

Vibrations and Oscillatory Instabilities of Gap Solitons

I. V. Barashenkov,* D. E. Pelinovsky,† and E. V. Zemlyanaya‡

Department of Applied Mathematics, University of Cape Town, Rondebosch 7701, South Africa

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Stability of optical gap solitons is analyzed within a coupled-mode theory. Lower intensity solitons are shown to always possess a vibration mode responsible for their long-lived oscillations. As the intensity of the soliton is increased, the vibration mode falls into resonance with two branches of the long-wavelength radiation producing a cascade of oscillatory instabilities of higher intensity solitons. [S0031-9007(98)06265-6]

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In the late 1970s and early 1980s, the theory of elementary particles [1] and condensed matter physics (in particular, the Su-Schrieffer-Heeger polyacetylene model [2]) stimulated a wide interest in particlelike solutions of classical spinor field equations. Recently there has been a remarkable upsurge of the interest; the localized solutions of spinorlike systems have made a comeback under the new name of *gap solitons*.

Thanks to the gap in the linear spectrum, solitons in spinorlike systems can propagate without losing their energy to resonantly excited radiation waves [3,4]. An example of the gap-soliton bearing system is given by an optical fiber with periodically varying refractive index [3]; here the gap is produced by the Bragg reflection and resonance of the waves along the grating. Another class of gap solitons arises in two-wave resonant optical materials with a $\chi^{(2)}$ susceptibility and diatomic crystal lattices (see [4] and references therein). Finally, in the already mentioned polyacetylene model [2], the gap in the electron spectrum is due to the electron-phonon interaction and effective period doubling of the lattice.

The aim of this Letter is to analyze the *stability* of gap solitons. Previous analytical studies of the spinor soliton stability faced serious obstacles (cf. [5]), while results of computer simulations were contradictory (cf. [6,7]). As a result, no stability or instability criterion is available to date. The main difficulty of the previous analyses was that they were all based on a postulate that stable solutions must render the energy minimum. In the actual fact, however, the minimality of energy is not necessary for stability in systems with indefinite metrics [8]. As far as *optical* gap solitons are concerned, they have been commonly deemed stable following recent computer simulations carried out for certain particular parameter values [9]. In this Letter we demonstrate that the gap solitons *can* be unstable, elucidate the mechanism of instability, and demarcate the stability/instability regions on the plane of their parameters.

In nonlinear optics the gap solitons are usually analyzed within the coupled-mode theory [3] which reduces to a system of coupled equations for the amplitudes of the forward- and backward-propagating waves,

$$\begin{aligned} i(u_t + u_x) + v + (|v|^2 + \rho|u|^2)u &= 0, \\ i(v_t - v_x) + u + (|u|^2 + \rho|v|^2)v &= 0. \end{aligned} \quad (1)$$

In the periodic Kerr medium one typically has $\rho = 1/2$ [3]; in other problems of the fiber optics ρ may range up to infinity [10]. In the case $\rho = 0$ Eqs. (1) yield the massive Thirring model of the field theory. In this case Eqs. (1) are invariant with respect to the Lorentz transformations $X = (x - Vt)/\sqrt{1 - V^2}$, $T = (t - Vx)/\sqrt{1 - V^2}$, with u and v transforming as components of the Lorentz spinor [see Eq. (2) below]. Although in the general case ($\rho \neq 0$) the Lorentz symmetry is broken, its artifact is that the soliton solution is still written in terms of the boosted variables X and T [9],

$$\begin{aligned} u &= \alpha W(X) e^{y/2 + i\varphi(X) - i \cos \theta T}, \\ v &= -\alpha W^*(X) e^{-y/2 + i\varphi(X) - i \cos \theta T}, \end{aligned} \quad (2)$$

where $\alpha^{-2} = 1 + \rho \cosh(2y)$,

$$\begin{aligned} \varphi(X) &= 2\alpha^2 \rho \sinh(2y) \arctan \left\{ \tanh[(\sin \theta)X] \tan \frac{\theta}{2} \right\}, \\ W(X) &= \frac{\sin \theta}{\cosh[(\sin \theta)X - i\theta/2]}. \end{aligned}$$

