Generation of soliton oscillations in nonlinear quadratic materials

D. E. Pelinovsky,[1] J. E. Sipe,[2] and J. Yang[3]
[1]*Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 3G3*
[2]*Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7*
[3]*Department of Mathematics and Statistics, University of Vermont, Burlington, Vermont 05401*

(Received 13 January 1999)

We show analytically and numerically that the generation of long-lasting soliton oscillations in resonant $\chi^{(2)}$ optical materials possesses a threshold for the amplitude of a fundamental wave. The persistent oscillations of solitary waves reported by Etrich *et al.* [Phys. Rev. E **54**, 4321 (1996)] are found to appear for finite values of the wave amplitude. [S1063-651X(99)06506-X]

PACS number(s): 42.65.Tg, 42.60.Rn, 42.65.Sf

There have recently been a host of theoretical and experimental studies dealing with the mutual locking of fundamental and second harmonic beams in optical waveguides. Many of them have considered the existence, stability, and propagation of soliton excitations. In particular, nonlinear self-modulation of plane (harmonic) waves was shown [1] to lead to modulational instability and the breakup of a wave background into chains of coupled solitons [2]. The solitons exist due to the nonlinear coupling between the resonant harmonics, and they were generated numerically [3] and experimentally [4] from a small-intensity beam at a fundamental frequency. The coupled solitons were proved to be unstable in a narrow domain of their existence [5] and stable otherwise [6]. The dynamics of stable solitary waves, including their generation and interaction, were observed to be complicated [7,8], involving a range of persistent oscillations, unlike the nonresonant case governed by a one-dimensional nonlinear Schrödinger (NLS) equation. Studies of these oscillations showed [9] that the spectrum of an associated linear eigenvalue problem possesses discrete internal modes that cause the oscillatory excitations of the coupled solitary waves in nonlinear quadratic optical materials.

In this Brief Report we study the conditions under which these internal oscillations can be excited. We find surprisingly that, for fixed wave vector mismatch between the two harmonics, the small intensities of the fundamental wave do not support internal oscillations of solitons. This conclusion implies that pumping of a small fundamental wave weakly coupled to the second harmonics in the $\chi^{(2)}$ optical materials leads basically to the same dynamics as can be expected in a nonresonant system, i.e., the wave background splits into solitons without persistent oscillations or nontrivial soliton interactions.

Our analysis relies on a conventional system of coupled equations,

$$iw_z + w_{xx} + w^* v = 0, \tag{1}$$

$$i\sigma v_z + v_{xx} - \Delta v + \frac{1}{2} w^2 = 0, \tag{2}$$

where w and v stand for envelope functions of a fundamental and second harmonics, respectively, Δ is proportional to the wave vector mismatch between the harmonics, and σ describes either the ratio of the wave vectors in the case of spatial solitons (when $\sigma=2$) or the ratio of the group-velocity dispersions in the case of temporal solitons.

The continuous wave background is taken to be a stationary solution of this system,

$$w = W_s e^{i\Omega z}, \quad v = V_s e^{2i\Omega z},$$

where

$$V_s = \frac{W_s^2}{2(\Delta + 2\sigma\Omega)}$$

and the dependence $\Omega = \Omega(|W_s|^2, \Delta)$ is given by

$$\Omega = \Omega_{\pm} = \frac{1}{4\sigma}\left[\pm\sqrt{\Delta^2 + 4\sigma|W_s|^2} - \Delta\right]. \tag{3}$$

This stationary solution describes two branches of the plane waves that are weakly coupled in the limit of small intensities, i.e., for $|W_s|^2 \to 0$. In this limit, the branch with $\Omega = \Omega_+$ represents the fundamental wave for $\Delta > 0$, when $|V_s| \ll |W_s|$, while that with $\Omega = \Omega_-$ represents the second harmonics, when $|W_s| \ll |V_s|$. The experiments on soliton generation [4] involve typically the incident fundamental small-intensity beams, where essentially only the first branch is excited. Since the wave background is modulationally unstable, the plane wave leads to the formation of soliton spikes. The problem at the center of our analysis is to determine whether the soliton spike supported by a fundamental beam displays the oscillatory dynamics for small and finite intensities W_s of the fundamental wave.

