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# Orbital Stability of Dirac Solitons

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**Abstract.** We prove $H^1$ orbital stability of Dirac solitons in the integrable massive Thirring model by working with an additional conserved quantity which complements Hamiltonian, momentum and charge functionals of the general nonlinear Dirac equations. We also derive a global bound on the $H^1$ norm of the $L^2$-small solutions of the massive Thirring model.

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## 1. Introduction

Nonlinear Dirac equations are considered as a relativistic version of the nonlinear Schrödinger (NLS) equation. Compared to the NLS equation, proofs of global existence and orbital stability of solitary waves are complicated by the fact that the quadratic part of the Hamiltonian of the nonlinear Dirac equations is not bounded from neither above nor below. Similar situation occurs in a gap between two bands of continuous spectrum in the Schrödinger equations with a periodic potential, for which the nonlinear Dirac equations are justified rigorously as an asymptotic model (Chapter 2.2 in [39]).

Because orbital stability of solitary waves is not achieved by the standard energy arguments [19], researchers have studied spectral and asymptotic stability of solitary waves in many details. Spectral properties of linearized Dirac operators were studied by a combination of analytical methods and numerical approximations [3,4,7,10–13,17]. Asymptotic stability of small solitary waves in the general nonlinear Dirac equations was studied with dispersive estimates both in the space of one [30,31,40] and three [5,6,8] dimensions. Global existence and scattering to zero for small initial data were obtained again in one [20,21] and three [33,34] dimensions.

When nonlinear Dirac equations are considered in one spatial dimension, a particular attention is drawn to the massive Thirring model (MTM) [42], which is known to be integrable with the inverse scattering transform method [29,32]. In laboratory coordinates, this model takes the following form:

$$\begin{cases} i(u_t+u_x)+v=2|v|^2u, \\ i(v_t-v_x)+u=2|u|^2v, \end{cases} \tag{1}$$

where $(u,v)(x,t):\mathbb{R}\times\mathbb{R}_+\to\mathbb{C}^2$.

Selberg and Tesfahun [41] proved local well-posedness of the MTM system in $H^s(\mathbb{R})$ for $s>0$ and global well-posedness in $H^s(\mathbb{R})$ for $s>\frac{1}{2}$. Machihara et al. [35] proved for similar nonlinear Dirac equations with quadratic nonlinear terms that local well-posedness holds in $H^s(\mathbb{R})$ for $s>-\frac{1}{2}$ and that the Cauchy problem is ill-posed in $H^{-1/2}(\mathbb{R})$. Candy [9] proved local and global well-posedness of the MTM system in $L^2(\mathbb{R})$. Global solutions of the massless Thirring model in $L^2(\mathbb{R})$ were considered by Huh [23] and Zhang [43]. Recently, global solutions to the massive Gross–Neveu equation were constructed by Huh [24] who found an unexpected bound on the $L^\infty$ norm of the solutions (these results can be extended to the MTM system). The aforementioned works do not rely on the inverse scattering transform for the MTM system.

On the other hand, using the inverse scattering transform, spectral stability of the MTM solitons was established by Kaup and Lakoba [27]. The stationary MTM solitons are known in the exact analytical form:

$$\begin{cases} u=U_\omega(x+x_0)e^{i\omega t+i\alpha}, \\ v=\bar{U}_\omega(x+x_0)e^{i\omega t+i\alpha}, \end{cases} \tag{2}$$

with

$$U_\omega(x)=\frac{\sqrt{1-\omega^2}}{\sqrt{1+\omega}\cosh\left(\sqrt{1-\omega^2}x\right)+i\sqrt{1-\omega}\sinh\left(\sqrt{1-\omega^2}x\right)}, \tag{3}$$

where $\alpha$ and $x_0$ are real parameters related to the gauge and space translations, whereas $\omega\in(-1,1)$ is a parameter that determines where the MTM solitons are placed in the gap between two branches of the continuous spectrum of the linearized MTM system. Perturbations of the MTM system and the loss of spectral stability of solitary waves were consequently considered using the spectral representations [2,28] and the Evans function [25]. No results on the orbital or asymptotic stability of the MTM solitons (2) in the time evolution of the MTM system (1) have been obtained so far.

The idea for our work relies on the existence of an infinite set of conserved quantities in the MTM system, which has been known for quite some time [32]. The three standard conserved quantities for the nonlinear Dirac equations are related to the translational invariance of the system with respect to gauge, space, and time transformations. For the MTM system (1), these three conserved quantities are referred to as the charge $Q$, momentum $P$, and Hamiltonian $H$ functionals:

$$Q = \int_{\mathbb{R}} \left( |u|^2 + |v|^2 \right) \mathrm{d}x, \tag{4}$$

$$P = \frac{i}{2} \int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u} + v\bar{v}_x - v_x\bar{v})\, \mathrm{d}x, \tag{5}$$

and

$$H = \frac{i}{2} \int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u} - v\bar{v}_x + v_x\bar{v})\, \mathrm{d}x + \int_{\mathbb{R}} \left( -v\bar{u} - u\bar{v} + 2|u|^2|v|^2 \right) \mathrm{d}x. \tag{6}$$

One can use the charge $Q$ to establish global bound on the $L^2$ norm of solutions, as soon as the local existence in $L^2(\mathbb{R})$ is proven [9]. The other two functionals $P$ and $H$ are defined in $H^{1/2}(\mathbb{R})$, but they are not so useful because of the fact that the quadratic part of the Hamiltonian $H$ is not bounded from neither above nor below. Therefore, although the MTM solitons (3) are critical points of the functional $H + \omega Q + cP$, where $\omega \in (-1, 1)$ and $c = 0$ for stationary MTM solitons, it cannot serve as a Lyapunov functional for orbital stability or instability of the MTM solitons.

Nevertheless, we find another conserved quantity of the MTM system (1), thanks to the inverse scattering transform method:

$$R = \int_{\mathbb{R}} \Big[ |u_x|^2 + |v_x|^2 - \frac{i}{2}(u_x\bar{u} - \bar{u}_x u)(|u|^2 + 2|v|^2) + \frac{i}{2}(v_x\bar{v} - \bar{v}_x v)(2|u|^2 + |v|^2)$$
$$- (u\bar{v} + \bar{u}v)(|u|^2 + |v|^2) + 2|u|^2|v|^2(|u|^2 + |v|^2) \Big] \mathrm{d}x. \tag{7}$$

Derivation of the conserved quantity $R$ is reviewed in Appendix A. The conserved quantity $R$ is well defined in $H^1(\mathbb{R})$ and we shall use it to prove orbital stability of the MTM solitons in $H^1(\mathbb{R})$. The main result of this article is the following theorem.

THEOREM 1. *There is $\omega_0 \in (0, 1]$ such that for any fixed $\omega \in (-\omega_0, \omega_0)$, the MTM soliton $(u, v) = (U_\omega, \bar{U}_\omega)$ is a local nondegenerate minimizer of $R$ in $H^1(\mathbb{R}, \mathbb{C}^2)$ under the constraints of fixed values of $Q$ and $P$. Therefore, the MTM soliton is orbitally stable in $H^1(\mathbb{R}, \mathbb{C}^2)$ with respect to the time evolution of the MTM system* (1).