Here the rapidity y parametrizes the soliton's velocity: $V = \tanh y$, and θ determines its detuning frequency within the spectrum gap, $\Omega = \cos \theta$ ($0 < \theta < \pi$). At the upper edge of the gap (i.e., as $\theta \rightarrow 0$) and assuming $|V| \ll 1$, Eq. (2) approaches the small-amplitude nonlinear Schrödinger soliton [3]: $W(X) \rightarrow \theta \operatorname{sech}[\theta(X - i/2)]$. At the lower edge, i.e., in the limit $\theta \rightarrow \pi$, the gap soliton has a finite amplitude and decays as a power law: $W(X) = i/(X + i/2)$. These two limits are referred to as the "low intensity" and "high intensity" limits [3].

Linearizing Eq. (1) about the stationary soliton (2) and choosing the perturbation as

$$\begin{aligned} u &= [\alpha W(X) + z_1(X) e^{i\lambda T}] e^{y/2 + i\varphi(X) - i\Omega T}, \\ v &= [-\alpha W^*(X) + z_2(X) e^{i\lambda T}] e^{-y/2 + i\varphi(X) - i\Omega T}, \\ u^* &= [\alpha W^*(X) + z_3(X) e^{i\lambda T}] e^{y/2 - i\varphi(X) + i\Omega T}, \\ v^* &= [-\alpha W(X) + z_4(X) e^{i\lambda T}] e^{-y/2 - i\varphi(X) + i\Omega T} \end{aligned}$$

gives an eigenvalue problem

$$\hat{\mathcal{H}} \mathbf{z} = \lambda J \mathbf{z}, \quad J = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad (3)$$

where $\mathbf{z} = (z_1, z_2, z_3, z_4)^T$ and the Hermitian operator $\hat{\mathcal{H}}$ is defined by

$$\hat{H} = i \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \frac{d}{dX} + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} + (|W|^2 + \cos \theta) J^2 + \alpha^2 \begin{pmatrix} \rho e^{2y} |W|^2 & -W^2 & \rho e^{2y} W^2 & -|W|^2 \\ -W^{*2} & \rho e^{-2y} |W|^2 & -|W|^2 & \rho e^{-2y} W^{*2} \\ \rho e^{2y} W^{*2} & -|W|^2 & \rho e^{2y} |W|^2 & -W^{*2} \\ -|W|^2 & \rho e^{-2y} W^2 & -W^2 & \rho e^{-2y} |W|^2 \end{pmatrix}.$$

(Here σ_0 , σ_1 , and σ_3 are the Pauli matrices.) Equation (3) has four zero eigenvalues arising from symmetries of Eqs. (1) and four branches of the continuous spectrum pertaining to real λ . The associated eigenfunctions can be specified by their asymptotic behavior as $X \rightarrow -\infty$; in particular, the eigenfunctions associated with $\lambda_1(k) = \sqrt{1 + k^2} - \cos \theta$ and $\lambda_2(k) = \sqrt{1 + k^2} + \cos \theta$ satisfy

$$\mathbf{Z}^{(1)}(X, k) \rightarrow (0, 0, 1, -r)^T e^{ikX}, \quad (4)$$

$$\mathbf{Z}^{(2)}(X, k) \rightarrow (1, r, 0, 0)^T e^{ikX}, \quad (5)$$

respectively. Here $r = r(k) = \sqrt{1 + k^2} + k$. The continuous spectrum solutions describe radiations propagating on the solitonic background.

