Stability of discrete breathers in magnetic metamaterials

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Summary. We consider the discrete Klein–Gordon equation for magnetic metamaterials derived by Eleftheriou, Lazarides, and Tsironis [4, 7]. We obtain a general criterion for spectral stability of multi-site breathers for a small coupling constant. We show how this criterion differs from the one derived in the standard discrete Klein–Gordon equation [6, 13].

1 Introduction

We address space-localized and time-periodic breathers in the discrete Klein–Gordon equation describing magnetic metamaterials which consist of periodic arrays of split-ring resonators [4, 7]:

$$\ddot{q}_n + V'(q_n) = \epsilon(\ddot{q}_{n+1} + \ddot{q}_{n-1}), \quad n \in \mathbb{Z}, \tag{1}$$

where $t \in \mathbb{R}$ is the evolution time, $q_n(t) \in \mathbb{R}$ is the normalized charge stored in the capacitor of the n-th split-ring resonator, $V : \mathbb{R} \to \mathbb{R}$ is a smooth on-site potential for the voltage across the slit of the n-th resonator, and $\epsilon \in \mathbb{R}$ is the coupling constant from the mutual inductance. In particular, the voltage $u = f(q) = V'(q)$ is found by inverting the charge-voltage dependence near small charge:

$$q = u + \alpha u^3 \quad \Rightarrow \quad u = f(q) = q - \alpha q^3 + \mathcal{O}(q^5) \quad \text{as} \quad q \to 0, \tag{2}$$

where α is the parameter for the self-focusing ($\alpha > 0$) or self-defocusing ($\alpha < 0$) nonlinearity. These parameter values correspond to the soft and hard potentials V respectively, for sufficiently small values of q. Note that V is an even function of q.

Discrete breathers in both one-dimensional and two-dimensional lattices were approximated numerically in the limit of small coupling constant ϵ [4, 7]. Excitations of discrete breathers near the edge of a one-dimensional lattice created by a truncated array of nonlinear split-ring resonators were considered numerically in [8].

It is the purpose of this paper to consider spectral stability of multi-site discrete breathers in the limit of small coupling constant ϵ. This limit is referred usually as the *anti-continuum* limit and it has been considered before in the context of spectral stability of discrete breathers in the standard discrete Klein–Gordon equation [1, 3, 6, 10]. Recent works [13, 14, 18] were devoted to the derivation of the most general stability criterion for multi-site breathers in Klein–Gordon lattices. Our main result shows that the stability criterion for multi-site breathers in the discrete Klein–Gordon equation (1) differs from the one derived in the standard discrete Klein–Gordon equation [6, 13].

The paper is organized as follows. We formulate the discrete Klein–Gordon equation (1) as an evolution problem in Section 2. The existence and continuation results for multi-site discrete breathers in the limit of small coupling constant ϵ are reviewed in Section 3. Spectral stability of multi-site breathers for small coupling constants is considered in Section 4. Section 5 discusses application of the stability criterion to the multi-site breathers in magnetic metamaterials.

2 Formalism

In what follows, we shall use bold-faced notations for vectors in discrete space $l^p(\mathbb{Z})$ defined by their norms

$$\|\mathbf{q}\|_{l^p} := \left(\sum_{n \in \mathbb{Z}} |q_n|^p \right)^{1/p}, \quad p \geq 1.$$

Components of $\mathbf{q}$ are denoted by q_n for $n \in \mathbb{Z}$. These components can be functions of t, in which case they can be considered either in the space $C^2(0,T)$ of twice continuously differentiable functions on $(0,T)$ or in the L^2-based Sobolev space $H^s_{\rm per}(0,T)$ of T-periodic functions equipped with the norm,

$$\|f\|_{H^s_{\rm per}} := \left(\sum_{m \in \mathbb{Z}} (1+m^2)^s |c_m|^2 \right)^{1/2}, \quad s \geq 0,$$

where the coefficients $\{c_m\}_{m\in\mathbb{Z}}$ define the Fourier series of a T-periodic function f,

$$f(t) = \sum_{m \in \mathbb{Z}} c_m \exp\left(\frac{2\pi i m t}{T} \right), \quad t \in [0,T].$$

To start analysis, we set up the discrete Klein–Gordon equation (1) as an evolution problem in t in the phase space $C^2([0,T), l^2(\mathbb{Z}))$, where $T > 0$ is the maximal existence time (which may be infinite). Let us consider the bounded operator

$$M(\epsilon) = I - \epsilon(\sigma_+ + \sigma_-) : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}),$$

where the shift operators $\sigma_\pm : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ are defined by

$$(\sigma_\pm \mathbf{q})_n = q_{n\pm 1}, \quad n \in \mathbb{Z}. \tag{3}$$

For any $\epsilon \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, the operator $M(\epsilon) : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is diagonally dominant and hence invertible and the inverse operator $M^{-1}(\epsilon) : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is bounded. Moreover, the operator $M^{-1}(\epsilon)$ is analytic at $\epsilon = 0$ and admits the Taylor series,

$$M^{-1}(\epsilon) = I + \sum_{k=1}^{\infty} \epsilon^k (\sigma_+ + \sigma_-)^k, \quad \epsilon \in \left(-\frac{1}{2}, \frac{1}{2}\right). \tag{4}$$

The discrete Klein–Gordon equation (1) can be written in the operator form as follows:

$$M(\epsilon)\frac{d^2\mathbf{q}}{dt^2} + \mathbf{f}(\mathbf{q}) = \mathbf{0}, \tag{5}$$

where $(\mathbf{f}(\mathbf{q}))_n = V'(q_n)$. Inverting $M(\epsilon)$ for any $\epsilon \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, we obtain the evolution form of the discrete Klein–Gordon equation (5):

$$\frac{d^2\mathbf{q}}{dt^2} + M^{-1}(\epsilon)\mathbf{f}(\mathbf{q}) = \mathbf{0}. \tag{6}$$

With this formulation, we prove the first result on local existence of solutions of the Cauchy problem associated with the evolution equation (6).