From a technical point, we establish that the MTM solitons (3) are critical points of the functional $\Lambda_\omega = R + (1 - \omega^2)Q$, where $\omega \in (-1, 1)$ is the same parameter of the MTM solitons. By using operator calculus in constrained spaces, we prove that there is $\omega_0 \in (0, 1]$ such that for any fixed $\omega \in (-\omega_0, \omega_0)$, the functional $\Lambda_\omega$ is strictly convex at the MTM soliton $(u, v) = (U_\omega, \bar{U}_\omega)$ under the constraints of fixed values of $Q$ and $P$. Hence, $\Lambda_\omega$ can serve as a Lyapunov functional for orbital stability of the MTM solitons thanks to the conservation of $R$, $Q$, and $P$

and the standard analysis of orbital stability of solitary waves [19]. Appendix B states relevant technical results used in our work.

Note that the nondegenerate minimizer in Theorem 1 can be translated along two "trivial" parameters $\alpha$ and $x_0$ in (2). These parameters are related to the gauge and space translations and can be excluded by additional constraints on the perturbations to the MTM soliton $(u, v) = (U_\omega, \bar{U}_\omega)$. Whereas we have not succeeded in finding the exact value of $\omega_0$, we conjecture that $\omega_0 = 1$, that is, the result of Theorem 1 extends to the entire family of MTM solitons. This conjecture is verified using numerical approximations.

We also mention some recent relevant results. First, orbital stability of breathers of the modified KdV equation is proved in space $H^2(\mathbb{R})$ in the recent work of Alejo and Munoz [1]. An additional conserved quantity is introduced and used to complement conservation of the Hamiltonian and momentum of the modified KdV equation. This work is conceptually similar to the ideas of our paper with the following difference. It uses the known characterization of orbital stability of multi-solitons in the KdV and modified KdV equations with higher-order conserved quantities [22,36], whereas our work introduces a new concept of orbital stability of Dirac solitons.

Second, a different technique involving additional conserved quantities is proposed in the work of Deconinck and Kapitula [15], where no constraints are imposed to study orbital stability of periodic waves of the KdV equation with respect to perturbations of a multiple period. The lack of minimizing properties of the higher-order Hamiltonian is corrected by adding lower-order Hamiltonians with a carefully selected amplitude parameter.

As a bi-product of our work, we obtain a global a priori bound on the $H^1$ norm of the $L^2$-small solutions of the MTM system in $H^1(\mathbb{R})$. The standard apriori bounds on the $H^1$ norm of the corresponding solutions of a general nonlinear Dirac equation grow at a double-exponential rate [18,38]. At the present time, we do not know if global bounds on the $H^1$ norm can be proven for all (not $L^2$-small) solutions of the MTM system (1).

We also do not know if asymptotic stability of the MTM solitons can be proved using the inverse scattering transform methods, e.g., the auto–Bäcklund transformation, similar to what was done recently for NLS solitons [14,16,37]. (Asymptotic stability of the zero solution and a criterion for the absence of the MTM solitons were established earlier in [20,21,38], respectively.) These problems remain open for further studies.

The paper is organized as follows. Global bounds on the $H^1$ norm of the $L^2$-small solutions are obtained in Sect. 2. In Sect. 3, we prove that the MTM solitons are nondegenerate minimizers of $R$ in $H^1(\mathbb{R})$ under the constraints of fixed $Q$ and $P$ for $\omega \in (-\omega_0, \omega_0)$ with some $\omega_0 \in (0, 1]$. Appendix A reports derivation of the conserved quantity $R$. Appendix B lists a number of technical results without proofs.

## 2. Global $H^1$ Bound on the $L^2$-Small Solutions

Thanks to local existence of solutions of the MTM system (1) in $L^2(\mathbb{R})$ [9] and the conservation of $Q$ in (4), the $L^2$ solutions are extended globally for all $t\in\mathbb{R}$ with a global bound on the $L^2$ norm of the corresponding solutions. On the other hand, local existence of solutions of the MTM system (1) holds also in $H^n(\mathbb{R})$ for any integer $n\in\mathbb{N}$, but the apriori bounds on the $H^n$ norm grows at a super-exponential rate [18,38]. Here, we use the conservation of $R$ to obtain the global bound on the $H^1$ norm of the corresponding solutions with small $L^2$ norm. The following theorem gives the main result of this section.

THEOREM 2. *There is a $Q_0>0$ such that for all $(u_0,v_0)\in H^1(\mathbb{R},\mathbb{C}^2)$ with $\|u_0\|^2_{L^2}+\|v_0\|^2_{L^2}\le Q_0$, there is a positive $(u_0,v_0)$-dependent constant $C(u_0,v_0)$ such that*

$$\|u\|^2_{H^1}+\|v\|^2_{H^1}\le C(u_0,v_0), \tag{8}$$

*for all $t\in\mathbb{R}$.*

*Proof.* By Sobolev embedding of $H^1(\mathbb{R})$ into $L^p(\mathbb{R})$ for any $p\ge 2$, the values of $Q$ and $R$ are finite if $(u_0,v_0)\in H^1(\mathbb{R},\mathbb{C}^2)$. To obtain the assertion of the theorem, we need to show that the value of $R$ gives an upper bound for the value of $\|\partial_x u\|^2_{L^2}+\|\partial_x v\|^2_{L^2}$. Then, the standard approximation argument in Sobolev space $H^2(\mathbb{R})$ yields a conservation of $R$ from the balance equation [see Eq. (50) in Appendix A], whereas the standard continuation argument yields the global bound (8). Recall here that $Q=\|u\|^2_{L^2}+\|v\|^2_{L^2}$ is conserved in time $t$.

The lower bound for $R$ follows from two applications of the Gagliardo–Nirenberg inequality in one dimension. For any $p\ge 1$, there is a constant $C_p>0$ such that

$$\|u\|^{2p}_{L^{2p}}\le C_p\|\partial_x u\|^{p-1}_{L^2}\|u\|^{1+p}_{L^2},\quad u\in H^1(\mathbb{R}). \tag{9}$$

First, we note that quadratic and sixth-order terms of $R$ in (7) are positive definite. To control the last fourth-order terms of $R$ from below, we note that there is a positive constant $C$ (which may change from line to another line) such that

$$\begin{aligned}\left|\int_{\mathbb{R}}(|u|^2+|v|^2)(v\bar{u}+u\bar{v})\mathrm{d}x\right| &\le C\left(\|u\|^4_{L^4}+\|v\|^4_{L^4}\right)\\ &\le C\left(\|u\|^3_{L^2}+\|v\|^3_{L^2}\right)\left(\|\partial_x u\|_{L^2}+\|\partial_x v\|_{L^2}\right).\end{aligned}$$

Because the positive part of $R$ is quadratic in $\|\partial_x u\|_{L^2}$ and $\|\partial_x v\|_{L^2}$, the previous bound is sufficient to control the last fourth-order terms of $R$ from below. On the other hand, the first fourth-order terms of $R$ like

$$\left| \int_{\mathbb{R}} (u_x \overline{u} - \overline{u}_x u)|u|^2 \mathrm{d}x \right| \leq C\|u\|_{L^6}^3 \|\partial_x u\|_{L^2} \leq C\|u\|_{L^2}^2 \|\partial_x u\|_{L^2}^2$$

can only be controlled from below if $\|u\|_{L^2}^2$ is sufficiently small. This proves the assertion of the theorem. □

REMARK 1. *One can try to squeeze the first fourth-order terms of $R$ between the positive quadratic and sixth-order terms of $R$. For example, if reduction $v = \bar{u}$ is used, these terms of $R$ are estimated from below by*

$$2\|\partial_x u\|_{L^2}^2 + 4\|u\|_{L^6}^6 - 3i \int_{\mathbb{R}} (u_x \overline{u} - \overline{u}_x u)|u|^2 \mathrm{d}x \geq 2\|\partial_x u\|_{L^2}^2 + 4\|u\|_{L^6}^6 \\ -6\|\partial_x u\|_{L^2}\|u\|_{L^6}^3.$$