In the Thirring case ($\rho = 0$) the set of the neutral and continuum eigenfunctions is complete [11] so that any additional eigenvalues are absent. However, as ρ deviates from zero, new eigenvalues can detach from the edges of the continuous spectrum. To see whether this is indeed the case, we expand solutions to (3) over the complete set of the Thirring eigenfunctions,

$$\mathbf{z}(X) = \sum_{i=1}^2 \int_{-\infty}^{\infty} \frac{a_i(k) \mathbf{Z}^{(i)}(X, k)}{\lambda - \lambda_i(k)} dk + (\dots), \quad (6)$$

where (\dots) stands for terms which remain bounded as λ approaches $\lambda_1(0)$ and $\lambda_2(0)$. Using the orthogonality relations between the Thirring eigenfunctions [11], Eq. (3) can be reduced to a system of two integral equations

$$a_i(k) = \frac{(-1)^i \rho}{4\pi \sqrt{1 + k^2} r(k)} \sum_{j=1}^2 \int_{-\infty}^{\infty} \frac{K_{ij}(k, k') a_j(k')}{\lambda - \lambda_j(k')} dk' \quad (7)$$

($i = 1, 2$), with the kernel

$$K_{ij} = \cosh(2y) \int_{-\infty}^{+\infty} (p_i^* + q_i^*) (p_j + q_j) dX + \sinh(2y) \int_{-\infty}^{+\infty} (p_i^* p_j - q_i^* q_j) dX. \quad (8)$$

Here $p_m = W^* Z_1^{(m)} + W Z_3^{(m)}$, $q_m = W Z_2^{(m)} + W^* Z_4^{(m)}$ ($m = 1, 2$), and we have denoted $Z_n^{(i)*} = Z_n^{(i)*}(X, k)$ and $Z_n^{(j)} = Z_n^{(j)}(X, k')$. The edges of the continuum branches, $\lambda_1(0)$ and $\lambda_2(0)$, are well separated unless $\theta \approx \pi/2$. Consequently, if θ is not very close to $\pi/2$ we can get away with a single-mode approximation and disregard the nonresonant branch. First, let $\theta \ll \pi/2$ and assume that a new eigenvalue detaches from the edge of the (inner) branch λ_1 : $\lambda = \lambda_1(0) - \frac{1}{2} \kappa^2$. Sending $\rho \rightarrow 0$ and dis-

regarding the branch λ_2 , we obtain $|\kappa| = \frac{1}{2} \rho K_{11}(0, 0)$. The Thirring eigenfunctions pertaining to $k = 0$ satisfy $Z_1^{(1)}(X, 0) = -Z_2^{(1)*}(X, 0) \equiv z_1^{(1)}(X)$ and $Z_3^{(1)}(X, 0) = -Z_4^{(1)*}(X, 0) \equiv z_2^{(1)}(X)$; this follows from (3)–(5). Using these symmetries we finally arrive at

$$|\kappa| = -\frac{\rho}{2} \cosh(2y) \int_{-\infty}^{+\infty} [W(z_1^{(1)*} - z_2^{(1)}) - \text{c.c.}]^2 dX. \quad (9)$$

Since the right-hand side in Eq. (9) is positive, we conclude that a small deviation from the integrable case $\rho = 0$ does indeed bring about a new real eigenvalue $\lambda < \lambda_1(0)$. This additional eigenvalue represents a vibration mode of the gap soliton with $\theta \ll \pi/2$.

Next, let $\pi - \theta \ll \pi/2$ and assume that an eigenvalue $\lambda = \lambda_2(0) - \frac{1}{2} \kappa^2$ detaches from the branch λ_2 (which is now the inner branch). The same asymptotic procedure as above produces $|\kappa| = -\frac{1}{2} \rho K_{22}(0, 0)$. Making use of the symmetry relations $Z_1^{(2)}(X, 0) = Z_2^{(2)*}(X, 0) \equiv z_1^{(2)}(X)$ and $Z_3^{(2)}(X, 0) = Z_4^{(2)*}(X, 0) \equiv z_2^{(2)}(X)$, this becomes

$$|\kappa| = -\frac{\rho}{2} \cosh(2y) \int_{-\infty}^{+\infty} [W(z_1^{(2)*} + z_2^{(2)}) + \text{c.c.}]^2 dX. \quad (10)$$

Since the right-hand side is negative, we have arrived at a contradiction. Thus the birth of a vibration mode cannot occur for $\pi - \theta \ll \pi/2$.