Proposition 1. *Let $V \in C^2(\mathbb{R})$ and $\mathbf{q}_0, \mathbf{q}_1 \in l^2(\mathbb{Z})$. For any $\epsilon \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, there exist $T > 0$ and a unique local solution of the evolution problem (6) in the phase space $\mathbf{q} \in C^2([0,T), l^2(\mathbb{Z}))$ such that $\mathbf{q}(0) = \mathbf{q}_0$ and $\dot{\mathbf{q}}(0) = \mathbf{q}_1$.*

Proof. Because $V \in C^2(\mathbb{R})$ and $l^2(\mathbb{Z})$ is a Banach algebra with respect to pointwise multiplication, the map $\mathbf{f}(\mathbf{q}) : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is C^1. For any $\epsilon \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, there exists $C(\epsilon) > 0$ such that $\|M^{-1}(\epsilon)\|_{l^2 \to l^2} \leq C(\epsilon)$. Therefore, the vector field $M^{-1}(\epsilon)\mathbf{f}(\mathbf{q})$ is a bounded C^1 map from $l^2(\mathbb{Z})$ to $l^2(\mathbb{Z})$, hence, it is locally Lipschitz. The result of Proposition 1 follows from the standard existence theory of second-order evolution equations in Banach spaces [2, Chapter 2]. □

Remark 1. For the particular function V defined by (2), we note that the assumption $V \in C^2(\mathbb{R})$ is satisfied for any $\alpha > 0$ (in which case, $V \in C^\infty(\mathbb{R})$), because $1+3\alpha u^2 > 0$ for all $u \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is one-to-one and onto. However, for $\alpha < 0$, the function f is one-to-one and onto on $(-Q_0, Q_0)$ with the range in $(-U_0, U_0)$, where

$$Q_0 = \frac{2}{\sqrt{27|\alpha|}}, \quad U_0 = \frac{1}{\sqrt{3|\alpha|}}. \tag{7}$$

Therefore, in this case, we only have $V \in C^2(-Q_0, Q_0)$ (in fact, $V \in C^\infty(-Q_0, Q_0)$), so a unique local solution of the evolution problem (6) exists if $\mathbf{q}_0 \in l^2(\mathbb{Z})$ satisfies further restriction: $(\mathbf{q}_0)_n \in (-Q_0, Q_0)$ for all $n \in \mathbb{Z}$.

Remark 2. We shall only consider the Klein–Gordon lattice (1) with the nonlinear potential (2) for small values of q_n.

3 Existence of multi-site discrete breathers

We consider space-localized and time-periodic breathers of the discrete Klein–Gordon equation (1) in the space $\mathbf{q} \in H^2_{\text{per}}((0,T), l^2(\mathbb{Z}))$, where $T > 0$ represents the fundamental period.

Remark 3. Note that the space $H^2_{\rm per}((0,T),l^2(\mathbb{Z}))$ for discrete breathers is actually weaker than the space $C^2([0,T),l^2(\mathbb{Z}))$, for which the existence of a unique local solution is established in Proposition 1, but Sobolev's embedding of $H^2_{\rm per}(0,T)$ to $C_{\rm per}(0,T)$ and the bootstrapping arguments from the evolution equation (6) show that if $\mathbf{q} \in H^2_{\rm per}((0,T),l^2(\mathbb{Z}))$, then $\mathbf{q} \in C^2_{\rm per}((0,T),l^2(\mathbb{Z}))$ (the opposite is true immediately).

Accounting for symmetries, we shall work in the restriction of $H^2_{\rm per}(0,T)$ to the space of even T-periodic functions,

$$H^2_e(0,T) = \left\{ f \in H^2_{\rm per}(0,T) : \quad f(-t) = f(t), \quad t \in \mathbb{R} \right\}.$$

We shall also assume everywhere that the nonlinear potential V is an even function of q, which agrees with the potential defined by (2). This assumption is not very restrictive and is used to simplify the technical computations.

At $\epsilon = 0$, we have many possible configurations of multi-site breathers,

$$\mathbf{Q}^{(0)}(t) = \sum_{k \in S} \sigma_k \varphi(t) \mathbf{e}_k, \tag{8}$$

where $\mathbf{e}_k$ is the unit vector in $l^2(\mathbb{Z})$ associated with the site $k \in \mathbb{Z}$, $S \subset \mathbb{Z}$ is a finite set of excited sites of the lattice, $\sigma_k \in \{+1,-1\}$ encodes the phase factor of the k-th oscillator, and $\varphi \in H^2_e(0,T)$ is an even solution of the nonlinear oscillator equation at the energy level E,

$$\ddot{\varphi} + V'(\varphi) = 0 \quad \Rightarrow \quad E = \frac{1}{2}\dot{\varphi}^2 + V(\varphi). \tag{9}$$

Remark 4. Note that if φ is a solution of (9), then $-\varphi$ is also a solution of (9) because V' is an odd function of φ. This motivates the notations in (8) due to the technical simplification that V is even. If V is of general type, we would need to modify the representation formula (8) and the subsequent analysis.