*Unfortunately, the lower bound is not positive definite. Therefore, we do not know if the global bound (8) can be extended to all (not necessarily $L^2$-small) solutions of the MTM system (1).*

## 3. $H^1$ Orbital Stability of Solitons

Critical points of the energy functional $H + \omega Q$ with fixed $\omega \in (-1, 1)$ satisfy the system of first-order differential equations

$$\begin{cases} +i\frac{\mathrm{d}u}{\mathrm{d}x} - \omega u + v = 2|v|^2 u, \\ -i\frac{\mathrm{d}v}{\mathrm{d}x} - \omega v + u = 2|u|^2 v. \end{cases} \tag{10}$$

The stationary MTM solitons (3) correspond to the reduction $u = U_\omega$ and $v = \bar{U}_\omega$, where $U_\omega$ is a solution of the first-order differential equation

$$i\frac{\mathrm{d}U}{\mathrm{d}x} - \omega U + \bar{U} = 2|U|^2 U. \tag{11}$$

Integrating this differential equation with the zero boundary conditions, we obtain MTM solitons in the explicit form (3).

REMARK 2. *Two translational parameters of the MTM solitons (2) are obtained from the gauge and space translations:*

$$u(x,t) \mapsto u(x + x_0, t)e^{i\alpha}, \quad v(x,t) \mapsto v(x + x_0, t)e^{i\alpha}, \tag{12}$$

*where $\alpha$ and $x_0$ are real-valued. On the other hand, more general moving MTM solitons must have another parameter of velocity $c \in (-1, 1)$, which can be recovered* *using the Lorentz transformation:*

$$\begin{cases} u(x,t) \mapsto \left(\frac{1+c}{1-c}\right)^{1/4} u\left(\frac{x-ct}{\sqrt{1-c^2}}, \frac{t-cx}{\sqrt{1-c^2}}\right), \\ v(x,t) \mapsto \left(\frac{1-c}{1+c}\right)^{1/4} v\left(\frac{x-ct}{\sqrt{1-c^2}}, \frac{t-cx}{\sqrt{1-c^2}}\right). \end{cases} \tag{13}$$

*In what follows, without the loss of generality, we simplify our consideration by working with the stationary MTM solitons for $c=0$.*

Critical points of the modified energy functional $R+\Omega Q$ satisfy the system of second-order differential equations

$$\begin{cases} \frac{d^2u}{dx^2} + 2i(|u|^2+|v|^2)\frac{du}{dx} + 2iuv\frac{d\bar{v}}{dx} - 2|v|^2(2|u|^2+|v|^2)u + (2|u|^2+|v|^2)v + u^2\bar{v} = \Omega u, \\ \frac{d^2v}{dx^2} - 2i(|u|^2+|v|^2)\frac{dv}{dx} - 2iuv\frac{d\bar{u}}{dx} - 2|u|^2(|u|^2+2|v|^2)v + (|u|^2+2|v|^2)u + v^2\bar{u} = \Omega v. \end{cases} \tag{14}$$

Using the reduction $u=U$ and $v=\bar{U}$, we obtain a second-order differential equation

$$\frac{d^2U}{dx^2} + 6i|U|^2\frac{dU}{dx} - 6|U|^4U + 3|U|^2\bar{U} + U^3 = \Omega U. \tag{15}$$

Substituting Eq. (11) to Eq. (15) yields the constraint

$$(1-\omega^2)U + \left(2|U|^4 + 2\omega|U|^2 - U^2 - \bar{U}^2\right)U = \Omega U,$$

which is satisfied by the MTM soliton $U=U_\omega$ in the explicit form (3) if $\Omega = 1-\omega^2$. Therefore, the MTM soliton (3) is a critical point of the modified energy functional

$$\Lambda_\omega := R + (1-\omega^2)Q, \quad \omega \in (-1,1) \tag{16}$$

in the energy space $H^1(\mathbb{R},\mathbb{C}^2)$.

We shall now prove that there is $\omega_0 \in (0,1]$ such that for any fixed $\omega \in (-\omega_0, \omega_0)$, the critical point of $\Lambda_\omega$ is a local nondegenerate minimizer in the constrained space $X_\omega$, which is defined as an orthogonal complement in $H^1(\mathbb{R},\mathbb{C}^2)$ of the following complex-valued constraints:

$$(u,v) \in \mathbb{C}^2: \quad \int_{\mathbb{R}} \left(\bar{U}_\omega u + U_\omega v\right) dx = 0, \tag{17}$$

$$(u,v) \in \mathbb{C}^2: \quad \int_{\mathbb{R}} \left(\bar{U}'_\omega u + U'_\omega v\right) dx = 0, \tag{18}$$

where the prime denotes the derivative of $U_\omega$ with respect to $x$.

The real part of the constraint (17) is equivalent to the condition that the conserved quantity $Q$ is fixed under the perturbation $(u,v)$ to the MTM soliton $(U_\omega, \bar{U}_\omega)$ at the first order. The imaginary part of the constraint (17) represents the orthogonality of the perturbation $(u, v, \bar{u}, \bar{v})$ to the following eigenvector of a linearization operator for the zero eigenvalue,

$$\mathbf{F}_g := \begin{bmatrix} iU_\omega \\ i\bar{U}_\omega \\ -i\bar{U}_\omega \\ -iU_\omega \end{bmatrix}, \tag{19}$$

which is induced by the gauge translation of the MTM soliton $(U_\omega, \bar{U}_\omega, \bar{U}_\omega, U_\omega)$ related to the parameter $\alpha$ in the transformation (12).

Similarly, the imaginary part of the constraint (18) is equivalent to the condition that the conserved quantity $P$ is fixed under the perturbation $(u, v)$ to the MTM soliton $(U_\omega, \bar{U}_\omega)$ at the first order. The real part of the constraint (18) represents the orthogonality of the perturbation $(u, v, \bar{u}, \bar{v})$ to the following eigenvector of a linearization operator for the zero eigenvalue,

$$\mathbf{F}_s := \begin{bmatrix} U'_\omega \\ \bar{U}'_\omega \\ \bar{U}'_\omega \\ U'_\omega \end{bmatrix}, \tag{20}$$

which is induced by the space translation of the MTM soliton $(U_\omega, \bar{U}_\omega, \bar{U}_\omega, U_\omega)$ related to the parameter $x_0$ in the transformation (12).

The following theorem gives the main result of this section.

