v\|_{L^2}.$$

Recall that if $p < 1$, then $\epsilon^{4p-2} \gg \epsilon^{2p} \gg \epsilon^2$ as $\epsilon \to 0$. Let $v = v^*(u)$, where $v^*(u) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is uniquely expressed in terms of $u \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ by Lemma 3.1. Then, we obtain

$$\frac{1}{\sqrt{2\pi}}\|\hat{H}_\epsilon(\hat{u}, \hat{v}^*(\hat{u}))\|_{L^2} \leqslant C_{R,r}\epsilon^{4p-2}\left(1 + \epsilon^{2-2p}\|u\|_{L^2}\right)^{1+\epsilon^2}\|u\|_{L^2}, \tag{3.20}$$

where the positive constant $C_{R,r}$ is independent of ϵ and $\|u\|_{L^2}$. Since $\mathcal{I}_p$ is compact, we also have

$$\frac{1}{2\pi}\|\hat{H}_\epsilon(\hat{u}, \hat{v}^*(\hat{u}))\|_{L^1} \leqslant \frac{\epsilon^{p/2}}{\sqrt{2\pi}}\|\hat{H}_\epsilon(\hat{u}, \hat{v}^*(\hat{u}))\|_{L^2}. \tag{3.21}$$

Let us define the set

$$B_{R,r,C} := \left\{u \in B_{R,r} : \quad \|u\|_{L^2} \leqslant C\epsilon^{-1/2}\right\}$$

for some ϵ-independent constant $C > \|W_{\mathrm{stat}}\|_{L^2}$. If p belongs to the bounds (3.15) and u belongs to $B_{R,r,C}$, then the term $\epsilon^{2-2p}\|u\|_{L^2}$ is bounded by a small constant as $\epsilon \to 0$. For convenience, we will simply omit this term in the upper bounds. In what follows, we use a fixed-point argument in $B_{R,r,C}$, which ensures that u satisfies (3.18).

Let $H_\epsilon(u,v) := \mathcal{F}^{-1}\hat{H}_\epsilon(\hat{u},\hat{v})$. From (3.20) and (3.21) for $u \in B_{R,r,C}$, we have

$$\|H_\epsilon(u, v^*(u))\|_{L^2\cap L^\infty} \leqslant C_{R,r}\epsilon^{4p-2}\|u\|_{L^2}. \tag{3.22}$$

Since the mapping $u \mapsto v$ is differentiable and all nonlinear functions in $H_\epsilon(u,v)$ are differentiable with respect to both u and v, the remainder term $H_\epsilon(u,v^*(u))$ is differentiable with respect to u in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Next, we study the left-hand side of (3.19). We write it as $\hat{F}_\epsilon(\hat{u}) + \hat{G}_\epsilon(\hat{u},\hat{v})$, where

$$\hat{F}_\epsilon(\hat{u})(k) := \left(\lambda + \frac{k^2}{12\epsilon^2}\right)\hat{u}(k) - \chi_{\mathcal{I}_p}(k)\mathcal{F}((1+u)\log(1+u))(k)$$

and

$$\hat{G}_\epsilon(\hat{u},\hat{v})(k) := -\chi_{\mathcal{I}_p}(k)\mathcal{F}((1+u+v)\log(1+u+v) - (1+u)\log(1+u))(k).$$

Since the function $f(w) := (1+w)\log(1+w)$ is differentiable for any $w > -1$ with $f'(w) = 1 + \log(1+w)$, we have the bound

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\|\hat{G}_\epsilon(\hat{u},\hat{v})\|_{L^2} &\leqslant \|(1+u+v)\log(1+u+v) - (1+u)\log(1+u)\|_{L^2} \\
&\leqslant (1 + C_r\|\log(1+u+v)\|_{L^\infty})\|v\|_{L^2} \\
&\leqslant (1 + C_r\|u+v\|_{L^\infty})\|v\|_{L^2}.
\end{aligned}$$

Using the bound (3.16) from Lemma 3.1 and a similar bound for $\|\hat{G}_\epsilon(\hat{u},\hat{v})\|_{L^1}$, we hence have for $u \in B_{R,r,C}$,

$$\frac{1}{\sqrt{2\pi}}\|\hat{G}_\epsilon(\hat{u},\hat{v}^*(\hat{u}))\|_{L^2\cap L^1} \leqslant C_{R,r}\epsilon^{2-2p}\|u\|_{L^2}. \tag{3.23}$$

Let $G_\epsilon(u,v) := \mathcal{F}^{-1}\hat{G}_\epsilon(\hat{u},\hat{v})$. From (3.23), we have

$$\|G_\epsilon(u,v^*(u))\|_{L^2\cap L^\infty} \leqslant C_{R,r}\epsilon^{2-2p}\|u\|_{L^2}. \tag{3.24}$$

Again, $G_\epsilon(u,v^*(u))$ is differentiable with respect to u in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

It remains to study the leading-order part $\hat{F}_\epsilon(\hat{u})$, where we apply arguments based on the implicit function theorem. Let us define $F_\epsilon(u) := \mathcal{F}^{-1}(\hat{F}_\epsilon(\hat{u}))$. For any $\epsilon > 0$, the nonlinear operator $F_\epsilon(u)$ is a bounded operator from a subset of $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ to $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ thanks to the bounds

$$\frac{1}{\sqrt{2\pi}}\|\hat{F}_\epsilon(\hat{u})\|_{L^2} \leqslant \left(\lambda + \frac{1}{12\epsilon^{2(1-p)}} + C_r(1 + \|u\|_{L^\infty})\right)\|u\|_{L^2}$$

and a similar bound for $\|\hat{F}_\epsilon(\hat{u})\|_{L^1}$. The map $u \mapsto F_\epsilon(u)$ is C^∞ thanks to the smoothness of the function $u \mapsto \log(1+u)$ on $(-1,\infty)$.

Consider the solitary wave solution W_{stat} to the stationary log–KdV equation (2.7) from Proposition 3.2, and let $w_{\text{lead}} = W_{\text{stat}}(\epsilon\cdot)$ be the corresponding solution to the same equation written in the Fourier form (3.5). We have the relationship between the Fourier transforms of w_{lead} and W_{stat}:

$$\widehat{w_{\text{lead}}}(k) = \int_{-\infty}^{\infty} W_{\text{stat}}(\epsilon z)\, e^{-ikz} dz = \frac{1}{\epsilon}\widehat{W_{\text{stat}}}\left(\frac{k}{\epsilon}\right). \tag{3.25}$$

We further define an approximation of $w_{\rm lead}$ by truncating the Fourier transform $\widehat{w_{\rm lead}}$ on $\mathcal{I}_p$, that is,

$$\begin{aligned} w_{\rm app}(z) &:= \mathcal{F}^{-1}(\chi_{\mathcal{I}_p}\widehat{w_{\rm lead}})(z) = \frac{1}{2\pi}\int_{\mathcal{I}_p}\widehat{w_{\rm lead}}(k)e^{ikz}dk \\ &= \frac{1}{2\pi}\int_{-\epsilon^{p-1}}^{\epsilon^{p-1}}\widehat{W_{\rm stat}}(\kappa)e^{i\kappa\epsilon z}d\kappa. \end{aligned} \tag{3.26}$$