The unique even solution $\varphi(t)$ satisfies the initial condition,

$$\varphi(0) = a, \quad \dot{\varphi}(0) = 0, \tag{10}$$

where a is the smallest positive root of $V(a) = E$ for a fixed value of E. Period of oscillations T is uniquely defined by the energy level E, according to the following formula:

$$T = \sqrt{2} \int_{-a}^{a} \frac{d\varphi}{\sqrt{E - V(\varphi)}}. \tag{11}$$

Remark 5. All nonlinear oscillators at the excited sites of $S \subset \mathbb{Z}$ in the configuration (8) have the same period T. Two oscillators at the j-th and k-th sites are said to be in-phase if $\sigma_j \sigma_k = 1$ and anti-phase if $\sigma_j \sigma_k = -1$.

Persistence of the limiting configuration (8) as a space-localized and time-periodic breather of the discrete Klein–Gordon equation (1) for small values of ϵ is established by MacKay & Aubry [9]. Using this theory, we prove the next result on the existence and continuation of the multi-site discrete breathers.

Proposition 2. *Fix the period T and the solution $\varphi \in H^2_e(0,T)$ of the nonlinear oscillator equation (9) with an even function $V \in C^\infty(\mathbb{R})$ such that $V''(0) = 1$. Assume that $T \neq 2\pi n$, $n \in \mathbb{N}$ and $T'(E) \neq 0$. Define $\mathbf{Q}^{(0)}$ by the representation (8) with fixed finite $S \subset \mathbb{Z}$ and $\{\sigma_k\}_{k\in S}$. There are $\epsilon_0 \in \left(0, \frac{1}{2}\right)$ and $C > 0$ such that for all $\epsilon \in (-\epsilon_0, \epsilon_0)$, there exists a unique solution $\mathbf{Q}^{(\epsilon)} \in H^2_e((0,T), l^2(\mathbb{Z}))$ of the discrete Klein–Gordon equation (1) satisfying*

$$\|\mathbf{Q}^{(\epsilon)} - \mathbf{Q}^{(0)}\|_{H^2_{\rm per}((0,T),l^2(\mathbb{Z}))} \leq C|\epsilon|. \tag{12}$$

Moreover, the map $(-\epsilon_0, \epsilon_0) \ni \epsilon \mapsto \mathbf{Q}^{(\epsilon)} \in H^2_e((0,T), l^2(\mathbb{Z}))$ is C^∞.

Proof. We shall write $M^{-1}(\epsilon) = I + \epsilon K(\epsilon)$, where $K(\epsilon) : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is a bounded analytic operator for all $\epsilon \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Then, the discrete Klein–Gorodn equation (1) for T-periodic solutions $\mathbf{Q}$ can be rewritten in the perturbed form:

$$\frac{d^2\mathbf{Q}}{dt^2} + \mathbf{f}(\mathbf{Q}) = -\epsilon K(\epsilon)\mathbf{f}(\mathbf{Q}), \tag{13}$$

where $(\mathbf{f}(\mathbf{Q}))_n = V'(Q_n)$.

Substituting $\mathbf{Q} = \mathbf{Q}^{(0)} + \mathbf{W}$, where $\mathbf{Q}^{(0)}$ is given by (8), we obtain the coupled system of differential-difference equations in the form:

$$L_e W_n = -\epsilon \left(K(\epsilon)\mathbf{f}(\mathbf{Q}^{(0)})\right)_n + \left(\mathbf{N}(\mathbf{W},\epsilon)\right)_n, \quad n \in S \tag{14}$$

and

$$L_0 W_n = -\epsilon \left(K(\epsilon)\mathbf{f}(\mathbf{Q}^{(0)})\right)_n + \left(\mathbf{N}(\mathbf{W},\epsilon)\right)_n, \quad n \in \mathbb{Z}\backslash S, \tag{15}$$

where the linear operators are

$$\begin{aligned} L_e &= \partial_t^2 + V''(\varphi(t)) : & H^2_e(0,T) \to L^2_e(0,T), \\ L_0 &= \partial_t^2 + 1 : & H^2_e(0,T) \to L^2_e(0,T) \end{aligned}$$

and the nonlinear vector field is

$$\begin{aligned} \left(\mathbf{N}(\mathbf{W},\epsilon)\right)_n = &-\epsilon \left(K(\epsilon)(\mathbf{f}(\mathbf{Q}^{(0)} + \mathbf{W}) - \mathbf{f}(\mathbf{Q}^{(0)})\right)_n \\ &+ V'(Q_n^{(0)}) + V''(Q_n^{(0)})W_n - V'(Q_n^{(0)} + W_n). \end{aligned}$$

We have used here that V is even and $V''(0) = 1$.

Under the condition $T'(E) \neq 0$, the operator L_e is invertible, because the only eigenvector $\dot{\varphi}$ of $L = \partial_t^2 + V''(\varphi(t)) : H^2_{\rm per}(0,T) \to L^2_{\rm per}(0,T)$ is odd in t. Similarly, operator L_0 is invertible if $T \neq 2\pi n$, $n \in \mathbb{N}$.

Thanks to Banach algebra of $H^2_e((0,T), l^2(\mathbb{Z}))$ and the assumption $V \in C^\infty(\mathbb{R})$, the map $\mathbf{N}(\mathbf{Q}, \epsilon) : H^2_e((0,T), l^2(\mathbb{Z})) \times \mathbb{R} \to H^2_e((0,T), l^2(\mathbb{Z}))$ is C^∞, hence it is locally Lipschitz. Thanks to the invertibility of the linearized operators L_e and L_0 on $L^2_e(0,T)$, the result of the theorem follows from the Implicit Function Theorem and the map $\epsilon \mapsto \mathbf{Q}^{(\epsilon)}$ is C^∞ for small ϵ (Theorem 4.E in [19]).□

Remark 6. Although persistence of other breather configurations, where oscillators are neither in-phase nor anti-phase, can not be apriori excluded, we restrict our studies to the breather configurations covered by Proposition 2.