THEOREM 3. *There is $\omega_0 \in (0, 1]$ such that for any fixed $\omega \in (-\omega_0, \omega_0)$, the Lyapunov functional $\Lambda_\omega$ defined by* (16) *is strictly convex at $(u, v) = (U_\omega, \bar{U}_\omega)$ in the orthogonal complement of the complex-valued constraints* (17) *and* (18) *in $H^1(\mathbb{R}, \mathbb{C}^2)$.*

To prove Theorem 3, we use a perturbation $(u, v, \bar{u}, \bar{v})$ to the MTM soliton $(U_\omega, \bar{U}_\omega, \bar{U}_\omega, U_\omega)$ and expand the Lyapunov functional $\Lambda_\omega$ to the quadratic form in $(u, v, \bar{u}, \bar{v})$, which is defined by the matrix operator

$$L = \begin{bmatrix} L_1 & 2L_2 & L_2 & L_3 \\ 2\bar{L}_2 & \bar{L}_1 & \bar{L}_3 & \bar{L}_2 \\ \bar{L}_2 & \bar{L}_3 & \bar{L}_1 & 2\bar{L}_2 \\ L_3 & L_2 & 2L_2 & L_1 \end{bmatrix}, \tag{21}$$

where

$$\begin{aligned} L_1 &= -\frac{d^2}{dx^2} - 4i|U_\omega|^2\frac{d}{dx} - 4i\bar{U}_\omega\frac{dU_\omega}{dx} + 10|U_\omega|^4 - 2U_\omega^2 - 2\bar{U}_\omega^2 + 1 - \omega^2, \\ L_2 &= -2iU_\omega\frac{dU_\omega}{dx} + 4U_\omega^2|U_\omega|^2 - 2|U_\omega|^2, \\ L_3 &= -2i|U_\omega|^2\frac{d}{dx} - 2i\bar{U}_\omega\frac{dU_\omega}{dx} + 8|U_\omega|^4 - U_\omega^2 - \bar{U}_\omega^2. \end{aligned}$$

Similarly in the case of linearized Dirac equations [10], the $4\times 4$ matrix operator $L$ is diagonalized by two $2\times 2$ matrix operators $L_\pm$ by means of the orthogonal similarity transformation

$$S^T L S = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix}, \quad \text{where} \quad S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

The matrix operators $L_\pm$ are found from the reduction of $L$ under the constraints $v = \pm\bar{u}$:

$$L_+ = \begin{bmatrix} \ell_+ & -6\omega U_\omega^2 \\ -6\omega \bar{U}_\omega^2 & \bar{\ell}_+ \end{bmatrix}, \quad L_- = \begin{bmatrix} \ell_- & 2\omega U_\omega^2 \\ 2\omega \bar{U}_\omega^2 & \bar{\ell}_- \end{bmatrix}, \tag{22}$$

where

$$\ell_+ = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 6i|U_\omega|^2 \frac{\mathrm{d}}{\mathrm{d}x} + 6|U_\omega|^4 - 3U_\omega^2 + 3\bar{U}_\omega^2 - 6\omega|U_\omega|^2 + 1 - \omega^2,$$
$$\ell_- = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2i|U_\omega|^2 \frac{\mathrm{d}}{\mathrm{d}x} - 2|U_\omega|^4 - U_\omega^2 + \bar{U}_\omega^2 - 2\omega|U_\omega|^2 + 1 - \omega^2,$$

and Eq. (11) for $U_\omega$ has been used. Thanks to the exponential decay of $U_\omega$ to 0 at infinity, by Weyl's Lemma, the continuous spectrum of operators $L_\pm$ is located on the semi-infinite interval $[1-\omega^2, \infty)$ with $1-\omega^2 > 0$. The following results characterize the discrete spectrum of operators $L_\pm$.

PROPOSITION 1. *For any* $\omega \in (-1,1)$, *we have*

$$L_+ \begin{bmatrix} U_\omega' \\ \bar{U}_\omega' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad L_- \begin{bmatrix} U_\omega \\ -\bar{U}_\omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{23}$$

*which represent the eigenvectors* (19) *and* (20). *In addition, for* $\omega = 0$, *the zero eigenvalue of* $L_\pm$ *has multiplicity two and is associated with the eigenvectors in*

$$\omega = 0: \quad L_+ \begin{bmatrix} U_0' \\ -\bar{U}_0' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad L_- \begin{bmatrix} U_0 \\ \bar{U}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{24}$$

*in addition to the eigenvectors in* (23).

*Proof.* The eigenvectors of $L_+$ and $L_-$ in (23) for any $\omega \in (-1,1)$ are verified by direct substitution with the help of Eqs. (11) and (15). Because operators $L_\pm$ are diagonal for $\omega = 0$, the existence of the eigenvectors in (24) follows from the existence of the eigenvectors in (23) for $\omega = 0$. □

LEMMA 1. *For any* $\omega \in (-1,1)$, *operator* $L_-$ *has exactly two eigenvalues below the continuous spectrum. Besides the zero eigenvalue associated with the eigenvector in*

(23), *$L_-$ also has a positive eigenvalue for $\omega \in (0,1)$ and a negative eigenvalue for $\omega \in (-1,0)$, which is associated with the eigenvector in* (24) *for $\omega = 0$.*

*Proof.* Let us consider the eigenvalue problem $L_- \mathbf{u} = \mu \mathbf{u}$, where $\mathbf{u} = (u, \bar{u})$ is an eigenvector and $\mu$ is the spectral parameter. Using the transformation

$$u(x) = \varphi(x) e^{-i\int_0^x |U_\omega(x')|^2 \mathrm{d}x'}$$

where $\varphi$ is a new eigenfunction, we obtain an equivalent spectral problem:

$$\begin{bmatrix} -\partial_x^2 + 1 - \omega^2 - 2\omega|U_\omega|^2 - 3|U_\omega|^4 & 2\omega|U_\omega|^2 \\ 2\omega|U_\omega|^2 & -\partial_x^2 + 1 - \omega^2 - 2\omega|U_\omega|^2 - 3|U_\omega|^4 \end{bmatrix} \begin{bmatrix} \varphi \\ \bar{\varphi} \end{bmatrix} = \mu \begin{bmatrix} \varphi \\ \bar{\varphi} \end{bmatrix},$$

thanks to the fact that

$$U_\omega^2 e^{2i\int_0^x |U_\omega(x')|^2 \mathrm{d}x'} = \frac{1-\omega^2}{\omega + \cosh(2\sqrt{1-\omega^2}x)} = |U_\omega|^2.$$

Because the off-diagonal entries are real, we set

$$\psi_\pm := \varphi(x) \pm \bar{\varphi}(x), \quad z := \sqrt{1-\omega^2}x, \quad \mu := (1-\omega^2)\lambda$$

to diagonalize the spectral problem into two uncoupled spectral problems associated with the linear Schrödinger operators:

$$-\frac{\mathrm{d}^2\psi_+}{\mathrm{d}z^2} + \left[1 - \frac{3(1-\omega^2)}{(\omega+\cosh(2z))^2}\right]\psi_+ = \lambda\psi_+ \tag{25}$$

and

$$-\frac{\mathrm{d}^2\psi_-}{\mathrm{d}z^2} + \left[1 - \frac{3(1-\omega^2)}{(\omega+\cosh(2z))^2} - \frac{4\omega}{\omega+\cosh(2z)}\right]\psi_- = \lambda\psi_-. \tag{26}$$

The spectral problem (26) for $\lambda = 0$ admits the eigenfunction

$$\psi_0(z) = \frac{1}{(\omega+\cosh(2z))^{1/2}},$$

thanks to the existence of the eigenvector of $L_-$ in (23). Because the eigenfunction $\psi_0$ is positive definite, Sturm's Nodal Theorem (Theorem A in Appendix B) states that the simple zero eigenvalue of the spectral problem (26) is at the bottom of the spectrum of the Schrödinger operator for any $\omega \in (-1,1)$. Furthermore, the function

$$\psi_c(z) = \frac{\sinh(2z)}{\omega+\cosh(2z)}$$

corresponds to the end-point resonance at $\lambda = 1$ for the spectral problem

$$-\frac{d^2\psi}{\mathrm{d}z^2} + \left[1 - \frac{8(1-\omega^2)}{(\omega+\cosh(2z))^2} - \frac{4\omega}{\omega+\cosh(2z)}\right]\psi = \lambda\psi. \tag{27}$$

Because the function $\psi_c$ has exactly one zero, there is only one isolated eigenvalue below the continuous spectrum for the spectral problem (27) (Theorem A in Appendix B). Now the difference between the potentials of the spectral problems (26) and (27) is

$$\Delta V(z) = \frac{5(1-\omega^2)}{(\omega + \cosh(2z))^2},$$

where $\Delta V > 0$ for all $z \in \mathbb{R}$ and $\omega \in (-1, 1)$. By Sturm's Comparison Theorem (Theorem B in Appendix B), a solution of the spectral problem (26) for $\lambda = 1$, which is bounded as $z \to -\infty$, has exactly one zero. Therefore, the spectral problem (26) has exactly one isolated eigenvalue for all $\omega \in (-1, 1)$ and this is the zero eigenvalue with the eigenfunction $\psi_0$.