Since $W_{\rm stat}\in H^\infty(\mathbb{R})$ by Proposition 3.2, Sobolev's embedding implies that $W_{\rm stat}\in C^\infty(\mathbb{R})$, which then implies that $\widehat{W_{\rm stat}}$ decays faster than any power at infinity. It follows from (3.26) for $p<1$ that the integration interval extends to the entire line as $\epsilon\to 0$. As a result, for any $s>0$, we have an ϵ-independent positive constant C_s such that for all sufficiently small $\epsilon>0$,

$$\|w_{\rm app}-w_{\rm lead}\|_{L^2\cap L^\infty}\leqslant C_s\epsilon^s. \tag{3.27}$$

The nonlinear operator $F_\epsilon(u)$ evaluated at $u=w_{\rm app}$ is given in the Fourier form by

$$\begin{aligned} &\mathcal{F}[F_\epsilon(w_{\rm app})](k) \\ &\quad=\chi_{\mathcal{I}_p}(k)\mathcal{F}((1+w_{\rm app})\log(1+w_{\rm app}))(k)-\chi_{\mathcal{I}_p}(k)\mathcal{F}((1+w_{\rm lead})\log(1+w_{\rm lead}))(k). \end{aligned}$$

Consequently, thanks to the smoothness of the map $u\mapsto F_\epsilon(u)$ in $L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$ and the bound (3.27), we obtain

$$\|F_\epsilon(w_{\rm app})\|_{L^2\cap L^\infty}\leqslant C_s\epsilon^s \tag{3.28}$$

for any $s>0$ and sufficiently small ϵ.

We rewrite equation (3.19) as the implicit equation

$$f_\epsilon(u)=h_\epsilon(u,v), \tag{3.29}$$

where

$$\begin{aligned} f_\epsilon(u) &:= \mathcal{F}^{-1}\chi_{\mathcal{I}_p}\left(\lambda-1+\frac{k^2}{12\epsilon^2}\right)^{-1}\left(\hat{F}_\epsilon(\hat{u})-\hat{F}_\epsilon(\widehat{w_{\rm app}})\right), \\ h_\epsilon(u,v) &:= \mathcal{F}^{-1}\chi_{\mathcal{I}_p}\left(\lambda-1+\frac{k^2}{12\epsilon^2}\right)^{-1}\left(\hat{H}_\epsilon(\hat{u},\hat{v})-\hat{G}_\epsilon(\hat{u},\hat{v})-\hat{F}_\epsilon(\widehat{w_{\rm app}})\right). \end{aligned}$$

Since $\lambda>1$, we infer from the bounds (3.22), (3.24), and (3.28) that for $u\in B_{R,r,C}$,

$$\|h_\epsilon(u,v^*(u))\|_{L^2\cap L^\infty}\leqslant C_{R,r,\lambda}\max\{\epsilon^{4p-2},\epsilon^{2-2p}\}\|u\|_{L^2}, \tag{3.30}$$

where the positive constant $C_{R,r,\lambda}$ is independent of ϵ and $\|u\|_{L^2}$. Therefore, the right-hand side of (3.29) is small in $L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$ norm, if p satisfies the bounds (3.15) and u belongs to $B_{R,r,C}$. The left-hand side of (3.29) is zero at $u=w_{\rm app}$.

Let us now consider the linearized operator $\partial_u f_\epsilon(w_{\rm app})$. In the Fourier form, the linearized operator acting on U is given by

$$\mathcal{F}[\partial_u f_\epsilon(w_{\rm app})U](k):=\hat{U}(k)-\chi_{\mathcal{I}_p}(k)\left(\lambda-1+\frac{k^2}{12\epsilon^2}\right)^{-1}\mathcal{F}(\log(1+w_{\rm app})U)(k).$$

Note that $w_{\rm app}(z)$ is an even function of z if $W_{\rm stat}(x)$ is an even function of x because $\widehat{W_{\rm stat}}(k)$ is an even function of k and the truncation in the Fourier domain is taken symmetrically around $k=0$. Also note that the fixed-point problem (3.2) preserves the parity property in the space of even functions. Therefore, we can consider u or $U := u - w_{\rm app}$ in the space of even functions.

Recall the unbounded Schrödinger operator L_λ given by (3.7) and the bounded operator S_λ given by (3.8). Let us now define the bounded operator $S_{\lambda,p}$ in the Fourier form by

$$[\hat{S}_{\lambda,p}\hat{U}](k) := \chi_{\mathcal{I}_p}(k)\left(\lambda - 1 + \frac{k^2}{12\epsilon^2}\right)^{-1}\mathcal{F}(\log(1+w_{\rm lead})U)(k),$$

where we recall the correspondence between the Fourier transforms in x and z variables; see (3.25). Therefore, we obtain the bound

$$\begin{aligned}\frac{1}{2\pi}\|(\hat{S}_\lambda - \hat{S}_{\lambda,p})\hat{U}\|_{L^2}^2 &= \frac{1}{2\pi}\int_{\mathcal{J}_p}\frac{1}{(\lambda - 1 + \frac{k^2}{12\epsilon^2})^2}\left|\mathcal{F}(\log(1+w_{\rm lead})U)(k)\right|^2 dk\\ &\leqslant (12\epsilon^{2-2p})^2\|\log(1+w_{\rm lead})U\|_{L^2}^2,\end{aligned}$$

which yields, thanks to the positivity of $w_{\rm lead}$,

$$\|(S_\lambda - S_{\lambda,p})U\|_{L^2} \leqslant 12\epsilon^{2-2p}\|w_{\rm lead}\|_{L^\infty}\|U\|_{L^2}. \tag{3.31}$$

By Proposition 3.4, the linear operator $I - S_\lambda$ is invertible with bounded inverse on the subspace of even functions in $L^2(\mathbb{R})$. Thanks to the bound (3.31), the linear operator $I - S_{\lambda,p}$ is also invertible with bounded inverse on the subspace of even functions in $L^2(\mathbb{R})$. Finally, thanks to the bound (3.27), the linearized operator $\partial_u f_\epsilon(w_{\rm app})$ is also invertible with bounded inverse on the subspace of even functions in $L^2(\mathbb{R})$. In other words, there is a positive ϵ-independent constant C_λ such that for any sufficiently small ϵ and any even function h in $L^2(\mathbb{R})$, we have

$$\|\left[\partial_u f_\epsilon(w_{\rm app})\right]^{-1} h\|_{L^2} \leqslant C_\lambda\|h\|_{L^2}.$$

Since $\mathcal{I}_p$ is compact, we then have

$$\|\left[\partial_u f_\epsilon(w_{\rm app})\right]^{-1} h\|_{L^2\cap L^\infty} \leqslant C_\lambda\|h\|_{L^2}. \tag{3.32}$$

Writing $u = w_{\rm app} + U$, we can now apply the standard implicit function theorem to obtain a unique solution U to the implicit equation (3.29) in $L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$ close to the zero solution for small $\epsilon > 0$. In view of the bounds (3.30) and (3.32), the solution satisfies the bound

$$\|U\|_{L^2\cap L^\infty} \leqslant C_{R,r,\lambda}\max\{\epsilon^{4p-2},\epsilon^{2-2p}\}\|w_{\rm app}(\epsilon\cdot)\|_{L^2},$$

where the positive constant $C_{R,r,\lambda}$ is ϵ-independent. This bound yields (3.18) thanks to the proximity between $w_{\rm lead} = W_{\rm stat}(\epsilon\cdot)$ and $w_{\rm app}$ given by the bound (3.27). ∎

3.6. Proof of the bound (2.9). By bootstrapping arguments, applied to the differential advance-delay equation (2.4), a solution in $L^2(\mathbb{R})$ belongs to $H^\infty(\mathbb{R})$ because $\tilde{V}_\epsilon'$ is smooth on $(-1,\infty)$. Here we prove the bound (2.9) for $k=1$. The proof extends to every $k\in\mathbb{N}$ by similar arguments and by induction.