4 Stability of multi-site breathers

Let $\mathbf{Q} \in H^2_e((0,T), l^2(\mathbb{Z}))$ be a multi-site breather in Proposition 2. To study the spectral stability of multi-site breathers, we substitute the decomposition $\mathbf{q}(t) = \mathbf{Q}(t) + \mathbf{w}(t)$ to the discrete Klein–Gordon equation (1), neglect quadratic and higher-order terms in $\mathbf{w}$, and obtain the linearized discrete Klein–Gordon equation,

$$\ddot{w}_n + V''(Q_n) w_n = \epsilon \left(\ddot{w}_{n+1} + \ddot{w}_{n-1} \right), \quad n \in \mathbb{Z}. \tag{16}$$

Using the abstract evolution form (6) and the decomposition $M^{-1}(\epsilon) = I + \epsilon K(\epsilon)$, we can rewrite the linearized equations (16) in the equivalent form:

$$\frac{d^2 \mathbf{w}}{dt^2} + \mathbf{f}'(\mathbf{Q}) \mathbf{w} = -\epsilon K(\epsilon) \mathbf{f}'(\mathbf{Q}) \mathbf{w}, \tag{17}$$

where $\mathbf{f}'(\mathbf{Q})$ is the diagonal operator with entries $V''(Q_n)$, $n \in \mathbb{Z}$.

Because $\mathbf{Q}(t+T) = \mathbf{Q}(t)$, an infinite-dimensional analogue of the Floquet theorem applies and the Floquet monodromy matrix $\mathcal{M}$ is defined by $\mathbf{w}(T) = \mathcal{M} \mathbf{w}(0)$. We say that the breather is stable if all eigenvalues of $\mathcal{M}$, called Floquet multipliers, are located on the unit circle and it is unstable if there is at least one Floquet multiplier outside the unit disk. Because the linearized system (16) is reversible, Floquet multipliers come in pairs μ_1 and μ_2 with $\mu_1 \mu_2 = 1$.

To consider Floquet multipliers, we can introduce the characteristic exponent λ in the decomposition $\mathbf{w}(t) = \mathbf{W}(t) e^{\lambda t}$. If $\mu = e^{\lambda T}$ is the Floquet multiplier of the monodromy operator $\mathcal{M}$, then $\mathbf{W} \in H^2_{\rm per}((0,T), l^2(\mathbb{Z}))$ is a solution of the eigenvalue problem,

$$\frac{d^2 \mathbf{W}}{dt^2} + 2\lambda \frac{d\mathbf{W}}{dt} + \lambda^2 \mathbf{W} + \mathbf{f}'(\mathbf{Q}) \mathbf{W} = -\epsilon K(\epsilon) \mathbf{f}'(\mathbf{Q}) \mathbf{W}. \tag{18}$$

In particular, Floquet multiplier $\mu = 1$ corresponds to the characteristic exponent $\lambda = 0$. The generalized eigenvector $\mathbf{Z}_0 \in H^2_{\rm per}((0,T), l^2(\mathbb{Z}))$ of the eigenvalue problem (18) for $\lambda = 0$ solves the inhomogeneous problem,

$$\frac{d^2 \mathbf{Z}_0}{dt^2} + \mathbf{f}'(\mathbf{Q}) \mathbf{Z}_0 = -\epsilon K(\epsilon) \mathbf{f}'(\mathbf{Q}) \mathbf{Z}_0 - 2 \frac{d\mathbf{W}_0}{dt}, \tag{19}$$

where $\mathbf{W}_0$ is the eigenvector of (18) for $\lambda = 0$. To normalize $\mathbf{Z}_0$ uniquely, we add a constraint that $\mathbf{Z}_0$ is orthogonal to $\mathbf{W}_0$ with respect to the natural inner product

$$\langle \mathbf{W}_0, \mathbf{Z}_0 \rangle_{L^2_{\rm per}((0,T), l^2(\mathbb{Z}))} := \int_0^T \sum_{n \in \mathbb{Z}} (\bar{\mathbf{Z}}_0)_n(t) (\mathbf{W}_0)_n(t) dt.$$

At $\epsilon = 0$, the eigenvector $\mathbf{W}_0$ of the eigenvalue problem (18) for $\lambda = 0$ is spanned by the linear combination of N fundamental solutions,

$$\mathbf{W}^{(0)}(t) = \sum_{k \in S} c_k \dot{\varphi}(t) \mathbf{e}_k, \tag{20}$$

where N is the number of sites in the set S. The generalized eigenvector $\mathbf{Z}_0$ is spanned by the linear combination of N solutions,

$$\mathbf{Z}^{(0)}(t) = -\sum_{k \in S} c_k v(t) \mathbf{e}_k, \quad v := 2 L_e^{-1} \ddot{\varphi}, \tag{21}$$

where $L_e = \partial_t^2 + V''(\varphi(t)) : H_e^2(0,T) \to L_e^2(0,T)$ is invertible and $\ddot{\varphi} \in L_e^2(0,T)$. Note that $\langle \dot{\varphi}, v \rangle_{L^2_{\rm per}(0,T)} = 0$ because $\dot{\varphi}$ is odd and v is even in t.

We proceed now with perturbation expansions for particular configurations S of the limiting breather (8). Perturbation expansions are different depending if the set S has no holes (the excited oscillators are adjacent) or includes some holes (oscillators at rest are located between excited oscillators).