The difference between the potentials of the spectral problems (25) and (26) is given by

$$\Delta V(z) = \frac{4\omega}{\omega + \cosh(2z)}.$$

Since $\Delta V > 0$ for $\omega \in (0, 1)$, the spectral problem (25) has precisely one isolated eigenvalue for $\omega \in (0, 1)$ (Theorem B in Appendix B) and this eigenvalue is positive (Theorem C in Appendix B). On the other hand, since $\Delta V < 0$ for $\omega \in (-1, 0)$ and $\psi_0 > 0$ is an eigenfunction of the spectral problem (26) for $\lambda = 0$, the spectral problem (25) has at least one negative eigenvalue for $\omega \in (-1, 0)$ (Theorem C in Appendix B). To show that this nonzero eigenvalue is the only isolated eigenvalue of the spectral problem (25), we note that

$$\omega + \cosh(2z) \geq \omega + 1 + 2z^2, \quad z \in \mathbb{R}$$

and consider the spectral problem

$$-\frac{d^2\psi}{dz^2} + \left[1 - \frac{3(1-\omega^2)}{(\omega + 1 + 2z^2)^2}\right]\psi = \lambda\psi. \tag{28}$$

Rescaling the independent variable $z := \frac{\sqrt{1+\omega}}{\sqrt{2}} y$ and denoting $\psi(z) := \tilde{\psi}(y)$, we rewrite (28) in the equivalent form

$$-\frac{d^2\tilde{\psi}}{dy^2} - \frac{3}{(1+y^2)^2}\left(1 - \frac{1+\omega}{2}\right)\tilde{\psi} = \frac{(\lambda-1)(1+\omega)}{2}\tilde{\psi}. \tag{29}$$

It follows that the function

$$\tilde{\psi}_c(y) = \frac{y}{\sqrt{1+y^2}}$$

corresponds to the endpoint resonance at $\lambda = 1$ for the spectral problem

$$-\frac{d^2\tilde{\psi}}{dy^2} - \frac{3}{(1+y^2)^2}\tilde{\psi} = \frac{(\lambda-1)(1+\omega)}{2}\tilde{\psi}. \tag{30}$$

Because the function $\tilde{\psi}_c$ has exactly one zero, there is only one isolated eigenvalue below the continuous spectrum for the spectral problem (30). Because the difference between potentials of the spectral problems (29) and (30) as well as those of the spectral problems (25) and (28) is strictly positive for all $\omega \in (-1,1)$, the spectral problem (25) has exactly one isolated eigenvalue for all $\omega \in (-1,1)$ and this eigenvalue is negative for $\omega \in (-1,0)$, zero for $\omega = 0$, and positive for $\omega \in (0,1)$. □

LEMMA 2. *There is $\omega_0 \in (0,1]$ such that for any fixed $\omega \in (-\omega_0, \omega_0)$, operator $L_+$ has exactly two eigenvalues below the continuous spectrum. Besides the zero eigenvalue associated with the eigenvector in (23), $L_+$ also has a negative eigenvalue for $\omega \in (0,\omega_0)$ and a positive eigenvalue for $\omega \in (-\omega_0, 0)$, which is associated with the eigenvector in (24) for $\omega = 0$.*

*Proof.* Because the double zero eigenvalue of $L_+$ at $\omega = 0$ is isolated from the continuous spectrum located for $[1,\infty)$, the assertion of the lemma will follow by the perturbation theory if we can show that the zero eigenvalue is the only eigenvalue of $L_+$ at $\omega = 0$ and the endpoint of the continuous spectrum does not admit a resonance.

To develop the perturbation theory, we consider the eigenvalue problem $L_+\mathbf{u} = \mu\mathbf{u}$, where $\mathbf{u} = (u, \bar{u})$ is an eigenvector and $\mu$ is the spectral parameter. Using the transformation

$$u(x) = \varphi(x) e^{-3i\int_0^x |U_\omega(x')|^2 \mathrm{d}x'}$$

where $\varphi$ is a new eigenfunction, we obtain an equivalent spectral problem:

$$\begin{bmatrix} -\partial_x^2 + 1 - \omega^2 - 6\omega|U_\omega|^2 - 3|U_\omega|^4 & -6\omega W \\ -6\omega\bar{W} & -\partial_x^2 + 1 - \omega^2 - 6\omega|U_\omega|^2 - 3|U_\omega|^4 \end{bmatrix} \begin{bmatrix} \varphi \\ \bar{\varphi} \end{bmatrix} = \mu \begin{bmatrix} \varphi \\ \bar{\varphi} \end{bmatrix},$$

where

$$\begin{aligned} W &= U_\omega^2 e^{6i\int_0^x |U_\omega(x')|^2 \mathrm{d}x'} \\ &= (1-\omega^2) \frac{\left(1 + \omega\cosh\left(2\sqrt{1-\omega^2}x\right) + i\sqrt{1-\omega^2}\sinh\left(2\sqrt{1-\omega^2}x\right)\right)^2}{\left(\omega + \cosh\left(2\sqrt{1-\omega^2}x\right)\right)^3}. \end{aligned}$$

Setting now $z := \sqrt{1-\omega^2}x$ and $\mu := (1-\omega^2)\lambda$, we rewrite the spectral problem in the form

$$\begin{bmatrix} -\partial_z^2+1+V_1(z) & V_2(z) \\ \bar{V}_2(z) & -\partial_z^2+1+V_1(z) \end{bmatrix} \begin{bmatrix} \varphi \\ \bar{\varphi} \end{bmatrix} = \lambda \begin{bmatrix} \varphi \\ \bar{\varphi} \end{bmatrix}, \tag{31}$$

where

$$V_1(z) := -\frac{3(1-\omega^2)}{(\omega+\cosh(2z))^2} - \frac{6\omega}{\omega+\cosh(2z)}$$

and

$$V_2(z) := -6\omega \frac{\left(1+\omega\cosh(2z)+i\sqrt{1-\omega^2}\sinh(2z)\right)^2}{(\omega+\cosh(2z))^3}.$$

The spectral problem (31) for $\lambda=0$ admits the eigenvector $(\varphi_0, \bar{\varphi}_0)$ with

$$\varphi_0(z) = \frac{\omega\sinh(2z)+i\sqrt{1-\omega^2}\cosh(2z)}{(\omega+\cosh(2z))^{3/2}},$$

thanks to the existence of the eigenvector of $L_+$ in (23). Now, for $\omega=0$, $\lambda=0$ is a double zero eigenvalue of the spectral problem (31). The other eigenvector is $(\varphi_0, -\bar{\varphi}_0)$ and it corresponds to the eigenvector of $L_+$ in (24). The endpoint $\lambda=1$ of the continuous spectrum of the spectral problem (31) does not admit a resonance for $\omega=0$, which follows from the comparison results in Lemma 1. No other eigenvalues exist for $\omega=0$.