In the small-intensity limit, when $\Delta > 0$ and $\Omega = \Omega_+ \approx |W_s|^2/2\Delta$ as $|W_s|^2 \to 0$, the underlying system (1) and (2) reduces to the NLS equation at the leading order approximation. In order to analyze this limit and extend its applicability, we assume the following scaling transformation:

$$w = \epsilon W(X,Z), \quad v = \epsilon^2 V(X,Z), \tag{4}$$

where $X = \epsilon x$, $Z = \epsilon^2 z$, and $\epsilon \ll 1$. The complex functions $W(X,Z)$ and $V(X,Z)$ satisfy the coupled system,

$$iW_Z + W_{XX} + W^* V = 0, \tag{5}$$

$$\epsilon^2(i\sigma V_Z + V_{XX}) - \Delta V + \frac{1}{2}W^2 = 0. \tag{6}$$

The NLS equation follows from this system in negligence of the O(ϵ^2) terms. The soliton solutions have no internal (oscillatory) modes within the framework of the NLS equation. However, for a perturbed NLS equation [10,11] it was recently shown that a certain class of perturbations can deform the spectrum of linear excitations of solitons and lead to the appearance of an internal (oscillatory) eigenmode from the edge of the wave continuum. It was assumed in numerical studies of the model (1) and (2) [9] that this bifurcation does take place for the coupled solitons. But the analytic criterion for this bifurcation was not checked, and the numerical data did not confirm its appearance. Here we recover the bifurcation criterion by extending the underlying NLS equation into the next-order approximation,

$$V = \frac{1}{2\Delta}W^2 + \frac{\epsilon^2}{\Delta^2}\left[(1-\sigma)WW_{XX} + W_X^2 - \frac{\sigma}{2\Delta}|W|^2W^2\right] + \mathrm{O}(\epsilon^4). \tag{7}$$

The function $U = (2\sqrt{\Delta})^{-1}W(X,Z)$ satisfies the perturbed NLS equation in the form

$$iU_Z + U_{XX} + 2|U|^2U + 4\mu[(1-\sigma)|U|^2U_{XX} + U_X^2U^* - 2\sigma|U|^4U] + \mathrm{O}(\mu^2) = 0, \tag{8}$$

where $\mu = \epsilon^2/\Delta \ll 1$. Following to Pelinovsky *et al.* [11], we extend the stationary soliton solutions in the asymptotic series,

$$U = \Phi_\mu(X)e^{iZ}, \quad \Phi_\mu = \Phi_0(X) + \mu\Phi_1(X) + \mathrm{O}(\mu^2),$$

where $\Phi_0 = \mathrm{sech}\,X$ and

$$\Phi_1 = (\sigma+2)\mathrm{sech}\,X - 2\,\mathrm{sech}^3X.$$

Perturbations to the soliton solutions can be written in the form

$$U = [\Phi_\mu(X) + (a(X) - b(X))e^{i\lambda Z} + (a^*(X) + b^*(X))e^{-i\lambda^* Z}]e^{iZ},$$

where λ is an eigenvalue and $a(X)$ and $b(X)$ satisfy the linear eigenvalue problem,

$$\mathcal{L}_1 a = \lambda b + 4\mu\,\delta\mathcal{L}_1 a,$$

$$\mathcal{L}_0 b = \lambda a + 4\mu\,\delta\mathcal{L}_0 b.$$

Here $\mathcal{L}_0 = -\partial_X^2 + 1 - 2\,\mathrm{sech}^2X$, $\mathcal{L}_1 = -\partial_X^2 + 1 - 6\,\mathrm{sech}^2X$ and the operators of the perturbative terms are given by

$$\delta\mathcal{L}_0 = \Phi_0\Phi_1 - 2\sigma\Phi_0^4 - \Phi_{0X}^2 + 2\Phi_0\Phi_{0X}\partial_X + (1-\sigma)\Phi_0^2\partial_X^2,$$

$$\delta\mathcal{L}_1 = 3\Phi_0\Phi_1 - 10\sigma\Phi_0^4 + 2(1-\sigma)\Phi_0\Phi_{0XX} + \Phi_{0X}^2 + 2\Phi_0\Phi_{0X}\partial_X + (1-\sigma)\Phi_0^2\partial_X^2.$$

According to the bifurcation criterion derived in Ref. [11], the internal mode detaches from the wave continuum for $\mu > 0$ and has the oscillation frequency $\lambda = \lambda_{osc} = 1 - \mu^2\kappa^2$, if the parameter κ is *positive*, where

$$\kappa = \int_{-\infty}^{\infty} dX[a_0(X)\,\delta\mathcal{L}_1 a_0(X) + b_0(X)\,\delta\mathcal{L}_0 b_0(X)]. \tag{9}$$