4.1 Adjacent excited oscillators

We consider here the set $S = \{1, 2, ..., N\}$ of N adjacent sites with excited oscillators at $\epsilon = 0$. By Proposition 2, the breather solution $\mathbf{Q}^{(\epsilon)}$ can be expanded into the power series

$$\mathbf{Q}^{(\epsilon)} = \mathbf{Q}^{(0)} + \sum_{m=1}^{\infty} \epsilon^m \mathbf{Q}^{(m)}, \tag{22}$$

where $\mathbf{Q}^{(0)}(t) = \sum_{k=1}^{N} \sigma_k \varphi(t) \mathbf{e}_k$ and the correction terms are computed recursively from the system of linear inhomogeneous equations. In particular, for the first-order correction term, we write the linear inhomogeneous problem explicitly as follows:

$$\left(\frac{d^2}{dt^2} + V''(Q_n^{(0)}) \right) Q_n^{(1)} = - \left((\sigma_+ + \sigma_-) \mathbf{f}(\mathbf{Q}^{(0)}) \right)_n, \quad n \in \mathbb{Z}. \tag{23}$$

where again $(\mathbf{f}(\mathbf{Q}))_n = V'(Q_n)$.

Let φ be an even T-periodic solution of the nonlinear oscillator equation (9) subject to the initial conditions (10). Let ψ and ϕ be even T-periodic solutions of the linear inhomogeneous equations

$$\ddot{\psi} + V''(\varphi)\psi = V'(\varphi), \tag{24}$$

and

$$\ddot{\phi} + \phi = V'(\varphi). \tag{25}$$

Note that the unique even solutions exist for the linear equations (24) and (25) under the conditions $T'(E) \neq 0$ and $T \neq 2\pi m$, $m \in \mathbb{N}$ because of invertibility of operators L_e and L_0 defined in the proof of Proposition 2. By using solutions ψ and ϕ, we can write the first-order correction term $\mathbf{Q}^{(1)}$ explicitly as follows:

$$\mathbf{Q}^{(1)}(t) = -\sigma_1 \phi(t) \mathbf{e}_0 - \sum_{k=1}^{N} (\sigma_{k-1} + \sigma_{k+1}) \psi(t) \mathbf{e}_k - \sigma_N \phi(t) \mathbf{e}_{N+1}, \tag{26}$$

where we have used the convention: $\sigma_0 = \sigma_{N+1} = 0$. The following theorem represents the main result of the perturbation computations.

Theorem 1. *Under assumptions of Proposition 2, let* $\mathbf{Q}^{(0)} = \sum_{k=1}^{N} \sigma_k \varphi \mathbf{e}_k$ *yield a solution* $\mathbf{Q}^{(\epsilon)} \in H_e^2((0,T), l^2(\mathbb{Z}))$ *of the discrete Klein–Gordon equation (1) for small* $\epsilon > 0$*. Then the eigenvalue problem (18) for small* $\epsilon > 0$ *has* $2N$ *small eigenvalues,*

$$\lambda = \epsilon^{1/2} \Lambda + \mathcal{O}(\epsilon),$$

where Λ *is an eigenvalue of the matrix eigenvalue problem*

$$\frac{T^2(E)}{T'(E)M_1}\Lambda^2 \mathbf{c} = \mathcal{S}\mathbf{c}, \quad \mathbf{c} \in \mathbb{C}^N. \tag{27}$$

Here M_1 is a positive numerical coefficient given by

$$M_1 = \int_0^T \ddot{\varphi}^2 dt > 0$$

and the $N \times N$ matrix $\mathcal{S}$ is given by

$$\mathcal{S} = \begin{bmatrix} -\sigma_1\sigma_2 & 1 & 0 & \dots & 0 & 0 \\ 1 & -\sigma_2(\sigma_1+\sigma_3) & 1 & \dots & 0 & 0 \\ 0 & 1 & -\sigma_3(\sigma_2+\sigma_4) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\sigma_{M-1}(\sigma_{M-2}+\sigma_M) & 1 \\ 0 & 0 & 0 & \dots & 0 & -\sigma_M\sigma_{M-1} \end{bmatrix}.$$

Proof. At $\epsilon = 0$, the eigenvalue problem (18) admits eigenvalue $\lambda = 0$ of geometric multiplicity N and algebraic multiplicity $2N$, which is isolated from the rest of the spectrum. Perturbation theory in ϵ applies thanks to the smoothness of $\mathbf{Q}^{(\epsilon)}$ in ϵ and V' in u. Perturbation expansions (so-called Puiseux series, see Chapter 2 in [5]) are smooth in powers of $\epsilon^{1/2}$ thanks to the Jordan block decomposition at $\epsilon = 0$.

We need to find out how the eigenvalue $\lambda = 0$ of algebraic multiplicity $2N$ split for small $\epsilon > 0$. Therefore, we are looking for the eigenvectors of the eigenvalue problem (18) in the subspace associated with the eigenvalue $\lambda = 0$ using the substitution $\lambda = \epsilon^{1/2}\tilde{\lambda}$ and the decomposition

$$\mathbf{W} = \mathbf{W}^{(0)} + \epsilon^{1/2}\tilde{\lambda}\mathbf{W}^{(1)} + \epsilon\tilde{\mathbf{W}},$$

where

$$\mathbf{W}^{(0)} = \sum_{k=1}^{N} c_k \dot{\varphi} \mathbf{e}_k$$

and

$$\mathbf{W}^{(1)} = -\sum_{k=1}^{N} c_k v(t) \mathbf{e}_k, \quad v := 2L_e^{-1}\ddot{\varphi}.$$

The error term $\tilde{\mathbf{W}}$ satisfies a residual equation, which we only write on the active sites $k \in S$ in the perturbation form:

$$\begin{aligned} \ddot{\tilde{W}}_k + V''(\varphi)\tilde{W}_k = & -\Big((\sigma_+ + \sigma_-)\mathbf{f}'(\mathbf{Q}^{(0)})\mathbf{W}^{(0)}\Big)_k - c_k\sigma_k V'''(\varphi)Q_k^{(1)}\dot{\varphi} \\ & + \tilde{\lambda}^2 c_k(2\dot{v} - \dot{\varphi}) + \mathcal{O}(\epsilon^{1/2}), \end{aligned} \tag{28}$$

where again $\mathbf{f}'(\mathbf{Q})$ is the diagonal operator with entries $V''(Q_n)$, $n \in \mathbb{Z}$.