To control the derivative of the solution $w = u + v$ of the fixed-point equation (3.1) in the $L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$ norm, we multiply the two equations (3.10) and (3.11) by k. After multiplication by k, (3.17) for $k\in\mathcal{J}_p$ is rewritten in the form

$$k\hat{v}(k) = \frac{1}{1+\epsilon^2\lambda}\hat{\Lambda}_{\mathcal{J}_p}(k)\left(k\hat{v}(k) + k\mathcal{F}(N_\epsilon(u+v))(k)\right), \quad k\in\mathcal{J}_p. \tag{3.33}$$

We repeat the estimates in the proof of Lemma 3.1, after commutation of the Fourier transform and of the multiplication operator by k, which becomes the derivative operator with respect to z applied to the nonlinear function $N_\epsilon(w)$. The nonlinearity $w \mapsto N_\epsilon(w)$ is smooth since $w \geqslant r > -1$.

Let $B'_{R,r}$ be the set

$$B'_{R,r} := \{u \in B_{R,r} : \ \partial_z u \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \ \|\partial_z u\|_{L^\infty} \leqslant R\}. \tag{3.34}$$

By the same technique as in the proof of Lemma 3.1, we obtain from (3.33) for sufficiently small ϵ that for any fixed $\lambda > 1$, $p \in (0,1)$, $R > 0$, $r \in (-1,0)$, and any $u \in B'_{R,r}$, the unique solution to (3.17) satisfies, in addition to the bound (3.16),

$$\|\partial_z v\|_{L^2 \cap L^\infty} \leqslant C_{R,r} \epsilon^{2-2p} \|\partial_z u\|_{L^2}, \tag{3.35}$$

where the positive constant $C_{R,r}$ is independent of ϵ and $\|u\|_{H^1}$.

We then proceed with analysis of (3.19), which we also multiply by k. From the same arguments as in the proof of Lemma 3.2, we obtain, in addition to the bounds (3.22) and (3.24),

$$\|\partial_z H_\epsilon(u, v^*(u))\|_{L^2 \cap L^\infty} \leqslant C_{R,r} \epsilon^{4p-2} \|\partial_z u\|_{L^2} \tag{3.36}$$

and

$$\|\partial_z G_\epsilon(u, v^*(u))\|_{L^2 \cap L^\infty} \leqslant C_{R,r} \epsilon^{2-2p} \|\partial_z u\|_{L^2}, \tag{3.37}$$

where $v^*(u)$ is obtained from (3.17) to satisfy the bounds (3.16) and (3.35). Applying now the derivative with respect to z to the implicit equation (3.29), we obtain

$$\partial_z f_\epsilon(u) = \partial_z h_\epsilon(u, v), \tag{3.38}$$

where the right-hand side satisfies the bound

$$\|\partial_z h_\epsilon(u, v^*(u))\|_{L^2 \cap L^\infty} \leqslant C_{R,r,\lambda} \max\{\epsilon^{4p-2}, \epsilon^{2-2p}\} \|\partial_z u\|_{L^2}. \tag{3.39}$$

The derivative of the linearized operator $\partial_u f_\epsilon(w_{\text{app}})$ applied to U is, by the product rule,

$$\partial_z \left(\partial_u f_\epsilon(w_{\text{app}}) U\right) = \partial_u f_\epsilon(w_{\text{app}}) \partial_z U + \left(\partial_z \partial_u f_\epsilon(w_{\text{app}})\right) U,$$

where the second term is bounded as

$$\| \left(\partial_z \partial_u f_\epsilon(w_{\text{app}})\right) U\|_{L^2} \leqslant C_{R,r,\lambda} \|\partial_z w_{\text{app}}\|_{L^\infty} \|U\|_{L^2}. \tag{3.40}$$

Using the bounds (3.32), (3.39), and (3.40), we obtain from (3.38) for sufficiently small ϵ that for any fixed $\lambda > 1$, the unique solution to (3.29) satisfies, in addition to the bound (3.18),

$$\|\partial_z u - \partial_z w_{\text{app}}\|_{L^2 \cap L^\infty} \leqslant C_{R,r,\lambda} \max\{\epsilon^{4p-2}, \epsilon^{2-2p}\} \|\partial_z w_{\text{app}}\|_{L^2}, \tag{3.41}$$

where the positive constant $C_{R,r,\lambda}$ is independent of ϵ. Thanks to the proximity of derivatives of $w_{\text{lead}} = W_{\text{stat}}(\epsilon \cdot)$ and w_{app}, the bound (3.41) yields (2.9) for $k = 1$, since the error bound is optimal for $p = \frac{2}{3}$ and

$$\|\partial_z W_{\text{stat}}(\epsilon \cdot)\|_{L^2} \leqslant C \epsilon^{1-1/2},$$

where the positive constant C is independent of ϵ.

4. Stability of FPU traveling waves near the log–KdV limit. This section presents the proof of Theorem 2.

Let $(w_{\rm trav}, p_{\rm trav}) \in C^1(\mathbb{R}, l^2(\mathbb{Z}))$ denote the traveling wave solution to the FPU lattice (2.10) with the squared speed $c^2 = 1 + \epsilon^2\lambda$. The amplitudes $(w_{\rm stat}, p_{\rm stat})$ of the traveling wave (2.13) are solutions to the system of advance equations

$$\begin{cases} -cw'_{\rm stat}(z) = p_{\rm stat}(z+1) - p_{\rm stat}(z), \\ -cp'_{\rm stat}(z) = \tilde{V}'_\epsilon(w_{\rm stat}(n-ct)) - \tilde{V}'_\epsilon(w_{\rm stat}(n-1-ct)), \end{cases} \quad z \in \mathbb{R}. \tag{4.1}$$

Properties of $w_{\rm stat}$ are described by Theorem 1 for sufficiently small ϵ.

For any fixed c, we decompose

$$w(t) = w_{\rm trav}(t) + \mathcal{W}(t), \quad p(t) = p_{\rm trav}(t) + \mathcal{P}(t),$$

and rewrite the system of FPU lattice equations (2.10) in the perturbed form

$$\begin{cases} \dot{\mathcal{W}}_n = \mathcal{P}_{n+1} - \mathcal{P}_n, \\ \dot{\mathcal{P}}_n = \tilde{V}''_\epsilon(w_{\rm stat}(n-ct))\mathcal{W}_n - \tilde{V}''_\epsilon(w_{\rm stat}(n-1-ct))\mathcal{W}_{n-1} \\ \qquad + \frac{1}{2}\tilde{V}'''_\epsilon(w_{\rm stat}(n-ct))\mathcal{W}_n^2 - \frac{1}{2}\tilde{V}'''_\epsilon(w_{\rm stat}(n-1-ct))\mathcal{W}_{n-1}^2 + R_n(\mathcal{W}), \end{cases} \tag{4.2}$$

where the remainder term R is cubic in $\mathcal{W}$ thanks to the smoothness of $\tilde{V}_\epsilon$ on $(-1,\infty)$. It is assumed in the perturbed form (4.2) that the solution w remains within the a priori bounds (2.12), which happens if $\mathcal{W}_n$ is sufficiently small for every $n \in \mathbb{Z}$.