Finally, the case $\theta \approx \pi/2$ has to be analyzed within the full two-mode system (7); in this case *both* continuous branches are resonant. We let $\theta = \pi/2 + \epsilon$ and look for a new eigenvalue as $\lambda = \min\{\lambda_1(0), \lambda_2(0)\} - \frac{1}{2} \kappa^2$. Assuming $\text{Re } \kappa > 0$ and $\rho \rightarrow 0$, the system (7) can be reduced to an algebraic equation for κ ,

$$\sqrt{\kappa^2 + 4\epsilon} (4\kappa + 2\rho K_{22}) - 2\rho K_{11} \kappa - \rho^2 D = 0, \quad (11)$$

where $D = K_{11} K_{22} - K_{12} K_{21}$ and $K_{ij} = K_{ij}(0, 0)$. [Here we have assumed $\epsilon \geq 0$; for $\epsilon < 0$ one should simply transpose κ and $\sqrt{\kappa^2 + 4|\epsilon|}$ in Eq. (11).] To find the coefficients in (11), we first derive the edge eigenfunctions $z_n^{(i)}(X)$ for $\theta = \pi/2$,

$$z_1^{(1)}(X) = -i \text{sech}(2X) \tanh(X - i\pi/4);$$

$$z_2^{(1)}(X) = \tanh(2X) \tanh(X + i\pi/4);$$

$z_1^{(2)}(X) = (z_2^{(1)})^*$; $z_2^{(2)}(X) = -(z_1^{(1)})^*$. Equation (8) yields

$$K_{nn}(0, 0) = 2[\pi + (-1)^n 2] \cosh(2y), \quad n = 1, 2; \\ K_{12}(0, 0) = K_{21}(0, 0) = -4 \sinh(2y), \quad (12)$$

and the analysis of the roots of Eq. (11) becomes straightforward. When $\epsilon < \epsilon_{cr} = [\rho D/4K_{22}]^2$, there are 4 real roots, $\kappa_1 < \kappa_4 < \kappa_2 < 0 < \kappa_3$. The positive root κ_3 corresponds to the above-mentioned vibration mode that continues from $\theta = 0$ (see Fig. 1). The negative root κ_2 becomes positive for ϵ between ϵ_{cr} and some ϵ_{osc} where κ_2 merges with κ_3 . That is, in this narrow region the gap soliton has two vibration modes. At $\epsilon = \epsilon_{osc}$ the two modes resonate, κ and λ become complex, and the oscillatory instability sets in (curve 1 in Fig. 1).

The numerical analysis of the eigenvalue problem (3) shows that the above bifurcation pattern persists for finite ρ . In Fig. 2(a) we have demarcated the boundary of the stability domain in the (Ω, V) plane for $\rho = 1/2$. The asymptotic approximation for the oscillatory bifurcation curve, $\Omega = \cos[\epsilon_{osc}(V) + \pi/2]$, is also shown for comparison (dashed curve). Figure 2(b) is a similar bifurcation and stability chart for $\rho = \infty$.