Expanding $\tilde{\lambda} = \Lambda + \mathcal{O}(\epsilon^{1/2})$, projecting the system of linear inhomogeneous equations (28) to $\dot{\varphi} \in H^2_{\rm per}(0,T)$, the kernel of $L = \partial_t^2 + V''(\varphi) : H^2_{\rm per}(0,T) \to L^2_{\rm per}(0,T)$, and truncating at the leading order, we obtain the system of difference equations for $k \in S$:

$$\Lambda^2 c_k \int_0^T \left(\dot{\varphi}^2 + 2v\ddot{\varphi}\right) dt = -(c_{k+1} + c_{k-1}) \int_0^T V''(\varphi)\dot{\varphi}^2 dt$$
$$+\sigma_k(\sigma_{k+1} + \sigma_{k-1})c_k \int_0^T V'''(\varphi)\dot{\varphi}^2 \psi dt, \qquad (29)$$

where the explicit expression (26) has been used, as well as the convention: $c_0 = c_{N+1} = 0$.

It is proved with the standard computation [13] that

$$\int_0^T \left(\dot{\varphi}^2 + 2v\ddot{\varphi}\right) dt = -\frac{T^2(E)}{T'(E)}. \qquad (30)$$

On the other hand, differentiating the linear inhomogeneous equation (24) and projecting it to $\dot{\varphi}$, we infer that

$$\int_0^T V'''(\varphi)\dot{\varphi}^2 \psi dt = \int_0^T V''(\varphi)\dot{\varphi}^2 dt, \qquad (31)$$

Using now the equation

$$\dddot{\varphi} + V''(\varphi)\dot{\varphi} = 0, \qquad (32)$$

we finally obtain

$$\int_0^T V''(\varphi)\dot{\varphi}^2 dt = -\int_0^T \dot{\varphi}\dddot{\varphi} dt = \int_0^T \ddot{\varphi}^2 dt = M_1.$$

Combining all together, the system of difference equations (29) yields the matrix eigenvalue problem (27), which defines $2N$ small eigenvalues that bifurcate from $\lambda = 0$ for small $\epsilon > 0$. □

We recall the result obtained by Sandstede in Lemma 5.4 and Appendix C [15], which we reproduce here without a proof.

Proposition 3. *Let n_0 be the number of negative elements in $\{\sigma_j \sigma_{j+1}\}_{j=1}^{N-1}$. Matrix $\mathcal{S}$ in Theorem 1 has exactly n_0 positive and $N-1-n_0$ negative eigenvalues counting their multiplicities, in addition to the simple zero eigenvalue.*

Remark 7. Because $M_1 > 0$, the matrix eigenvalue problem (27) differs from the similar reduction for the standard Klein–Gordon equation in [13] by the sign change in front of the matrix $\mathcal{S}$. In particular, if $T'(E) < 0$, the matrix eigenvalue problem (27) has n_0 pairs of purely imaginary eigenvalues Λ and $N-1-n_0$ pairs of purely real eigenvalues Λ counting their multiplicities, in addition to the double zero eigenvalue. If $T'(E) > 0$, the conclusion changes to the opposite.

4.2 Oscillators at rest between excited oscillators

We consider here the set $S = \{1, 3, ..., 2N-1\}$ of N sites with excited oscillators separated by exactly one oscillator at rest at $\epsilon = 0$. By using the power series expansions (22) in Proposition 2 with $\mathbf{Q}^{(0)}(t) = \sum_{k=1}^N \sigma_{2k-1}\varphi(t)\mathbf{e}_{2k-1}$, we compute a different explicit solution of the linear inhomogeneous equation (23):

$$\mathbf{Q}^{(1)}(t) = -\sum_{k=0}^{N}(\sigma_{2k-1} + \sigma_{2k+1})\phi(t)\mathbf{e}_{2k}, \tag{33}$$

where ϕ is an even T-periodic solution of the linear inhomogeneous equation (25) and we have used the convention: $\sigma_{-1} = \sigma_{2N+1} = 0$.

To find the second-order correction term, we write the linear inhomogeneous problem explicitly as follows:

$$\left(\frac{d^2}{dt^2} + V''(Q_n^{(0)})\right) Q_n^{(2)} = -\left((\sigma_+ + \sigma_-)^2 \mathbf{f}(\mathbf{Q}^{(0)})\right)_n - \left((\sigma_+ + \sigma_-)\mathbf{f}'(\mathbf{Q}^{(0)})\mathbf{Q}^{(1)}\right)_n,$$

where we have used the fact that $V'''(Q_n^{(0)})(Q_n^{(1)})^2 = 0$ for all $n \in \mathbb{Z}$.

Let θ and ζ be even T-periodic solutions of the linear inhomogeneous equation

$$\ddot{\theta} + V''(\varphi)\theta = \phi, \quad t \in \mathbb{R} \tag{34}$$

and

$$\ddot{\zeta} + \zeta = \phi, \quad t \in \mathbb{R}, \tag{35}$$

which exist and are unique under the conditions that $T'(E) \neq 0$ and $T \neq 2\pi m$, $m \in \mathbb{N}$. By using these solutions, we can write the second-order correction term $\mathbf{Q}^{(2)}$ explicitly as follows:

$$\begin{aligned}\mathbf{Q}^{(2)}(t) = & -\sum_{k=1}^{N}(\sigma_{2k-3} + 2\sigma_{2k-1} + \sigma_{2k+1})(\psi(t) - \theta(t))\mathbf{e}_{2k-1} \\ & -\sigma_1(\phi(t) - \zeta(t))\mathbf{e}_{-1} - \sigma_{2N-1}(\phi(t) - \zeta(t))\mathbf{e}_{2N+1}.\end{aligned} \tag{36}$$

The following theorem represents the main result of the perturbation computations.