Let B_ρ denote the ball in $l^2(\mathbb{Z})$ centered at zero with radius $\rho > 0$. Thanks to the embedding of $l^2(\mathbb{Z})$ into $l^\infty(\mathbb{Z})$, for any $\rho > 0$ small enough, there is a positive constant C_ρ such that the remainder term satisfies the bound

$$\|R(\mathcal{W})\|_{l^2} \leqslant C_\rho \sup_{z\in\mathbb{R}} |\tilde{V}''''_\epsilon(w_{\rm stat}(z))| \, \|\mathcal{W}\|^3_{l^2}, \quad \mathcal{W} \in B_\rho. \tag{4.3}$$

In what follows, such a number ρ is fixed, and C_ρ denotes a positive constant that depends only on ρ. Similarly to (4.2), we expand the energy (2.11) near the traveling wave

$$H = H_0 + H_1 + H_2 + H_R, \tag{4.4}$$

where

$$\begin{aligned} H_0 &= \frac{1}{2}\sum_{n\in\mathbb{Z}} p^2_{\rm stat}(n-ct) + \sum_{n\in\mathbb{Z}} \tilde{V}_\epsilon(w_{\rm stat}(n-ct)), \\ H_1 &= \sum_{n\in\mathbb{Z}} p_{\rm stat}(n-ct)\mathcal{P}_n + \sum_{n\in\mathbb{Z}} \tilde{V}'_\epsilon(w_{\rm stat}(n-ct))\mathcal{W}_n, \\ H_2 &= \frac{1}{2}\sum_{n\in\mathbb{Z}} \mathcal{P}_n^2 + \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{V}''_\epsilon(w_{\rm stat}(n-ct))\mathcal{W}_n^2, \end{aligned}$$

and the remainder term H_R satisfies the bound

$$|H_R| \leqslant C_\rho \sup_{z\in\mathbb{R}} |\tilde{V}'''_\epsilon(w_{\rm stat}(z))| \, \|\mathcal{W}\|^3_{l^2}. \tag{4.5}$$

From the time conservation of H, it follows that H_0 is independent of t. This can be checked by explicit differentiation, using the system (4.1),

$$\begin{aligned}\frac{dH_0}{dt} &= \sum_{n\in\mathbb{Z}} p_{\rm stat}(n-ct)\left[-cp'_{\rm stat}(n-ct)+\tilde{V}'_\epsilon(w_{\rm stat}(n-1-ct))-\tilde{V}'_\epsilon(w_{\rm stat}(n-ct))\right]\\ &= 0.\end{aligned}$$

On the other hand, H_1 is no longer constant. Using (4.1) and (4.2), we obtain

$$\begin{aligned}\frac{dH_1}{dt} &= \sum_{n\in\mathbb{Z}}\Big[p_{\rm stat}(n-ct)\dot{\mathcal{P}}_n+\tilde{V}''_\varepsilon(w_{\rm stat}(n-ct))(p_{\rm stat}(n+1-ct)\\ &\qquad - p_{\rm stat}(n-ct))\mathcal{W}_n\Big]\\ &= \frac{c}{2}\sum_{n\in\mathbb{Z}} w'_{\rm stat}(n-ct)\tilde{V}'''_\epsilon(w_{\rm stat}(n-ct))\mathcal{W}_n^2+S_R,\end{aligned} \tag{4.6}$$

where the remainder term satisfies the bound

$$|S_R| \leqslant C_\rho \sup_{z\in\mathbb{R}} |\tilde{V}''''_\epsilon(w_{\rm stat}(z))w'_{\rm stat}(z)|\, \|\mathcal{W}\|^3_{l^2}, \tag{4.7}$$

which follows from the bound (4.3).

We shall now recall that

$$\tilde{V}''_\epsilon(w) = (1+\epsilon^2)(1+w)^{\epsilon^2}, \quad \tilde{V}'''_\epsilon(w) = \epsilon^2(1+\epsilon^2)(1+w)^{\epsilon^2-1},$$

and so on. Since the negative part of $w_{\rm stat}$ is as small as $\mathcal{O}(\epsilon^{1/6})$ (see Remark 2.2), H_2 is a convex quadratic form with the lower bound

$$H_2 \geqslant \frac{1}{2}\|\mathcal{P}\|^2_{l^2}+\frac{1}{2}\|\mathcal{W}\|^2_{l^2}. \tag{4.8}$$

Using the bound (2.9) for $k=1$, bounds (4.7) and (4.8), we can estimate the balance equation (4.6) as follows:

$$\left|\frac{dH_1}{dt}\right| \leqslant C_\rho\epsilon^3(1+\rho)\|\mathcal{W}\|^2_{l^2} \leqslant 2C_\rho\epsilon^3(1+\rho)H_2$$

as long as $\|W\|_{l^2}\leqslant \rho$, where the positive constant C_ρ is independent of ϵ. As a result, we obtain the lower bound

$$H_1(t)-H_1(0) \geqslant -2C_\rho\epsilon^3(1+\rho)\int_0^{|t|} H_2(t')dt'. \tag{4.9}$$

Now, using the energy expansion (4.4) as well as the bounds (4.5) and (4.9), we can write

$$H-H_0-H_1(0) \geqslant -2C_\rho\epsilon^3(1+\rho)\int_0^{|t|} H_2(t')dt' + H_2(t)(1-C_\rho\epsilon^2\rho). \tag{4.10}$$

By Gronwall's inequality, we obtain

$$H_2(t) \leqslant \frac{H-H_0-H_1(0)}{1-C_\rho\epsilon^2\rho}e^{\tilde{C}_\rho\epsilon^3|t|}, \tag{4.11}$$

where $\tilde{C}_\rho$ is another positive ϵ-independent constant. Since $H - H_0 - H_1(0)$ is t-independent, we can express it at $t = 0$ by

$$H - H_0 - H_1(0) = H_2(0) + H_R(0) \leqslant \tilde{\tilde{C}}_\rho^2 \delta^2, \tag{4.12}$$

where $\tilde{\tilde{C}}_\rho^2$ is yet another positive ϵ-independent constant and the initial bound (2.14) is used. When $\tau_0 > 0$ is given, the bounds (4.8), (4.11), and (4.12) imply the stability bound (2.15) with

$$C_0 > 2\tilde{\tilde{C}}_\rho e^{\frac{1}{2}\tilde{C}_\rho \tau_0}$$

for $\epsilon \in (0, \epsilon_0)$ and $\tau \in [-\tau_0, \tau_0]$, with sufficiently small constants $\epsilon_0 > 0$ and $\delta_0 \in (0, 1)$. Possibly decreasing δ_0, so that δ_0 and $C_0\delta_0$ become less than ρ, Theorem 2 is proved.

5. Justification analysis for time-dependent solutions. This section presents the proof of Theorem 3. In fact, it is a modification of the arguments in the proof of Theorem 2. The arguments follow quite closely to the method described by Schneider and Wayne [27], where the interaction of counterpropagating waves has also been included. We add this section for completeness, as well as for comparison with stability theory of traveling waves in FPU lattices as described by KdV-type equations.