To study the splitting of the double zero eigenvalue if $\omega\neq 0$, we compute the quadratic form of the operator on the left-hand side of the spectral problem (31) at the vector $(\varphi_0, -\bar{\varphi}_0)$ to obtain

$$-2\int_{\mathbb{R}} \left(V_2+\bar{V}_2\right)|\varphi_0|^2 dz = -12\omega \int_{\mathbb{R}} \frac{-3+2\omega^2+\cosh(4z)}{(\omega+\cosh(2z))^4} dz.$$

Since the integral is positive for $\omega=0$, the perturbation theory (Theorem D in Appendix B) implies that the zero eigenvalue of the spectral problem (31) becomes negative for $\omega<0$ and positive for $\omega>0$ with sufficiently small $|\omega|$. □

CONJECTURE 1. *The spectral problem* (31) *has exactly two isolated eigenvalues and no endpoint resonances for all* $\omega\in(-1,1)$. *The nonzero eigenvalue is positive for all* $\omega\in(-1,0)$ *and negative for all* $\omega\in(0,1)$.

To illustrate Conjecture 1, we approximate eigenvalues of the spectral problem (31) numerically. We use the second–order central difference scheme for the second derivatives and the periodic boundary conditions. Figure 1 shows the only two isolated eigenvalues of the spectral problem (31) (asterisks) and the edge of the continuous spectrum at $\lambda=1$ (dashed line) versus parameter $\omega$. We confirm that the only nonzero isolated eigenvalue of $L_+$ is positive for $\omega\in(-1,0)$ and negative for $\omega\in(0,1)$.

We shall now consider eigenvalues of operators $L_\pm$ in the appropriately constrained spaces.

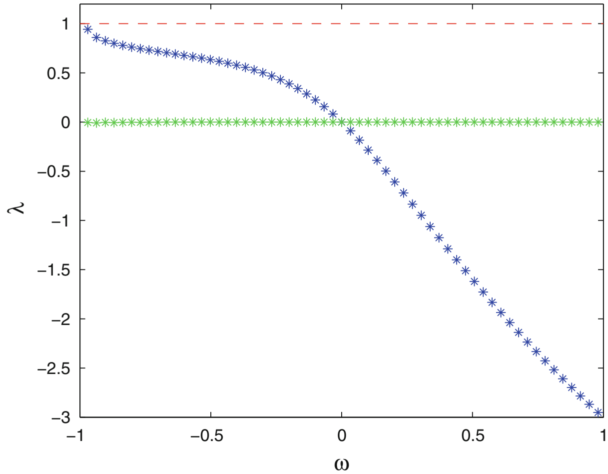


*Figure 1.* Isolated eigenvalues λ (*asterisks*) and the edge of the continuous spectrum $\lambda = 1$ (*dashed line*) versus parameter $\omega$ in the spectral problem (31).

LEMMA 3. *There is $\omega_0 \in (0, 1]$ such that for any $\omega \in (-\omega_0, \omega_0)$, operators $L_\pm$ have no negative eigenvalues and a simple zero eigenvalue in the constrained spaces $X_\pm$ defined by*

$$X_+ := \left\{ u \in L^2(\mathbb{R}) : \quad \int_{\mathbb{R}} \left( \bar{U}_\omega u + U_\omega \bar{u} \right) \mathrm{d}x = 0 \right\}, \tag{32}$$

$$X_- := \left\{ u \in L^2(\mathbb{R}) : \quad \int_{\mathbb{R}} \left( \bar{U}'_\omega u - U'_\omega \bar{u} \right) \mathrm{d}x = 0 \right\}. \tag{33}$$

*For operator $L_-$, the result extends to all $\omega \in (-1, 1)$.*

*Proof.* We use Theorems E in Appendix B and compute the value of $\sigma$ in this theorem explicitly both for operators $L_+$ and $L_-$.

For operator $L_+$, the constraint (32) yields the vector $\mathbf{s} = (U_\omega, \bar{U}_\omega)$. By taking derivative of Eq. (15) with respect to $\Omega$, we obtain

$$L_+ \begin{bmatrix} \partial_\Omega U_\omega \\ \partial_\Omega \bar{U}_\omega \end{bmatrix} = - \begin{bmatrix} U_\omega \\ \bar{U}_\omega \end{bmatrix}, \tag{34}$$

hence

$$\sigma := - \int_{\mathbb{R}} \left( \bar{U}_\omega \frac{\partial U_\omega}{\partial \Omega} + U_\omega \frac{\partial \bar{U}_\omega}{\partial \Omega} \right) \mathrm{d}x = \frac{1}{2\omega} \frac{\mathrm{d}}{\mathrm{d}\omega} \int_{\mathbb{R}} |U_\omega|^2 \mathrm{d}x = - \frac{1}{2\omega\sqrt{1-\omega^2}},$$

where we have used the exact expressions $\Omega = 1 - \omega^2$ and $\|U_\omega\|^2 = \arccos(\omega)$. We verify that $\sigma > 0$ for $\omega \in (-1, 0)$ and $\sigma < 0$ for $\omega \in (0, 1)$. By Lemma 2, $L_+$ has no negative eigenvalues for $\omega \in (-\omega_0, 0)$ and has one negative eigenvalue for $\omega \in (0, \omega_0)$, whereas the eigenvector $(U'_\omega, \bar{U}'_\omega)$ for zero eigenvalue of $L_+$ is orthogonal to the vector $\mathbf{s} = (U_\omega, \bar{U}_\omega)$. Conditions of Theorem E in Appendix B are satisfied and $L_+$ has no negative eigenvalues and a simple zero eigenvalue in the constrained space $X_+$ for all $\omega \in (-\omega_0, \omega_0)$. Note that the result holds also for $\omega = 0$, since the eigenvector $(U'_0, -\bar{U}'_0)$ in (24) does not belong to the constrained space $X_+$ because

$$\omega = 0: \quad \int_{\mathbb{R}} \left(\bar{U}_0 U'_0 - U_0 \bar{U}'_0\right) \mathrm{d}x = -i \int_{\mathbb{R}} \left(4|U_0|^4 - U_0^2 - \bar{U}_0^2\right) \mathrm{d}x = -2i \neq 0. \tag{35}$$

For operator $L_-$, the constraint (33) yields the vector $\mathbf{s} = (U'_\omega, -\bar{U}'_\omega)$. By using Eqs. (11) and (15), we obtain

$$L_- \left( -\frac{1}{2} x \begin{bmatrix} U_\omega \\ -\bar{U}_\omega \end{bmatrix} + \frac{1}{4i\omega} \begin{bmatrix} U_\omega \\ \bar{U}_\omega \end{bmatrix} \right) = \begin{bmatrix} U'_\omega \\ -\bar{U}'_\omega \end{bmatrix}, \tag{36}$$

hence

$$\begin{aligned} \sigma &:= \int_{\mathbb{R}} \left( \frac{1}{2} |U_\omega|^2 - \frac{1}{4i\omega} \left(\bar{U}_\omega U'_\omega - U_\omega \bar{U}'_\omega\right) \right) \mathrm{d}x \\ &= \frac{1}{4\omega} \int_{\mathbb{R}} \left( 4|U_\omega|^4 - U_\omega^2 - \bar{U}_\omega^2 + 4\omega |U_\omega|^2 \right) \mathrm{d}x \\ &= \frac{1-\omega^2}{2\omega} \int_{\mathbb{R}} \frac{1 + \omega \cosh(2\sqrt{1-\omega^2}x)}{(\omega + \cosh(2\sqrt{1-\omega^2}x))^2} \mathrm{d}x \\ &= \frac{\sqrt{1-\omega^2}}{2\omega}. \end{aligned}$$