Theorem 2. *Under assumptions of Proposition 2, let* $\mathbf{Q}^{(0)} = \sum_{k=1}^{N} \sigma_{2k-1}\varphi\mathbf{e}_{2k-1}$ *yield a solution* $\mathbf{Q}^{(\epsilon)} \in H_e^2((0,T), l^2(\mathbb{Z}))$ *of the discrete Klein–Gordon equation (1) for small* $\epsilon > 0$. *Then the eigenvalue problem (18) for small* $\epsilon > 0$ *has* $2N$ *small eigenvalues,*

$$\lambda = \epsilon\Lambda + \mathcal{O}(\epsilon^2),$$

where Λ *is an eigenvalue of the matrix eigenvalue problem*

$$\frac{T^2(E)}{T'(E)M_2}\Lambda^2\mathbf{c} = \mathcal{S}\mathbf{c}, \quad \mathbf{c} \in \mathbb{C}^N, \tag{37}$$

associated with the same matrix $\mathcal{S}$ *and a different numerical coefficient* M_2 *given by*

$$M_2 = -\int_0^T \ddot{\varphi}\ddot{\phi}\,dt.$$

Proof. Similarly to the proof of Theorem 1, we are looking for the eigenvectors of the eigenvalue problem (18) in the subspace associated with the eigenvalue $\lambda = 0$ using the substitution $\lambda = \epsilon\tilde{\lambda}$ and the decomposition

$$\mathbf{W} = \mathbf{W}^{(0)} + \epsilon\tilde{\lambda}\mathbf{W}^{(1)} + \epsilon\mathbf{W}^{(2)} + \epsilon^2\tilde{\mathbf{W}},$$

where

$$\mathbf{W}^{(0)} = \sum_{k=1}^{N} c_{2k-1}\dot{\varphi}\mathbf{e}_{2k-1}, \tag{38}$$

$$\mathbf{W}^{(1)} = -\sum_{k=1}^{N} c_{2k-1}v\mathbf{e}_{2k-1}, \tag{39}$$

$$\mathbf{W}^{(2)} = -\sum_{k=0}^{N} (c_{2k-1} + c_{2k+1})\dot{\phi}\mathbf{e}_{2k}, \tag{40}$$

subject to the convention: $c_{-1} = c_{2N+1} = 0$. The error term $\tilde{\mathbf{W}}$ satisfies a residual equation, which we only write on the active sites $2k-1 \in S$ in the perturbation form:

$$\begin{aligned}\ddot{\tilde{W}}_{2k-1} + V''(\varphi)\tilde{W}_{2k-1} = &-\left((\sigma_+ + \sigma_-)^2\mathbf{f}'(\mathbf{Q}^{(0)})\mathbf{W}^{(0)}\right)_{2k-1} \\ &-\left((\sigma_+ + \sigma_-)\mathbf{f}'(\mathbf{Q}^{(0)})\mathbf{W}^{(2)}\right)_{2k-1} \\ &-c_{2k-1}\sigma_{2k-1}V'''(\varphi)Q_k^{(2)}\dot{\varphi} + \tilde{\lambda}^2 c_{2k-1}(2\dot{v} - \dot{\varphi}) \\ &+\mathcal{O}(\epsilon),\end{aligned} \tag{41}$$

where we have used properties of the explicit solutions (33), (36), and (38)–(40).

Expanding $\tilde{\lambda} = \Lambda + \mathcal{O}(\epsilon)$, projecting the system of linear inhomogeneous equations (41) to $\dot{\varphi} \in H^2_{\rm per}(0,T)$, and truncating at the leading order, we obtain the system of difference equations for $2k-1 \in S$:

$$\begin{aligned}\Lambda^2 c_{2k-1}\int_0^T \left(\dot{\varphi}^2 + 2v\ddot{\varphi}\right) dt = &-(c_{2k+1} + 2c_{2k-1} + c_{2k-3})\int_0^T (V''(\varphi)\dot{\varphi}^2 - \dot{\varphi}\dot{\phi})dt \\ &+ \sigma_{2k-1}(\sigma_{2k+1} + 2\sigma_{2k-1} + \sigma_{2k-3})c_{2k-1}\int_0^T V'''(\varphi)\dot{\varphi}^2(\psi - \theta)dt.\end{aligned} \tag{42}$$

Differentiating the linear inhomogeneous equation (34) and projecting it to $\dot{\varphi}$, we infer that

$$\int_0^T V'''(\varphi)\dot{\varphi}^2\theta dt = \int_0^T \dot{\varphi}\dot{\phi}dt. \tag{43}$$

Combining (30), (31), and (43), the system of difference equations (42) yields the matrix eigenvalue problem (37) with

$$M_2 = \int_0^T (V''(\varphi)\dot{\varphi}^2 - \dot{\varphi}\dot{\phi})dt = \int_0^T \ddot{\varphi}(\ddot{\varphi} + \phi)dt = \int_0^T \ddot{\varphi}(\phi - V'(\varphi))dt = -\int_0^T \ddot{\varphi}\ddot{\phi}dt,$$

where we have used equations (25) and (32), as well as integration by parts. The matrix eigenvalue problem (37) defines $2N$ small eigenvalues that bifurcate from $\lambda = 0$ for small $\epsilon > 0$. □

Remark 8. The matrix eigenvalue problem (37) differs from the similar reduction for the standard Klein–Gordon equation in [13] by the sign change in front of the matrix $\mathcal{S}$ and by the replacement of the quantity M_2 with the quantity

$$M_2^{[13]} = \int_0^T \dot{\varphi}\dot{\phi}dt,$$

where ϕ is now a solution of the linear inhomogeneous equation $\ddot{\phi} + \phi = \varphi$ instead of equation (25).