From the assumptions of Theorem 3, we know there exist constants r_W and R_W such that

$$-1 < r_W \leqslant W(\xi, \tau) \leqslant R_W, \quad \xi \in \mathbb{R},\ \tau \in [-\tau_0, \tau_1]. \tag{5.1}$$

For $\epsilon_0 > 0$ small enough, for all $\epsilon \in (0, \epsilon_0)$, initial data $(w_{\text{ini},\epsilon}, p_{\text{ini},\epsilon})$ satisfying the bound (2.18) are such that all the terms in the sequence $w_{\text{ini},\epsilon}$ are greater than some $r > -1$ independent of ϵ. Thus there exists a solution $(w, p) \in C^1([-T_0, T_1], l^2(\mathbb{Z}))$ to the FPU lattice equations (2.10), at least for small times $T_0, T_1 > 0$. We show that, with ϵ_0 small enough, we can ensure $T_0 \geqslant \tau_0\epsilon^{-3}$ and $T_1 \geqslant \tau_1\epsilon^{-3}$, together with the approximation (2.19).

Let us use the decomposition

$$w_n(t) = W(\epsilon(n-t), \epsilon^3 t) + \mathcal{W}_n(t), \quad p_n(t) = P_\epsilon(\epsilon(n-t), \epsilon^3 t) + \mathcal{P}_n(t), \quad n \in \mathbb{Z}, \tag{5.2}$$

where $W(\xi, \tau)$ is the considered smooth solution to the log–KdV equation (2.16) (and thus W is ϵ-independent), whereas the ϵ-dependent function $P_\epsilon(\xi, \tau)$ is found from the truncation of the first equation of the system (2.10) rewritten as

$$P_\epsilon(\xi + \epsilon, \tau) - P_\epsilon(\xi, \tau) = -\epsilon\partial_\xi W(\xi, \tau) + \epsilon^3 \partial_\tau W(\xi, \tau). \tag{5.3}$$

We look for an approximate solution P_ϵ to this equation, under the form

$$P_\epsilon := P^{(0)} + \epsilon P^{(1)} + \epsilon^2 P^{(2)} + \epsilon^3 P^{(3)}, \tag{5.4}$$

with functions $P^{(j)}$ decaying to zero as ξ goes to infinity. Plug this ansatz into (5.3) and collect together the powers of ϵ:

$$\mathcal{O}(\epsilon) : \partial_\xi P^{(0)} = -\partial_\xi W, \qquad \text{satisfied when} \quad P^{(0)} = -W;$$

$$\mathcal{O}(\epsilon^2) : \partial_\xi P^{(1)} + \frac{1}{2}\partial_\xi^2 P^{(0)} = 0, \qquad \text{satisfied when} \quad P^{(1)} = \frac{1}{2}\partial_\xi W;$$

$$\mathcal{O}(\epsilon^3) : \partial_\xi P^{(2)} + \frac{1}{2}\partial_\xi^2 P^{(1)} + \frac{1}{6}\partial_\xi^3 P^{(0)} = -\frac{1}{24}\partial_\xi^3 W - \frac{1}{2}\partial_\xi g(W),$$

$$\text{satisfied when} \quad P^{(2)} = -\frac{1}{8}\partial_\xi^2 W - \frac{1}{2} g(W);$$

$$\mathcal{O}(\epsilon^4) : \partial_\xi P^{(3)} + \frac{1}{2}\partial_\xi^2 P^{(2)} + \frac{1}{6}\partial_\xi^3 P^{(1)} + \frac{1}{24}\partial_\xi^4 P^{(0)} = 0,$$

$$\text{satisfied when} \quad P^{(3)} = \frac{1}{48}\partial_\xi^3 W + \frac{1}{4}\partial_\xi g(W),$$

where $g(w) := (1+w)\log(1+w)$. Note that (5.3) is satisfied by the expansion (5.4) only approximately up to the terms of formal order $\mathcal{O}(\epsilon^5)$.

Recall from the proof of Theorem 1 that we can write the nonlinear potential in the perturbed form

$$\tilde{V}_\epsilon'(w) = w + \epsilon^2 g(w) + M_\epsilon(w),$$

where

$$M_\epsilon(w) = \log^2(1+w) \int_0^{\epsilon^2} \left(\int_0^x (1+w)^{1+y} dy \right) dx.$$

Substituting the decompositions (5.2) and (5.4) into the FPU lattice equations (2.10), we obtain the evolution problem for the error terms

$$(5.5) \quad \begin{cases} \dot{\mathcal{W}}_n(t) = \mathcal{P}_{n+1}(t) - \mathcal{P}_n(t) + \mathrm{Res}_n^{(1)}(t), \\ \dot{\mathcal{P}}_n(t) = \mathcal{W}_n(t) - \mathcal{W}_{n-1}(t) \\ \qquad + \epsilon^2 g'(W(\epsilon(n-t), \epsilon^3 t))\mathcal{W}_n(t) - \epsilon^2 g'(W(\epsilon(n-1-t), \epsilon^3 t))\mathcal{W}_{n-1}(t) \\ \qquad + \mathcal{R}_n(W, \mathcal{W})(t) + \mathrm{Res}_n^{(2)}(t), \end{cases}$$

where

$$\begin{aligned} &\mathcal{R}_n(W, \mathcal{W}) \\ &\quad := \epsilon^2 \big(g(W(\epsilon(n-\cdot), \epsilon^3 \cdot) + \mathcal{W}_n) - g(W(\epsilon(n-\cdot), \epsilon^3 \cdot)) - g'(W(\epsilon(n-\cdot), \epsilon^3 \cdot))\mathcal{W}_n \big) \\ &\quad - \epsilon^2 \big(g(W(\epsilon(n-1-\cdot), \epsilon^3 \cdot) + \mathcal{W}_{n-1}) - g(W(\epsilon(n-1-\cdot), \epsilon^3 \cdot)) \\ &\qquad - g'(W(\epsilon(n-1-\cdot), \epsilon^3 \cdot))\mathcal{W}_{n-1} \big) \\ &\quad + M_\epsilon(W(\epsilon(n-\cdot), \epsilon^3 \cdot) + \mathcal{W}_n) - M_\epsilon(W(\epsilon(n-1-\cdot), \epsilon^3 \cdot) + \mathcal{W}_{n-1}) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Res}_n^{(1)}(t) &:= \epsilon \partial_\xi W(\epsilon(n-t), \epsilon^3 t) - \epsilon^3 \partial_\tau W(\epsilon(n-t), \epsilon^3 t) \\ &\quad + P_\epsilon(\epsilon(n+1-t), \epsilon^3 t) - P_\epsilon(\epsilon(n-t), \epsilon^3 t), \\ \mathrm{Res}_n^{(2)}(t) &:= \epsilon \partial_\xi P_\epsilon(\epsilon(n-t), \epsilon^3 t) - \epsilon^3 \partial_\tau P_\epsilon(\epsilon(n-t), \epsilon^3 t) \\ &\quad + W(\epsilon(n-t), \epsilon^3 t) - W(\epsilon(n-1-t), \epsilon^3 t) \\ &\quad + \epsilon^2 g(W(\epsilon(n-t), \epsilon^3 t)) - \epsilon^2 g(W(\epsilon(n-1-t), \epsilon^3 t)). \end{aligned}$$

Lemma 5.2 below deals with estimating the nonlinear and residual terms of the system (5.5). Its proof relies on the following lemma, which is an improvement of Lemma 3.9 from [27].

LEMMA 5.1. *There exists $C > 0$ such that for all $X \in H^1(\mathbb{R})$ and $\epsilon \in (0, 1]$,*

$$\|x\|_{l^2} \leqslant C\epsilon^{-1/2}\|X\|_{H^1},$$

where $x_n := X(\epsilon n)$, $n \in \mathbb{Z}$.