We verify that $\sigma < 0$ for $\omega \in (-1, 0)$ and $\sigma > 0$ for $\omega \in (0, 1)$. By Lemma 1, $L_-$ has one negative eigenvalue for $\omega \in (-1, 0)$ and no negative eigenvalues for $\omega \in (0, 1)$, whereas the eigenvector $(U_\omega, -\bar{U}_\omega)$ for zero eigenvalue of $L_-$ is orthogonal to the vector $\mathbf{s} = (U'_\omega, -\bar{U}'_\omega)$. Conditions of Theorem E in Appendix B are satisfied and $L_-$ has no negative eigenvalues and a simple zero eigenvalue in the constrained space $X_-$ for all $\omega \in (-1, -1)$. Again, the result holds also for $\omega = 0$, since the eigenvector $(U_0, \bar{U}_0)$ of $L_-$ in (24) does not belong to the constrained space $X_-$ because of the same computation (35). □

REMARK 3. *Solutions of Eqs.* (34) *and* (36) *define the so-called generalized eigenvectors for the zero eigenvalue of the spectral stability problem associated with the MTM system* (1)*. These solutions are related to translation of the MTM solitons with respect to parameters ω and c. Indeed, from the Lorentz transformation* (13)*, we realize that the solution* (36) *is related to the derivative of the MTM soliton with respect to parameter c at* $c = 0$.

The proof of Theorem 3 follows from Lemma 3 and the fact that the eigenvectors of $L_+$ and $L_-$ in (23) are removed by adding the constraints

$$\tilde{X}_+ := \left\{ u \in L^2(\mathbb{R}) : \quad \int_{\mathbb{R}} \left( \bar{U}'_\omega u + U'_\omega \bar{u} \right) \mathrm{d}x = 0 \right\}, \tag{37}$$

$$\tilde{X}_- := \left\{ u \in L^2(\mathbb{R}) : \quad \int_{\mathbb{R}} \left( \bar{U}_\omega u - U_\omega \bar{u} \right) \mathrm{d}x = 0 \right\}. \tag{38}$$

Note that the constraints in $X_+$ and $\tilde{X}_-$ give real and imaginary parts of the complex-valued constraint (17), whereas the constraints in $\tilde{X}_+$ and $X_-$ give real and imaginary parts of the complex-valued constraint (18). Therefore, the matrix operator $L$ in (21) is strictly positive under the complex-valued constraints (17) and (18) for $\omega \in (-\omega_0, \omega_0)$ and the proof of Theorem 3 is complete.

The proof of Theorem 1 is based on the conservation of functionals $R$, $Q$, and $P$ for a solution of the MTM system (1) in $H^1(\mathbb{R}, \mathbb{C}^2)$ and the standard orbital stability arguments (Theorem F in Appendix B).

## Appendix A: Conserved Quantities by the Inverse Scattering Method

The MTM system (1) is a compatibility condition of the Lax system

$$\frac{\partial}{\partial x} \vec{\phi} = L \vec{\phi}, \quad \frac{\partial}{\partial t} \vec{\phi} = A \vec{\phi}, \tag{39}$$

where $\vec{\phi}(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2$ and $L$ is given by [27,32]:

$$L = \frac{i}{2}(|v|^2 - |u|^2)\sigma_3 - \frac{i\lambda}{\sqrt{2}} \begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{\sqrt{2}\lambda} \begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4} \left( \frac{1}{\lambda^2} - \lambda^2 \right) \sigma_3. \tag{40}$$

Let us consider a Jost function $\vec{\phi}(x;\lambda)$, which satisfies the boundary condition

$$\lim_{x \to -\infty} e^{-ik(\lambda)x} \vec{\phi}(x;\lambda) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{41}$$

where $k(\lambda) := \frac{1}{4}(\lambda^{-2} - \lambda^2)$ such that $k \in \mathbb{R}$ if $\lambda^2 \in \mathbb{R}$. This Jost function satisfies the scattering relation as $x \to +\infty$,

$$\vec{\phi}(x;\lambda) \sim a(\lambda) e^{ik(\lambda)x} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b(\lambda) e^{-ik(\lambda)x} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{42}$$

where $a(\lambda)$ and $b(\lambda)$ are spectral coefficients for $\lambda^2 \in \mathbb{R}$.

Setting

$$\vec{\phi}(x;\lambda) = \begin{bmatrix} 1 \\ \nu(x;\lambda) \end{bmatrix} \exp\left( ik(\lambda)x + \int_{-\infty}^{x} \chi(x';\lambda) dx' \right) \tag{43}$$

with two functions $\chi(x;\lambda)$ and $\nu(x;\lambda)$ satisfying the boundary conditions

$$\lim_{x\to-\infty}\chi(x;\lambda)=0 \quad \text{and} \quad \lim_{x\to-\infty}\nu(x;\lambda)=0$$

and taking a limit $x\to\infty$ in (43), we obtain

$$a(\lambda)=\exp\left(\int_{-\infty}^{\infty}\chi(x;\lambda)dx\right) \quad \Rightarrow \quad \log a(\lambda)=\int_{-\infty}^{\infty}\chi(x;\lambda)dx. \tag{44}$$

The scattering coefficient $a(\lambda)$ does not depend on time $t$, hence expansion of $\int_{-\infty}^{\infty}\chi(x;\lambda)\mathrm{d}x$ in powers of $\lambda$ yields conserved quantities with respect to $t$ [32]. Substituting Eq. (41) into the $x$-derivative part of the Lax system (39) and using Eq. (43), we obtain

$$\chi=\frac{i}{2}(|v|^2-|u|^2)-\frac{i}{\sqrt{2}}\left(\lambda\overline{v}+\frac{1}{\lambda}\overline{u}\right)\nu, \tag{45}$$

where $\nu$ satisfies a Ricatti equation

$$\nu_x+i\left(2k(\lambda)+|v|^2-|u|^2\right)\nu-\frac{i}{\sqrt{2}}\left(\lambda\overline{v}+\frac{1}{\lambda}\overline{u}\right)\nu^2+\frac{i}{\sqrt{2}}\left(\lambda v+\frac{1}{\lambda}u\right)=0. \tag{46}$$

To generate two hierarchies of conserved quantities, we consider the formal asymptotic expansion of $\chi(x;\lambda)$ in powers and inverse powers of $\lambda$:

$$\chi(x;\lambda)=\sum_{n=0}^{\infty}\lambda^n\chi_n(x), \quad \nu(x;\lambda)=\sum_{n=1}^{\infty}\lambda^n\nu_n(x) \tag{47}$$

and

$$\chi(x;\lambda)=\sum_{n=0}^{\infty}\frac{1}{\lambda^n}\widetilde{\chi}_n(x), \quad \nu(x;\lambda)=\sum_{n=1}^{\infty}\frac{1}{\lambda^n}\widetilde{\nu}_n(x). \tag{48}$$

We set

$$I_n:=\int_{-\infty}^{\infty}\chi_n(x)\mathrm{d}x, \quad I_{-n}:=\int_{-\infty}^{\infty}\widetilde{\chi}_n(x)\mathrm{d}x. \tag{49}$$