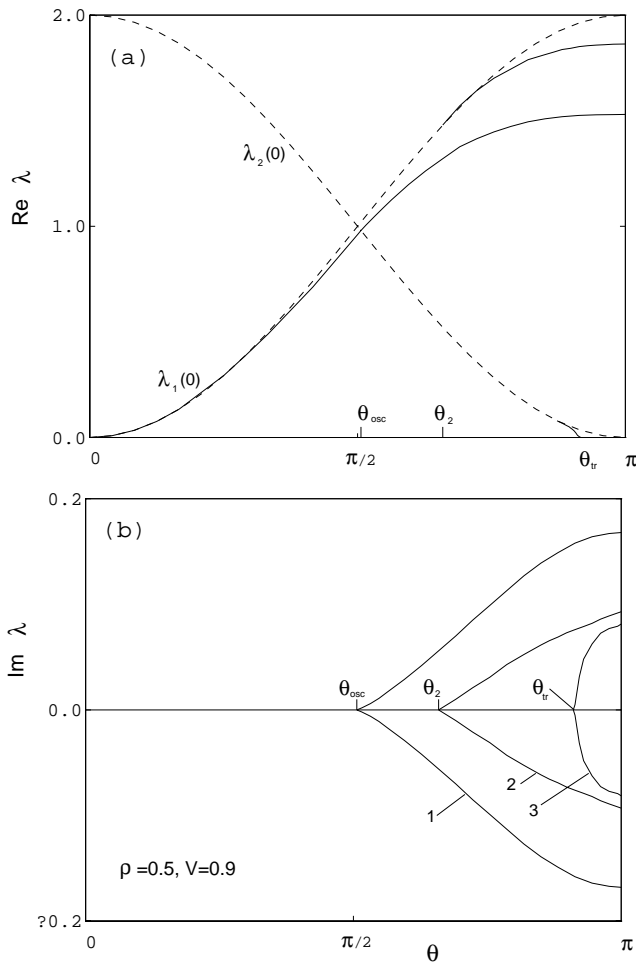


FIG. 1. Numerically found eigenvalues. Dashed lines indicate the edges of the continuous spectrum. A real eigenvalue detaches from λ_1 at $\theta = 0$, and another real eigenvalue detaches from λ_2 at $\theta = \theta_{cr} > \pi/2$ (not clearly visible). At $\theta = \theta_{osc}$ the two collide and the oscillatory instability sets in (curve 1). Another complex doublet (curve 2) emerges from λ_1 at $\theta = \theta_2$. Finally, one more real eigenvalue detaches from λ_2 and moves on the imaginary axis at θ_{tr} (curve 3).

As we increase θ further on, another pair of complex eigenvalues detaches from the edge of the continuous spectrum (curve 2 in Fig. 1). In contrast to the first bifurcation, this happens near the edge of the outer branch of the continuous spectrum. There is no intermediate region with two real discrete eigenvalues—the detaching eigenvalues are complex at once. The growth rate of this secondary instability is smaller than that of the primary one.

Here it is important to emphasize that the oscillatory instabilities cannot be detected within the variational approach. One notices that (2) is a stationary point of the functional $\mathcal{L} = H - VP - \omega N$, where the conserved Hamiltonian, momentum, and energy are given by

$$H = \frac{1}{2} \int_{-\infty}^{\infty} \left[i(uu_x^* + v^*v_x) - 2vu^* - |uv|^2 - \frac{\rho}{2} (|u|^4 + |v|^4) + c.c. \right] dx, \quad (13)$$

$$P = \frac{i}{2} \int_{-\infty}^{\infty} (uu_x^* + vv_x^* - c.c.) dx, \quad (14)$$

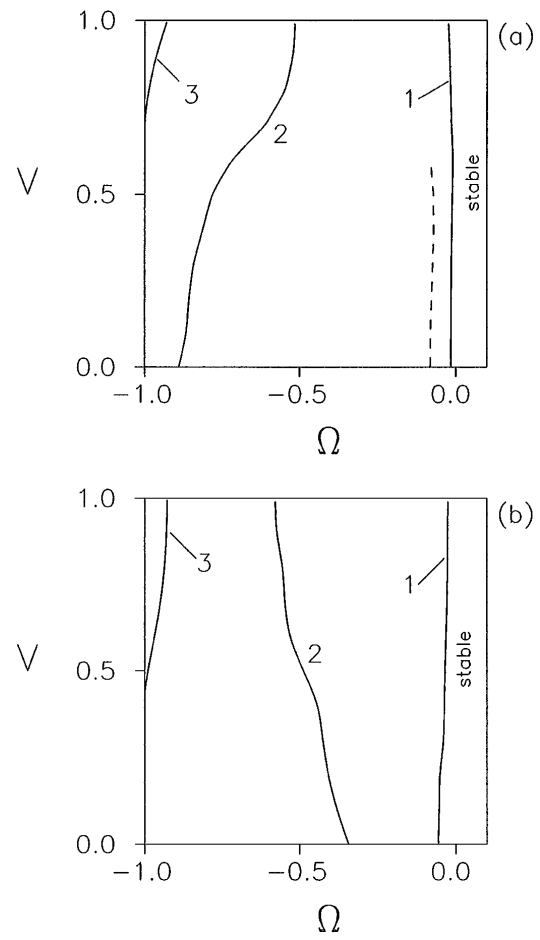


FIG. 2. Stability charts for (a) $\rho = 1/2$ and (b) $\rho = \infty$. Curves 1 and 2 are the lines of the primary and secondary oscillatory bifurcations, and the line 3 demarcates the onset of the translational instability.