Proof. We first prove the above inequality when X is in the Schwartz class. Denote $x_n := X(\epsilon n)$, and let $\hat{x} : \mathbb{R} \to \mathbb{C}$ be the 2π-periodic C^∞ function defined by

$$\hat{x}(\theta) := \sum_{n\in\mathbb{Z}} x_n e^{-in\theta},$$

so that

$$x_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{x}(\theta)\, e^{in\theta}\mathrm{d}\theta, \quad n \in \mathbb{Z}.$$

On the other hand, by the inverse Fourier transform applied to X, we have

$$\begin{aligned} x_n &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}(k)\, e^{ik\epsilon n}\mathrm{d}k \\ &= \frac{1}{2\pi\epsilon}\int_{-\infty}^{\infty} \hat{X}\left(\frac{p}{\epsilon}\right) e^{ipn}\mathrm{d}p \\ &= \frac{1}{2\pi\epsilon}\sum_{m\in\mathbb{Z}}\int_{(2m-1)\pi}^{(2m+1)\pi} \hat{X}\left(\frac{p}{\epsilon}\right) e^{ipn}\mathrm{d}p \\ &= \frac{1}{2\pi\epsilon}\sum_{m\in\mathbb{Z}}\int_{-\pi}^{\pi} \hat{X}\left(\frac{\theta+2\pi m}{\epsilon}\right) e^{in\theta}\mathrm{d}\theta. \end{aligned}$$

Due to the decay of $\hat{X}$, summation and integration can be interchanged. Then, the 2π-periodic C^∞ function $\theta \mapsto \frac{1}{\epsilon}\sum_{m\in\mathbb{Z}}\hat{X}(\frac{\theta+2\pi m}{\epsilon})$ has the same (inverse) Fourier coefficients as $\hat{x}$, so that they coincide:

$$\hat{x}(\theta) = \frac{1}{\epsilon}\sum_{m\in\mathbb{Z}} \hat{X}\left(\frac{\theta+2\pi m}{\epsilon}\right), \quad \theta \in \mathbb{R}.$$

Now, using Parseval's equality, we estimate the l^2 norm of x,

$$\begin{aligned} \|x\|_{l^2}^2 &= \frac{1}{2\pi\epsilon^2}\int_{-\pi}^{\pi}\left|\sum_{m\in\mathbb{Z}}\hat{X}\left(\frac{\theta+2\pi m}{\epsilon}\right)\right|^2\mathrm{d}\theta \\ &\leqslant \frac{1}{2\pi\epsilon^2}\int_{-\pi}^{\pi}\sum_{m_1,m_2\in\mathbb{Z}}\left|\hat{X}\left(\frac{\theta+2\pi m_1}{\epsilon}\right)\right|\left|\hat{X}\left(\frac{\theta+2\pi m_2}{\epsilon}\right)\right|\mathrm{d}\theta \\ &\leqslant \frac{1}{2\pi\epsilon^2}\sum_{m_1,m_2\in\mathbb{Z}}\int_{-\pi}^{\pi}\left|\hat{X}\left(\frac{\theta+2\pi m_1}{\epsilon}\right)\right|\left|\hat{X}\left(\frac{\theta+2\pi m_2}{\epsilon}\right)\right|\mathrm{d}\theta. \end{aligned}$$

Inserting the weights $(1+\pi^2m_1^2/\epsilon^2)^{-1}(1+\pi^2m_2^2/\epsilon^2)^{-1}$ and using the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
&\int_{-\pi}^{\pi} \left|\hat{X}\left(\frac{\theta+2\pi m_1}{\epsilon}\right)\right| \left|\hat{X}\left(\frac{\theta+2\pi m_2}{\epsilon}\right)\right| \mathrm{d}\theta \\
&\quad \leqslant \frac{1}{1+\pi^2 m_1^2/\epsilon^2}\frac{1}{1+\pi^2 m_2^2/\epsilon^2} \\
&\qquad \times \left(\frac{1}{2}\int_{-\pi}^{\pi}\left(1+\frac{\pi^2 m_1^2}{\epsilon^2}\right)^2 \left|\hat{X}\left(\frac{\theta+2\pi m_1}{\epsilon}\right)\right|^2 \mathrm{d}\theta \right. \\
&\qquad\quad \left. + \frac{1}{2}\int_{-\pi}^{\pi}\left(1+\frac{\pi^2 m_2^2}{\epsilon^2}\right)^2 \left|\hat{X}\left(\frac{\theta+2\pi m_2}{\epsilon}\right)\right|^2 \mathrm{d}\theta \right).
\end{aligned}$$

Summing with respect to m_1 and m_2, the two terms in the right-hand side above result in the same quantity, so that

$$\|x\|_{l^2}^2 \leqslant \frac{1}{2\pi\epsilon^2}\left(\sum_{m_1\in\mathbb{Z}}\frac{1}{1+\pi^2 m_1^2/\epsilon^2}\right)\left(\sum_{m_2\in\mathbb{Z}}\left(1+\frac{\pi^2 m_2^2}{\epsilon^2}\right)\int_{-\pi}^{\pi}\left|\hat{X}\left(\frac{\theta+2\pi m_2}{\epsilon}\right)\right|^2 \mathrm{d}\theta\right).$$

For $\epsilon \in (0,1]$, the first term in the product takes values between 1 and $\sum_{m\in\mathbb{Z}}(1+\pi^2 m^2)^{-1} < \infty$. The second term can be compared with the H^1 norm of X:

$$\begin{aligned}
\|X\|_{H^1}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(1+k^2)\left|\hat{X}(k)\right|^2 \mathrm{d}k \\
&= \frac{1}{2\pi\epsilon}\sum_{m\in\mathbb{Z}}\int_{-\pi}^{\pi}\left(1+\frac{(\theta+2\pi m)^2}{\epsilon^2}\right)\left|\hat{X}\left(\frac{\theta+2\pi m}{\epsilon}\right)\right|^2 \mathrm{d}\theta.
\end{aligned}$$

For any $m \in \mathbb{Z}$, $\theta \in [-\pi,\pi]$, we have $(\theta + 2\pi m)^2 \geqslant \pi^2 m^2$, so that the factor $(1 + (\theta + 2\pi m)^2/\epsilon^2)$ is bounded from below by $(1 + \pi^2 m^2/\epsilon^2)$. This gives the desired inequality with

$$C := \sum_{m\in\mathbb{Z}}\frac{1}{(1+\pi^2 m^2)^{1/2}}.$$

When X belongs to $H^1(\mathbb{R})$, we can consider a sequence $\{X^{(k)}\}_{k\in\mathbb{N}}$ of functions in the Schwartz class converging to X in H^1. For each $\epsilon \in (0,1]$ and $n \in \mathbb{Z}$, $X^{(k)}(\epsilon n)$ tends to $X(\epsilon n)$ as k tends to infinity, and Fatou's lemma concludes the proof. ∎

LEMMA 5.2. *Let $W \in C([-\tau_0,\tau_1], H^s(\mathbb{R}))$ be a solution to the log–KdV equation* (2.16)*, for an integer $s \geqslant 6$ and $\tau_0,\tau_1 \geqslant 0$. Assume that there exists $r_W > -1$ such that $W \geqslant r_W$. Then, there exists a positive constant C_W such that for all $t \in [-\tau_0\epsilon^{-3}, \tau_1\epsilon^{-3}]$ and $\epsilon \in (0,1]$,*

$$\|\mathrm{Res}^{(1)}(t)\|_{l^2} + \|\mathrm{Res}^{(2)}(t)\|_{l^2} \leqslant C_W \epsilon^{9/2}. \tag{5.6}$$