Here $a_0(X)$ and $b_0(X)$ are limiting (nonsecular) eigenfunctions for the edge of the wave continuum at $\lambda = 1$ for the unperturbed problem,

$$a_0 = 1 - 2\,\mathrm{sech}^2x, \quad b_0 = 1.$$

Calculating the integral (9), we find the simple result, $\kappa = -\frac{4}{3}(1+\sigma)$, which is negative for $\sigma > -1$. Therefore, the bifurcation of an internal mode does not occur for $\sigma > -1$. Although the relaxation oscillations of solitons induced by linear dispersive wave packets are still possible for intermediate time intervals, as in the NLS case [12], we conclude that the amplitude of the fundamental wave W_s must exceed a certain threshold for persistent oscillations of solitons to be supported by the existence of an internal mode. In the remainder of this paper, we find this threshold numerically.

The solitary waves of the model (1) and (2) have the form

$$w = w_0(x)w^{i\Omega z}, \quad v = v_0(x)e^{2i\Omega z}, \tag{10}$$

which exist for $-2\sigma\Omega < \Delta < \infty$ [3]. We employ a rescaling of variables

$$\bar{w} = w/\Omega, \quad \bar{v} = v/\Omega, \quad \bar{\Delta} = \Delta/\Omega, \quad \bar{x} = \sqrt{\Omega}x, \quad \bar{z} = \Omega z, \tag{11}$$

and drop the bars. Then the system (1) and (2) remains the same, but Ω in Eq. (10) is normalized to be 1. The functions $w_0(x)$ and $v_0(x)$ are real and single-humped for the fundamental solitons. They can be calculated by means of the shooting method [3]. We reproduce in Fig. 1(a) the profile of the soliton solutions at $\sigma = 2$ and $\Delta = 1/2$. The limit $\Delta \to \infty$ corresponds to the solitons supported solely by the fundamental wave, when $v_0(0) \ll w_0(0)$. According to the results above, this limit does not support the persistent oscillations of solitons. Therefore, we expect that the soliton oscillations may exist only for finite values of the wave speed mismatch $\Delta \leq \Delta_{thr}(\sigma) < \infty$, when the amplitudes $v(0)$ and $w(0)$ are comparable.

In order to study the internal modes of the solitary waves $(w_0(x), v_0(x))$, we impose the linear perturbation in the form

$$w = [w_0(x) + (w_r(x) - w_i(x))e^{i\lambda z} + (w_r^*(x) + w_i^*(x))e^{-i\lambda^* z}]e^{iz},$$

$$v = [v_0(x) + (v_r(x) - v_i(x))e^{i\lambda z} + (v_r^*(x) + v_i^*(x))e^{-i\lambda^* z}]e^{2iz},$$

where $w_r(x)$, $w_i(x)$, $v_r(x)$, and $v_i(x)$ satisfy the linear problem

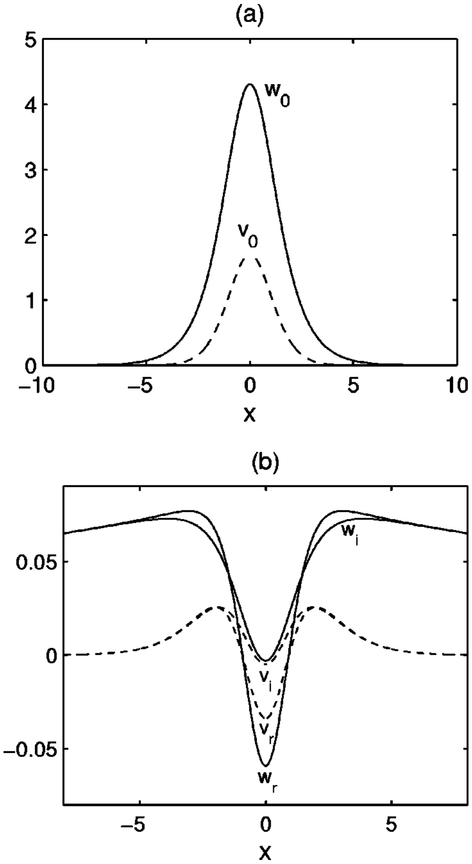

FIG. 1. The solitary wave (w_0, v_0) (a) and its internal mode (b) for $\sigma=2$ and $\Delta=0.5$.

$$\mathcal{L}_+\begin{pmatrix} w_r \\ v_r \end{pmatrix}=\lambda\begin{pmatrix} w_i \\ \sigma v_i \end{pmatrix}, \quad \mathcal{L}_-\begin{pmatrix} w_i \\ v_i \end{pmatrix}=\lambda\begin{pmatrix} w_r \\ \sigma v_r \end{pmatrix}. \tag{12}$$