Substitution of the asymptotic expansions (47) and (48) for $\nu$ into Eq. (46) allows one to determine each $\nu_n$ and $\widetilde{\nu}_n$ from which Eq. (45) is used to determine $\chi_n$ and $\widetilde{\chi}_n$. Let us explicitly write out first conserved quantities

$$I_0=\int_{\mathbb{R}}(|u|^2+|v|^2)\mathrm{d}x,$$

$$I_2=\int_{\mathbb{R}}(-2u_x\overline{u}+i\overline{v}u+i\overline{u}v-2i|u|^2|v|^2)\mathrm{d}x,$$

$$I_{-2}=\int_{\mathbb{R}}(-2v_x\overline{v}-i\overline{v}u-i\overline{u}v+2i|u|^2|v|^2)\mathrm{d}x,$$

$$I_4 = \int_{\mathbb{R}} [-4i\overline{u}u_{xx} - 2(u_x\overline{v} + \overline{u}v_x) + 4\overline{u}(u|v|^2)_x + 4u_x\overline{u}(|u|^2+|v|^2) + i(|u|^2+|v|^2) - 2iu\overline{v}(|u|^2+|v|^2) - 2iv\overline{u}(|u|^2+|v|^2) + 4i|u|^2|v|^2(|u|^2+|v|^2)]\mathrm{d}x,$$

and

$$I_{-4} = \int_{\mathbb{R}} [4i\overline{v}v_{xx} - 2(u_x\overline{v} + \overline{u}v_x) + 4\overline{v}(v|u|^2)_x + 4v_x\overline{v}(|u|^2+|v|^2) - i(|u|^2+|v|^2) + 2iu\overline{v}(|u|^2+|v|^2) + 2iv\overline{u}(|u|^2+|v|^2) - 4i|u|^2|v|^2(|u|^2+|v|^2)]\mathrm{d}x.$$

We note that $I_0$ corresponds to charge $Q$ in (4). After integration by parts, $I_2 + I_{-2}$ corresponds to momentum $P$ in (5) and $I_2 - I_{-2}$ corresponds to Hamiltonian $H$ in (6). The higher-order Hamiltonian $R$ in (7) is obtained from $I_4 - I_{-4}$ after integration by parts and dropping the conserved quantity $Q$ from the definition of $R$.

Using Wolfram's MATHEMATICA, we also obtain the balance equation for $R$:

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0, \tag{50}$$

where

$$\rho = |u_x|^2 + |v_x|^2 - \frac{i}{2}(u_x\overline{u} - \overline{u}_x u)(|u|^2 + 2|v|^2) + \frac{i}{2}(v_x\overline{v} - \overline{v}_x v)(2|u|^2 + |v|^2) - (u\overline{v} + \overline{u}v)(|u|^2+|v|^2) + 2|u|^2|v|^2(|u|^2+|v|^2)$$

and

$$j = |u_x|^2 - |v_x|^2 - \frac{i}{2}(u_x\overline{u} - \overline{u}_x u)(|u|^2 + 2|v|^2) - \frac{i}{2}(v_x\overline{v} - \overline{v}_x v)(2|u|^2 + |v|^2) - \frac{1}{2}(u\overline{v} + \overline{u}v)(|u|^2 - |v|^2).$$

## Appendix B: Auxiliary Results used in this Work

We are using the following technical results. In the next four theorems, $L : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ stands for the linear Schrödinger operator given by

$$L := -\partial_x^2 + c + V(x),$$

where $V \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $c > 0$ is fixed.

THEOREM A. [39, Lemma 4.2]. *There exists a unique solution of* $Lu_0 = \lambda_0 u_0$ *for any* $\lambda_0 \leq c$ *such that* $u_0 \in H^2_{\mathrm{loc}}(\mathbb{R})$ *and* $\lim_{x\to-\infty} e^{-\sqrt{c-\lambda_0}x} u_0(x) = 1$. *If* $u_0$ *has* $n(\lambda_0)$ *zeros on* $\mathbb{R}$, *then there exist exactly* $n(\lambda_0)$ *eigenvalues of* $L$ *for any* $\lambda < \lambda_0$.

THEOREM B. [39, Theorem B.10]. *Let* $u(x;V)$ *be a solution of* $Lu = cu$ *such that*

$$\lim_{x\to-\infty} u(x;V) = 1.$$

*Assume that* $V_1(x) > V_2(x)$ *for all* $x \in \mathbb{R}$ *and* $u(x;V_2)$ *has one zero on* $\mathbb{R}$. *Then,* $u(x;V_1)$ *has at most one zero on* $\mathbb{R}$.

THEOREM C. [26, § I.6.10]. *There exists the smallest eigenvalue* $\lambda_0 < c$ *of* $L$ *if and only if*

$$\lambda_0 = \inf_{u \in H^1(\mathbb{R}): \|u\|_{L^2}=1} \int_{\mathbb{R}} \frac{1}{2}\left[(\partial_x u)^2 + cu^2 + V(x)u^2\right] dx < c.$$

*In particular, if* $\lambda_0 = 0$ *for* $V = V_0$ *and* $\Delta V > 0$*, then* $\lambda_0 \gtrless 0$ *for* $V = V_0 \pm \Delta V$.

THEOREM D. [26, § VII.4.6]. *Let* $\lambda_0 < c$ *be an isolated eigenvalue of* $L$ *with the eigenfunction* $u_0 \in H^2(\mathbb{R})$*. Then, the perturbed operator* $\tilde{L} := L + \Delta V$ *with* $\Delta V \in L^\infty(\mathbb{R})$ *has a perturbed eigenvalue* $\tilde{\lambda}_0$ *near* $\lambda_0$ *and the sign of* $\tilde{\lambda}_0 - \lambda_0$ *coincides with the sign of the quadratic form* $\langle \Delta V u_0, u_0 \rangle_{L^2}$.

THEOREM E. [39, Theorem 4.1]. *Assume that* $H$ *is a Hilbert space equipped with the inner product* $\langle \cdot, \cdot \rangle$*. Fix a vector* $\mathbf{s} \in H$*. Assume that* $L$ *is a self-adjoint operator on* $H$ *such that the number of negative eigenvalues of* $L$ *is* $n(L)$*, the eigenvectors of* $L$ *for the zero eigenvalue are orthogonal to* $\mathbf{s}$*, and the rest of the spectrum of* $L$ *is bounded away from zero. Then, the number of negative eigenvalues of* $L$ *in the constrained space*

$$H_c := \{\mathbf{u} \in H : \quad \langle \mathbf{s}, \mathbf{u} \rangle = 0\}$$

*is defined by the sign of*

$$\sigma := \langle L^{-1}\mathbf{s}, \mathbf{s} \rangle.$$

*If* $\sigma > 0$*, then the number of negative eigenvalues of* $L$ *under the constraint is* $n(L)$*, whereas if* $\sigma < 0$*, then this number is* $n(L) - 1$.

THEOREM F. [39, Theorem 4.15]. *Let* $X = H^1(\mathbb{R}, \mathbb{C}^2)$ *be the energy space for the solution* $\vec{\psi} := (u, v)$ *of the MTM system* (1)*. Let* $\vec{\phi}_\omega := (U_\omega, \overline{U}_\omega)$ *be a local nondegenerate minimizer of* $R$ *in* $X$ *under the constraints of fixed* $Q$ *and* $P$ *for some* $\omega \in (-1, 1)$*. Then,* $\vec{\phi}_\omega$ *is orbitally stable in* $X$ *with respect to the time evolution of the MTM system* (1)*. In other words, for any* $\epsilon > 0$*, there is* $\delta > 0$ *such that if the initial datum satisfies*

$$\inf_{\alpha, \beta \in \mathbb{R}} \|\vec{\psi}|_{t=0} - e^{i\alpha}\vec{\phi}_\omega(\cdot + \beta)\|_X < \delta,$$

*then for all* $t > 0$*, we have*

$$\inf_{\alpha, \beta \in \mathbb{R}} \|\vec{\psi} - e^{i\alpha}\vec{\phi}_\omega(\cdot + \beta)\|_X < \epsilon.$$

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