and $N = \int_{-\infty}^{+\infty} (|u|^2 + |v|^2) dx$, respectively, and $\omega = \Omega\sqrt{1 - V^2}$. The idea of the variational (or energetic) approach to stability would be to prove that the soliton minimizes H for the fixed P and N or, equivalently, that $\delta^2\mathcal{L}$ is positive definite. However, writing $\delta^2\mathcal{L} = (\mathbf{z}, \hat{\mathcal{H}}\mathbf{z})$ with $\hat{\mathcal{H}}$ as in (3), and noting that the continuous spectrum of the eigenvalue problem $\hat{\mathcal{H}}\mathbf{z} = E\mathbf{z}$ extends from minus to plus infinity, one immediately concludes that the form $\delta^2\mathcal{L}$ is bounded neither from above nor from below.

The integrals of motion are not entirely useless though. They allow one to detect *translational* bifurcations where a real eigenvalue λ approaches zero and then passes on to the imaginary axis. According to the multiscale expansion method [12,13], the zero crossing occurs when

$$D(\omega, V) = \frac{\partial N_s}{\partial \omega} \frac{\partial P_s}{\partial V} - \frac{\partial N_s}{\partial V} \frac{\partial P_s}{\partial \omega} = 0, \quad (15)$$

where P_s, N_s are the invariants P, N computed on the soliton (2): $P_s = 4\alpha^4 V(1 - V^2)^{-1/2} \tilde{P}(V, \Omega)$, $N_s = 4\alpha^2 \tilde{N}(\Omega)$,

$$\tilde{P} = \left(1 + \rho \frac{5 + V^2}{1 - V^2}\right) \sqrt{1 - \Omega^2} - \frac{4\rho\Omega\tilde{N}(\Omega)}{1 - V^2},$$

$$\tilde{N} \equiv \frac{\pi}{2} - \arcsin \Omega.$$

The dependence $\Omega = \Omega_{\text{tr}}(V)$ defined by Eq. (15) can be found explicitly; we have plotted it in Fig. 2 (solid line 3). Consistently with conclusions of the multiscale analysis, the numerical study of Eq. (3) reveals that one more real eigenvalue detaches from the inner branch of continuous spectrum, reaches zero exactly at $\theta = \theta_{\text{tr}}$, and moves on to the imaginary axis (curve 3 in Fig. 1).

Finally, we make contact with two results available in literature. First, the stable gap solitons of Ref. [9] were observed in simulations with $\rho = 1/2$ and $\theta = \pi/2$; these values fall into the stability domain of Fig. 2(a). The observed soliton oscillations [9,14] are due to the excitation of the vibration mode. Second, stationary solutions *on finite intervals* are known to be unstable for high incoming intensities and exhibit self-pulsations and switchings from high- to low-transmissive stationary states [15,16]. The gap solitons correspond to the asymptotic limit of the stationary solutions in which the energy flow through the system vanishes [3]. Therefore, our present discussion should correspond to the $\gamma \rightarrow 0, L \rightarrow \infty$ limit of the stability analysis of Ref. [16]. However, our instability results cannot be deduced from the previous analysis as it was confined to the region $V = 0, \Omega > 0$ where the gap soliton is stable.

In conclusion, we have demonstrated that for any $\rho > 0$ the soliton solution of Eq. (1) becomes unstable as Ω is decreased beyond a (negative) critical value. The instability is caused by the resonance between the soliton's vibration mode and two branches of the long-wavelength radiation. This "triple-resonance" mechanism

is different from previously encountered mechanisms of oscillatory instability (cf. [17]).