Here,

$$\mathcal{L}_\pm=\begin{pmatrix} -\partial_x^2+1\mp v_0(x) & -w_0(x) \\ -w_0(x) & -\partial_x^2+\Delta+2\sigma \end{pmatrix}.$$

The linear system (12) has four neutral localized eigenmodes for $\lambda=0$ associated to symmetries of soliton solutions and four branches of the continuous spectrum located for $|\lambda|>1$ and $|\lambda|>2+\Delta/\sigma$ [5]. The internal mode can exist at $\lambda=\lambda_{osc}$, where

$$|\lambda_{osc}|<\min\left(1,2+\frac{\Delta}{\sigma}\right).$$

We find the internal eigenmode by solving Eq. (12) numerically and display the profile $(w_r(x), w_i(x), v_r(x), v_i(x))$ in Fig. 1(b) for $\sigma=2$ and $\Delta=1/2$. In this case, the eigenvalue $\lambda_{osc}=0.9989$. In Fig. 2, we present the dependence of λ_{osc} versus Δ for a fixed value of $\sigma=2$. It is clear that the internal mode merges the continuous spectrum at $\lambda=1$ when $\Delta\to\Delta_{thr}=1.1446$. This result should be compared with the previous numerical analysis of Etrich *et al.* [9], where the threshold on internal modes was overlooked. Revising Fig. 5 of those authors [9], we conclude that the internal mode does not exist for $\alpha\gtrsim 2.5723$, or in our notations, for $\Delta=2\alpha-4\gtrsim 1.1446$. The internal mode disappears for $\Delta\to\Delta_{stab}\approx -3.7887$ (Fig. 2). This bifurcation leads to the instability of the coupled soliton, as was shown earlier [5].

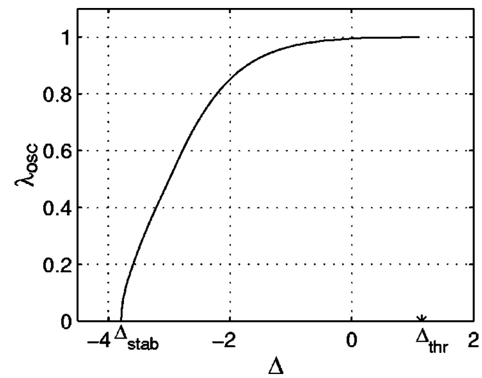

FIG. 2. The dependence of the internal eigenvalue λ_{osc} on Δ for $\sigma=2$.

In order to find the threshold $\Delta_{thr}(\sigma)$ for various σ values, we solve Eq. (12) numerically for $\lambda=1$ and look for a bounded eigenfunction. Generally, this limiting eigenfunction is secular since

$$\lim_{x\to\infty}(w_r+w_i)_x-\lim_{x\to-\infty}(w_r+w_i)_x=Q\neq 0,$$

where

$$Q=-\frac{1}{2}\int_{-\infty}^{\infty}dx[w_0(v_r+v_i)+v_0(w_r-w_i)].$$

When the quantity Q vanishes, the bifurcation of a new eigenvalue $\lambda=\lambda_{osc}<1$ may occur from the edge of the continuous spectrum. This criterion is satisfied in the asymptotic limit of the integrable NLS equation, when $\Delta\to\infty$,

$$v_0\to\frac{1}{2\Delta}w_0^2, \quad v_r\to\frac{1}{\Delta}w_0w_r, \quad v_i\to\frac{1}{\Delta}w_0w_i, \tag{13}$$

and

$$Q\to-\int_{-\infty}^{\infty}(3a_0(x)+b_0(x))\mathrm{sech}^2x \quad dx\equiv 0.$$

However, as we have checked above, the integrable NLS limit does not support a bifurcation of an internal mode. Therefore, we are looking for the bifurcation to occur in the non-integrable limit for a finite value of $\Delta=\Delta_{thr}(\sigma)$. We use the shooting method to find the bounded eigenfunction of Eq. (12) at $\lambda=1$, when $Q(\Delta,\sigma)=0$. The dependence $\Delta_{thr}(\sigma)$ is identified by this method and shown in Fig. 3. We notice that the coupled solitons supported by the two-wave interaction at an exact resonance ($\Delta=0$) also have an oscillatory mode except for a narrow range $0.4207<\sigma<0.5492$, where $\Delta_{thr}<0$. The condition $\Delta=0$ for the exact resonance can be achieved alternatively by a very large intensity W_s of the fundamental wave, when $|V_s|\sim|W_s|\gg 1$. This can be seen from the rescaling of variables (11) by letting $\Omega\to\infty$. Thus, the solitons supported by the large-intensity fundamental wave W_s display the persistent oscillations if $\sigma<0.4207$ or $\sigma>0.5492$. In conclusion, we have proved the existence of a threshold on the amplitude W_s of the funda-