Furthermore, for $\epsilon_0 \in (0,1]$ and for all $\epsilon \in (0,\epsilon_0]$, let $\mathcal{W}^\epsilon \in C([-\tau_0\epsilon^{-3},\tau_1\epsilon^{-3}], l^2(\mathbb{Z}))$ be such that, for some $r > -1$ and $R > 0$ independent of ϵ,

$$-1 < r \leqslant W(\epsilon(n-t),\epsilon^3 t) + \mathcal{W}_n^\epsilon(t) \leqslant R < \infty, \quad n\in\mathbb{Z}, \ t \in [-\tau_0\epsilon^{-3},\tau_1\epsilon^{-3}]. \tag{5.7}$$

Then, there exists a positive constant $C_{r,R,W}$ such that, for all $\epsilon \in (0, \epsilon_0]$, we have

$$\|\mathcal{R}(W, \mathcal{W}^\epsilon)(t)\|_{l^2} \leqslant C_{r,R,W}(\epsilon^2\|\mathcal{W}^\epsilon(t)\|_{l^2}^2 + \epsilon^4\|\mathcal{W}^\epsilon(t)\|_{l^2} + \epsilon^{9/2}), \quad t \in [-\tau_0\epsilon^{-3}, \tau_1\epsilon^{-3}]. \tag{5.8}$$

In addition, the constant $C_{r,R,W}$ may be kept with the same value when ϵ_0 is decreased.

Proof. To obtain the part of estimate (5.6) concerning $\mathrm{Res}^{(1)}(t)$, we use the definition (5.4) of P_ϵ, the expressions of the $P^{(j)}$'s as linear combinations of derivatives of W and $g(W)$, and Taylor expansions. The coefficients of $\epsilon^0, \dots, \epsilon^4$ vanish, due to the fact that W is a solution to the log–KdV equation (2.16). As a consequence, $\mathrm{Res}^{(1)}(t)$ is then expressed as a sum of integrals of the form

$$\epsilon^5 \int_0^1 (1-r)^k \partial_\xi^5 W(\epsilon(n-t+r), \epsilon^3 t)\mathrm{d}r \quad \text{and} \quad \epsilon^5 \int_0^1 (1-r)^l \partial_\xi^2 g(W)(\epsilon(n-t+r), \epsilon^3 t)\mathrm{d}r$$

with $0 \leqslant k \leqslant 4$ and $0 \leqslant l \leqslant 1$. The associated l^2 norm is then easily estimated in terms of $\|W\|_{H^6}$, thanks to Lemma 5.1. The proof of the rest of estimate (5.6) concerning $\mathrm{Res}^{(2)}(t)$ follows the same lines.

To prove the bound (5.8), we recall that for all $r > -1$, there exists $C_r > 0$ such that for all $w_1, w_2 \geqslant r$ and $\epsilon > 0$,

$$|M_\epsilon(w_1) - M_\epsilon(w_2)| \leqslant \epsilon^4 C_r \left(1 + \max\{|w_1|, |w_2|\}\right)^{1+\epsilon^2} \max\{|w_1|, |w_2|\}|w_1 - w_2|. \tag{5.9}$$

Then, using again Taylor expansions, we get

$$\begin{aligned} &\|\mathcal{R}(W, \mathcal{W})(t)\|_{l^2} \\ &\quad\leqslant C\left(\epsilon^2\|g''(W(\epsilon(\cdot - t), \epsilon^3 t))\|_{L^\infty}\|\mathcal{W}\|_{l^2}^2 + \epsilon^4\|W(\epsilon(\cdot - t), \epsilon^3 t)\|_{L^\infty}\|\mathcal{W}\|_{l^2} \right.\\ &\qquad\left. + \epsilon^5\|(\partial_\xi W(\epsilon(\cdot - t), \epsilon^3 t))_{n\in\mathbb{Z}}\|_{l^2}\right), \end{aligned}$$

which yields the bound (5.8). □

Thanks to Lemma 5.2, we complete the proof of Theorem 3 using energy estimates. When $\epsilon_0 > 0$ is given, we consider for each $\epsilon \in (0, \epsilon_0)$ initial data $(w_{\mathrm{ini},\epsilon}, p_{\mathrm{ini},\epsilon})$ satisfying the bound (2.18). Fixing

$$r := \frac{r_W - 1}{2} \in (-1, r_W) \quad \text{and} \quad R := 2R_W > R_W,$$

with ϵ_0 small enough, we can define for each $\epsilon \in (0, \epsilon_0)$ a local-in-time solution (w, p) to the FPU lattice equations (2.10), decomposed according to (5.2), and then set

$$T_0^\star(\epsilon) := \sup\left\{T_0 \in (0, \tau_0\epsilon^{-3}] : \quad r \leqslant W(\epsilon(n-t), \epsilon^3 t) + \mathcal{W}_n(t) \leqslant R, \quad n \in \mathbb{Z},\ t \in [-T_0, 0]\right\}$$

and

$$T_1^\star(\epsilon) := \sup\left\{T_1 \in (0, \tau_1\epsilon^{-3}] : \quad r \leqslant W(\epsilon(n-t), \epsilon^3 t) + \mathcal{W}_n(t) \leqslant R, \quad n \in \mathbb{Z},\ t \in [0, T_1]\right\}.$$

We shall prove that for ϵ_0 small enough, we have $T_0^\star(\epsilon) = \tau_0\epsilon^{-3}$ and $T_1^\star(\epsilon) = \tau_1\epsilon^{-3}$.

Let us define the energy-type quantity

$$\mathcal{E}(t) := \frac{1}{2}\sum_{n\in\mathbb{Z}}\left[\mathcal{P}_n^2(t) + \mathcal{W}_n^2(t) + \epsilon^2 g'(W(\epsilon(n-t),\epsilon^3 t))\mathcal{W}_n^2(t)\right]. \tag{5.10}$$

With $\epsilon_0 < \min(1, \|2g'\|^{-1/2}_{L^\infty(r_W,R_W)})$, from the bounds (5.1), we get, for $\epsilon \in (0,\epsilon_0)$,

$$\|\mathcal{P}(t)\|^2_{l^2} + \|\mathcal{W}(t)\|^2_{l^2} \leqslant 4\mathcal{E}(t), \quad t \in (-T_0^\star, T_1^\star).$$

Taking the derivative of $\mathcal{E}$ with respect to time t, we obtain

$$\begin{aligned}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}(t) = \sum_{n\in\mathbb{Z}}\Big[&\mathcal{P}_n(t)\mathcal{R}_n(W,\mathcal{W})(t) + \mathcal{P}_n(t)\mathrm{Res}_n^{(2)}(t)\\ &+ \mathcal{W}_n(t)[1+\epsilon^2 g'(W(\epsilon(n-t),\epsilon^3 t))]\mathrm{Res}_n^{(1)}(t)\\ &+ \frac{\epsilon^2}{2}g''(W(\epsilon(n-t),\epsilon^3 t))\mathcal{W}_n^2(t)(-\epsilon\partial_\xi + \epsilon^3\partial_\tau)W(\epsilon(n-t),\epsilon^3 t)\Big].\end{aligned}$$