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*Electronic address: igor@uctvms.uct.ac.za

†Current address: Department of Mathematics, University of Toronto, 100 St. George Street, Toronto, Ontario, Canada M5S 3G3.

Electronic address: dmpeli@math.toronto.edu

‡Permanent address: Laboratory for Computing Techniques and Automation, Joint Institute for Nuclear Research, Dubna, 141980 Russia.

Electronic address: zemel@cv.jinr.ru

- [1] See, e.g., S. Y. Lee, T. K. Kuo, and A. Gavrielides, *Phys. Rev. D* **12**, 2249 (1975); R. Friedberg and T. D. Lee, *Phys. Rev. D* **15**, 1694 (1977).
- [2] D. K. Campbell and A. R. Bishop, *Nucl. Phys.* **B200**, 297 (1982).
- [3] C. M. de Sterke and J. E. Sipe, in *Progress in Optics*, edited by E. Wolf (Elsevier, Amsterdam, 1994), Vol. XXXIII.
- [4] Yu. S. Kivshar, O. A. Chubykalo, O. V. Usatenko, and D. V. Grinyoff, *Int. J. Mod. Phys.* **9**, 2963 (1995).
- [5] J. Werle, *Phys. Lett.* **95B**, 391 (1980); *Acta Phys. Pol. B* **12**, 601 (1981); P. Mathieu and T. F. Morris, *Phys. Lett.* **126B**, 74 (1983); **155B**, 156 (1985); W. A. Strauss and L. Vazquez, *Phys. Rev. D* **34**, 641 (1986); P. Blanchard, J. Stubbe, and L. Vazquez, *Phys. Rev. D* **36**, 2422 (1987).
- [6] I. L. Bogolubsky, *Phys. Lett.* **73A**, 87 (1979).
- [7] A. Alvarez and B. Carreras, *Phys. Lett.* **85A**, 327 (1981); A. Alvarez and M. Soler, *Phys. Rev. Lett.* **50**, 1230 (1983); *Phys. Rev. D* **34**, 644 (1986).
- [8] I. V. Barashenkov, *Acta Phys. Austriaca* **55**, 155 (1983).
- [9] A. B. Aceves and S. Wabnitz, *Phys. Lett. A* **141**, 37 (1989).
- [10] M. Romangoli, S. Trillo, and S. Wabnitz, *Opt. Quantum Electron.* **24**, S1237 (1992).
- [11] D. J. Kaup and T. I. Lakoba, *J. Math. Phys. (N.Y.)* **37**, 308 (1996); **37**, 3442 (1996).
- [12] D. E. Pelinovsky and R. H. J. Grimshaw, in *Nonlinear Instability Analysis*, edited by L. Debnath and S. Choudhury (Computational Mechanics Publications, Southampton, 1997), Chap. 8.
- [13] D. E. Pelinovsky, A. V. Buryak, and Yu. S. Kivshar, *Phys. Rev. Lett.* **75**, 591 (1995); I. V. Barashenkov, *Phys. Rev. Lett.* **77**, 1193 (1996); A. V. Buryak, Yu. S. Kivshar, and S. Trillo, *Phys. Rev. Lett.* **77**, 5210 (1996).
- [14] B. A. Malomed and R. S. Tasgal, *Phys. Rev. E* **49**, 5787 (1994).
- [15] H. G. Winful and G. D. Cooperman, *Appl. Phys. Lett.* **40**, 298 (1982); C. M. de Sterke and J. E. Sipe, *Phys. Rev. A* **42**, 2858 (1990); H. G. Winful, R. Zamir, and S. Feldman, *Appl. Phys. Lett.* **58**, 1001 (1991).
- [16] C. M. de Sterke, *Phys. Rev. A* **45**, 8252 (1992).
- [17] I. V. Barashenkov, M. M. Bogdan, and V. I. Korobov, *Europhys. Lett.* **15**, 113 (1991); R. L. Pego, P. Smereka, and M. I. Weinstein, *Nonlinearity* **8**, 921 (1995).