5 Discussion

We consider the example of the discrete Klein–Gordon equation (1) related to the potential (2). Because V is even, the even solution $\varphi \in H^2_e(0,T)$ satisfies the symmetry

$$\varphi\left(\frac{T}{2} - t\right) = -\varphi(t), \quad t \in \mathbb{R}, \tag{44}$$

so that it can be expanded into the Fourier cosine series,

$$\varphi(t) = \sum_{n \in \mathbb{N}_{\rm odd}} c_n(T) \cos\left(\frac{2\pi n t}{T}\right), \tag{45}$$

with zero coefficients c_n for all even n. Because $V'(\varphi) = -\ddot{\varphi}$, a solution of the linear inhomogeneous equation (25) can also be found in the form of the Fourier cosine series:

$$\phi(t) = \sum_{n \in \mathbb{N}_{\rm odd}} \frac{4\pi^2 n^2 c_n(T)}{T^2 - 4\pi^2 n^2} \cos\left(\frac{2\pi n t}{T}\right). \tag{46}$$

Using Parseval's equality, we compute the numerical coefficient M_2 in Theorem 2 in the Fourier series form:

$$M_2 = -\int_0^T \ddot{\varphi}\ddot{\phi} dt = \sum_{n \in \mathbb{N}_{\rm odd}} \frac{(4\pi^2 n^2)^3 |c_n(T)|^2}{T^3 (4\pi^2 n^2 - T^2)}. \tag{47}$$

Consider now the dependence $T(E)$ defined by the integral formula (11). Because $V''(0) = 1$, we have $T(E) \to 2\pi$ as $E \to 0$. For small values of E, the cubic term in the expansion (2) shows that the case $\alpha < 0$ gives a hard potential with $T'(E) < 0$, whereas the case $\alpha > 0$ gives a soft potential with $T'(E) > 0$ for small E.

If $T'(E) < 0$ and $T(E) < 2\pi$, then $M_2 > 0$. Also recall that $M_1 > 0$. In this case, Proposition 3 implies that the only stable configuration of the multi-site breathers in Theorems 1 and 2 is the one with all alternating $\{\sigma_k\}_{k=1}^N$ or $\{\sigma_{2k-1}\}_{k=1}^N$ (anti-phase breathers). This conclusion is recorded in the first line of Table I.

If $T'(E) > 0$ and $T(E) > 2\pi$, then the situation is different between Theorems 1 and 2. Because $M_1 > 0$, the only stable configuration of the multi-site breathers in Theorem 1 is the one with all equal $\{\sigma_k\}_{k=1}^M$ (in-phase breathers).

On the other hand, the quantity M_2 changes sign in the interval $T(E)$ between two resonances at 2π and 6π, because the first negative term in the series (47) dominates if $T(E)$ is close to 2π whereas the second positive term dominates if $T(E)$ is close to 6π. Therefore, there exists a period $T_* \in (2\pi, 6\pi)$ such that $M_2 < 0$ for $T \in (2\pi, T_*)$ and $M_2 > 0$ for $T \in (T_*, 6\pi)$. Stable configurations of discrete solitons for $T'(E) > 0$ and $2\pi < T < 6\pi$ are recorded in the second line of Table I.

	Theorem 1	Theorem 2
$T'(E) < 0$ $0 < T < 2\pi$	anti-phase	anti-phase
$T'(E) > 0$ $2\pi < T < 6\pi$	in-phase	$2\pi < T < T_*$ anti-phase $T_* < T < 6\pi$ in-phase

Table I: Stable multi-site breathers in the hard and soft potentials.

We can now compare these analytical results with numerical simulations of one-dimensional discrete breathers in [4, 7]. Figure 1 in [7] and Figure 6 in [4] show stable propagation of the so-called fundamental breather ($N = 1$ in Theorem 1) for $\alpha > 0$. Profiles of stable breathers are also shown in Figure 4 for $\alpha > 0$ and Figure 5 for $\alpha < 0$ [4]. The stable fundamental breather corresponds to the sign-definite (positive) amplitudes for $\alpha > 0$ and sign-alternating amplitudes for $\alpha < 0$, which is in agreement with the results of Table I. The two-site twisted (sign-alternating) mode ($N = 2$ in Theorem 2) is reported to be stable both for $\alpha > 0$ and $\alpha < 0$, which is also in agreement with Table I for $2\pi < T < T_*$.

We note that the results of Table I apply only to the small-amplitude discrete breathers in the nonlinear potential (2) because the sign of $T'(E)$ may change for large amplitudes. In particular, the potential may become hard for large amplitudes in the case $\alpha > 0$ because $f(q) \sim q^{1/3}$ as $q \to \infty$. Similarly, the potential may become soft for large amplitudes in the case $\alpha < 0$ because $f(q)$ only exists for $q \in (-Q_0, Q_0)$, where Q_0 is given by (7).

We do not also know if any discrete breather in the nonlinear potential given by (2) can have the period close to the resonant value 6π, to observe additional phenomena such as pitchfork bifurcations of single-site and multi-site breathers [13]. These open questions will await further detailed numerical studies of the discrete Klein–Gordon equation (1).

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