Then, using Lemma 5.2 and the Cauchy–Schwarz inequality, we estimate

$$\begin{aligned}\left|\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}\right| &\leqslant \|\mathcal{P}\|_{l^2}\|\mathcal{R}(W,\mathcal{W})\|_{l^2} + \|\mathcal{P}\|_{l^2}\|\mathrm{Res}^{(2)}\|_{l^2} + \frac{3}{2}\|\mathcal{W}\|_{l^2}\left\|\mathrm{Res}^{(1)}\right\|_{l^2} + \epsilon^3 C_W\|\mathcal{W}(t)\|^2_{l^2}\\ &\leqslant C_W\mathcal{E}^{1/2}\left(\epsilon^{9/2} + \epsilon^3\mathcal{E}^{1/2} + \epsilon^2\mathcal{E}\right)\end{aligned}$$

with a new constant C_W. Choosing $\mathcal{Q} = \mathcal{E}^{1/2}$, we rewrite the energy balance equation in the form

$$\left|\frac{\mathrm{d}\mathcal{Q}}{\mathrm{d}t}\right| \leqslant C_W\left(\epsilon^{9/2} + \epsilon^3\mathcal{Q} + \epsilon^2\mathcal{Q}^2\right).$$

By Gronwall's inequality, we obtain

$$\mathcal{Q}(t) \leqslant (\mathcal{Q}(0) + C_W\epsilon^{9/2}|t|)\,e^{\epsilon^3 C_W|t|}, \quad t \in (-T_0^\star, T_1^\star).$$

Now, the bound (2.18) ensures that $\|(\mathcal{W},\mathcal{P})\|_{L^2}$ is $\mathcal{O}(\epsilon^{3/2})$ at $t=0$, so that from the definition (5.10) of $\mathcal{E}$, $\mathcal{Q}(0)$ is also $\mathcal{O}(\epsilon^{3/2})$, for ϵ_0 small enough. Thus, we get

$$\mathcal{Q}(t) \leqslant C_W(1+\max(\tau_0,\tau_1))\,\epsilon^{3/2}e^{C_W\max(\tau_0,\tau_1)}, \quad t \in (-T_0^\star, T_1^\star). \tag{5.11}$$

Finally, choosing ϵ_0 so that the right-hand side in (5.11) is so small that

$$-\frac{1+r_W}{2} \leqslant \mathcal{W}_n(t) \leqslant R_W$$

shows that for all $\epsilon \in (0,\epsilon_0)$, $T_0^\star(\epsilon) = \tau_0\epsilon^{-3}$ and $T_1^\star(\epsilon) = \tau_1\epsilon^{-3}$. Theorem 3 is proved.

Remark 5.1. Using instead of (5.2) an asymptotic expansion

$$\begin{aligned}w_n(t) &= W(\epsilon(n-t),\epsilon^3 t) + \sum_{k=1}^{K}\epsilon^k W^{(k)}(\epsilon(n-t),\epsilon^3 t) + \mathcal{W}_n(t),\\ p_n(t) &= \sum_{k=0}^{K+3}\epsilon^k P^{(k)}(\epsilon(n-t),\epsilon^3 t) + \mathcal{P}_n(t),\end{aligned}$$

at any order $K \in \mathbb{N}$, together with expansion of $\tilde{V}_\epsilon'(w)$ in powers of ϵ^2, we could improve the approximation (2.19), replacing $C_0\epsilon^{3/2}$ by $C_K\epsilon^{K+3/2}$, for any $K \in \mathbb{N}$. The approximation time remains $\mathcal{O}(\epsilon^{-3})$ in such an improved approximation. Here the correction terms $\{W^{(k)}\}_{k=1}^{K}$ satisfy a sequence of linearized log–KdV equations with source terms, whereas the correction terms $\{P^{(k)}\}_{k=0}^{K+3}$ are found from the expansion of (5.3) in powers of ϵ.

6. Discussion. The comparison of the two results given by Theorems 2 and 3 raises a serious concern on the validity of the KdV-type approximation for the stability theory of the traveling waves in the FPU lattices. On one hand, Theorem 2 yields nonlinear stability of the FPU traveling waves up to the time scale of $\mathcal{O}(\epsilon^{-3})$ at which the traveling waves are proved to satisfy the specific scaling leading to the KdV-type approximation. On the other hand, Theorem 3 shows that the nonlinear stability of the FPU traveling waves may depend on the orbital stability of the traveling waves in the KdV-type equations. It happens for the log–KdV equation (2.16) that the positive traveling waves are orbitally stable for all amplitudes [15]. However, it does not have to be the case for all KdV-type equations.

For instance, if we consider the FPU lattice (2.10) with the nonlinear potential

$$\tilde{V}_\epsilon(w) = \frac{1}{2}w^2 + \frac{\epsilon^2}{p+1}w^{p+1} \quad \text{for an integer } \; p \geqslant 2,$$

the results of Theorems 2 and 3 hold true but the generalized KdV equation takes the form

$$2W_\tau + \frac{1}{12}W_{\xi\xi\xi} + (W^p)_\xi = 0. \tag{6.1}$$

The generalized KdV equation (6.1) is known to have orbitally stable traveling waves for $p = 2, 3, 4$ and orbitally unstable traveling waves for $p \geqslant 5$ [1]. Thus, it may first appear that the results of Theorems 2 and 3 are in contradiction.

No contradiction arises as a matter of fact. The energy methods used in the proof of Theorems 2 and 3 give the upper bounds on the approximation errors (2.15) and (2.19) to be exponentially growing at the time scale of $\epsilon^3 t$, that is, on the same time scale of τ. The unstable eigenvalues of the linearized generalized KdV equation (6.1) at the traveling waves (if they exist) lead to the exponential divergence at the time scale of τ, which cannot be detected with the approximation results provided by Theorems 2 and 3.

Therefore, within the approximation results of Theorems 2 and 3, we are still left wondering if the traveling waves of the FPU lattice with the nonlinear potential $\tilde{V}_\varepsilon$ for $\epsilon > 0$ small enough are nonlinearly stable at the time scale of $\tau = \epsilon^3 t$. What the stability result of Theorem 2 rules out is the presence of the unstable eigenvalues of the size $\mathcal{O}(\epsilon^q)$ for any $q < 3$ in the linearized operator associated with the FPU lattice as in [9]. However, unstable eigenvalues of the size $\mathcal{O}(\epsilon^q)$ for $q \geqslant 3$ are still possible.

Note that the result of Theorem 2 does not depend on the nonlinear potential $\tilde{V}_\varepsilon$ as long as the latter provides the specific scaling leading to the KdV-type approximation. We did not have to construct the two-dimensional manifold of the traveling waves or use projections and modulation equations from the theory in [8, 9, 10]. Although the latter theory gives a complete proof of nonlinear orbital stability of FPU traveling waves of small amplitudes, it relies on the information about the spectral and asymptotic stabilities of the KdV traveling waves, which is only available in the case of the integrable KdV equation (6.1) with $p = 2$ (such information may also be available in the case $p = 3$, since the corresponding so-called "modified KdV" equation is integrable as well). It is not clear at the present time if any bits of the information needed to proceed with the theory in [8, 9, 10] can be obtained for the log–KdV equation (2.16), although the existing theory in [15] excludes unstable eigenvalues and guarantees nonlinear orbital stability of the traveling waves in the log–KdV equation.

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