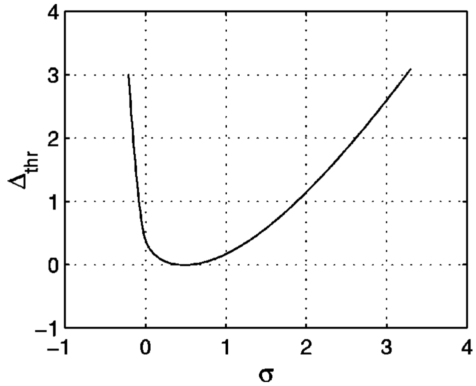

FIG. 3. The boundary curve Δ_{thr} of internal modes for arbitrary σ values.

mental wave to support the internal soliton's oscillations in resonant $\chi^{(2)}$ materials. For fixed wave vector mismatch Δ, small-intensity solitons do not display complicated oscillatory dynamics, while the large-intensity solitons do except in a narrow parameter window.

Lastly, we would like to make connections of our work to two recent results. In a similar problem of excitations of coupled solitons [13], a resonance of perturbations of two branches of the continuous spectrum can result in an oscillatory destabilization of a solitary wave. However, the resonance takes place only if the Hamiltonian of a system has sign-indefinite metrics at the continuous spectrum's eigenfunctions [13]. In the problem (1) and (2) under consideration, the Hamiltonian is sign-definite for small perturbations as it follows from the explicit representation,

$$H=\int_{-\infty}^{\infty} dx(|w_x|^2+|v_x|^2+\Delta|v|^2-\tfrac{1}{2}(w^2v^*+w^{*2}v)).$$

This feature implies that the solitary wave solutions may lose their stability only through a bifurcation of an internal mode at the origin of λ which occurs for $\Delta=\Delta_{stab}(\sigma)<0$ [5]. The solitary wave solutions are stable for $\Delta_{stab}(\sigma)<\Delta<\infty$ [6]. In another problem involving vector solitons in birefringent optical fibers, a similar pattern of internal oscillations was demonstrated [14]. In that case too, the region of existence of the internal mode does not cover the whole region of existence of the vector solitons, and the internal oscillations appear from the non-integrable limit at a special (threshold) value of the soliton parameters.

One of the authors (J.Y.) acknowledges partial financial support from the National Science Foundation under Grant No. DMS-9622802.

[1] A.E. Kaplan, Opt. Lett. **18**, 1223 (1993).
[2] S. Trillo and P. Ferro, Opt. Lett. **20**, 438 (1995).
[3] A.V. Buryak and Yu.S. Kivshar, Phys. Lett. A **197**, 407 (1995).
[4] R. Schiek, Y. Baek, and G.I. Stegeman, Phys. Rev. E **53**, 1138 (1996).
[5] D.E. Pelinovsky, A.V. Buryak, and Yu.S. Kivshar, Phys. Rev. Lett. **75**, 591 (1995).
[6] L. Berge, O. Bang, J.J. Rasmussen, and V.K. Mezentsev, Phys. Rev. E **55**, 3555 (1997).
[7] M.J. Werner and P.D. Drummond, J. Opt. Soc. Am. B **10**, 2390 (1993).
[8] C. Etrich, U. Peschel, F. Lederer, and B.A. Malomed, Phys. Rev. A **52**, R3444 (1995).
[9] C. Etrich, U. Peschel, F. Lederer, B.A. Malomed, and Yu.S. Kivshar, Phys. Rev. E **54**, 4321 (1996).
[10] Yu.S. Kivshar, D.E. Pelinovsky, T. Cretegny, and M. Peyrard, Phys. Rev. Lett. **80**, 5032 (1998).
[11] D.E. Pelinovsky, Yu.S. Kivshar, and V.V. Afanasjev, Physica D **116**, 121 (1998).
[12] E.A. Kuznetsov, A.V. Mikhailov, and I.A. Shimokhin, Physica D **87**, 201 (1995).
[13] I.V. Barashenkov, D.E. Pelinovsky, and E.V. Zemlyanaya, Phys. Rev. Lett. **80**, 5117 (1998).
[14] D.E. Pelinovsky and J. Yang (unpublished).