Gauge transformation and spectral decomposition for the Ishimori-II equations

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Abstract
We study the gauge transformation between spectral problems and their adjoints
for the Ishimori-II (IS-II) and Davey–Stewartson-II (DS-II) equations. The
commutativity between the gauge and adjoint transformations is proved. The
commutativity is used for spectral decomposition and surface representation
theorems for the Ishimori-II equations.

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1. Introduction

Spectral decomposition is a fundamental result in spectral theory of self-adjoint operators.
Many nonlinear partial differential equations in the inverse scattering transform method are
associated with non-self-adjoint linear operators [AC91]. Since the regular direct and inverse
scattering transforms are mutually reversible, non-self-adjoint linear operators must possess
complete sets of eigenfunctions which are orthogonal with respect to eigenfunctions of the
adjoint operators. If initial data evolve into algebraically decaying lump solutions of nonlinear
equations, the spectral decomposition involves both the continuous and discrete spectrum of
linear operators. Potentials of linear operators associated with lump solutions are defined in
a weaker function space, compared to that required for a regular inverse scattering transform.
Nevertheless, spectral decomposition theorems for time-dependent Schrödinger and Dirac
operators with lump potentials were proved in [PS00a, PS00b] by methods of complex analysis
based on the Riemann–Hilbert and $\bar{\partial}$ problems. It was shown that lumps of the Kadomtsev–
Petviashvili-I (KP-I) equation are structurally stable in the spectral decomposition [PS00a],
while lumps of the Davey–Stewartson-II (DS-II) equations are structurally unstable and decay
into continuous spectrum under generic perturbations of the initial data [GK96, PS00b].

In this paper, we study a complete set of eigenfunctions associated with the Ishimori-II
(IS-II) equations,

$$\mathbf{S}_t + \tfrac{1}{2}\mathbf{S} \times (\mathbf{S}_{xx} - \mathbf{S}_{yy}) + \phi_x \mathbf{S}_y + \phi_y \mathbf{S}_x = 0$$
$$\phi_{xx} + \phi_{yy} - \mathbf{S} \cdot (\mathbf{S}_x \times \mathbf{S}_y) = 0 \tag{1.1}$$

where $\mathbf{S} = (S_1, S_2, S_3)^T = \mathbf{S}(x, y, t)$ is a three-component unit vector on the plane (x, y) such that

$$\mathbf{S} \cdot \mathbf{S} = 1 \qquad \lim_{|x|,|y| \to \infty} \mathbf{S}(x, y, t) = (0, 0, 1)^T$$

while $\phi(x, y, t)$ is scalar potential function generated by the topological charge density $\mathbf{S} \cdot (\mathbf{S}_x \times \mathbf{S}_y)$.

The IS-II equations describe time evolution of the system of static spin vortices on the plane (x, y) [I84]. Static spin vortices are classified by the integer values of the topological charge:

$$Q = \frac{1}{4\pi} \iint_{\mathbb{R}^2} \mathbf{S} \cdot (\mathbf{S}_x \times \mathbf{S}_y) \, \mathrm{d}x \, \mathrm{d}y. \tag{1.2}$$

From mathematical point of view, the IS-II equations are remarkable because of their gauge equivalence to the DS-II equations [BC88, LS89]:

$$\begin{aligned} \mathrm{i}q_t + \tfrac{1}{2}(q_{xx} - q_{yy}) + 4(\varphi - |q|^2)q &= 0 \\ \varphi_{xx} + \varphi_{yy} - 2(|q|^2)_{xx} &= 0. \end{aligned} \tag{1.3}$$

Linear non-self-adjoint operators for the IS-II equations (1.1) have potentials multiplied by x-derivative operators, compared to linear operators for the DS-II equations (1.3). As a result, a proof of the spectral decomposition theorem is particularly difficult for the linear operators of the IS-II equations. We prove the orthogonality and completeness of eigenfunctions by using the inverse scattering transform for both IS-II and DS-II equations, combined with the gauge transformation between these equations. Our approach is different from the straightforward computations used in the preceding papers [PS00a, PS00b]. The key property in our analysis is a commutativity (theorem 5.3) between the gauge and adjoint transformations of linear operators. The commutativity theorem deeply intertwines with the scattering theory for the IS-II and DS-II equations.

The paper is organized as follows. In section 2, we review the scattering data of the IS-II equations. In section 3, we study the gauge transformation between the IS-II and DS-II equations. An adjoint problem for the IS-II equations is introduced in section 4. In section 5, we study the gauge transformation between the adjoint problems and prove commutativity of the gauge and adjoint transformations. Section 6 presents applications to the spectral decomposition and surface representation theorems.

2. Inverse scattering transform for the Ishimori-II equations

The IS-II equations are the compatibility conditions of the over-determined linear system,

$$(\mathrm{i}\partial_y + P\partial_x)\Psi = 0 \tag{2.1}$$

$$\left(2\partial_t - 2\mathrm{i}P\partial_x^2 - (\mathrm{i}P_x - P_y P - \mathrm{i}P\phi_x - \phi_y)\partial_x\right)\Psi = 0 \tag{2.2}$$

where $\Psi = (\Psi_1, \Psi_2)^T = \Psi(x, y, t)$, $P = S_1\sigma_1 + S_2\sigma_2 + S_3\sigma_3 = P(x, y, t)$ and $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

such that

$$P = \begin{pmatrix} S_3 & S_1 - \mathrm{i}S_2 \\ S_1 + \mathrm{i}S_2 & -S_3 \end{pmatrix}.$$

We introduce complex variables for elliptic problems on the plane (x, y):

$$z = \tfrac{1}{2}(y - ix) \qquad \bar{z} = \tfrac{1}{2}(y + ix) \tag{2.3}$$

such that the spectral problem (2.1) can be rewritten for $\Psi = \Psi(z)$:

$$\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \Psi = \frac{1}{2}(P(z) - \sigma_3)(\partial_z - \partial_{\bar{z}})\Psi \tag{2.4}$$

on $z \in \mathbb{C}$. Here and henceforth we suppress the dependence of Ψ from $\bar{z}$ and t.

The $\bar{\partial}$-method was used in [BC88, K93] to solve the inverse scattering transform for the linear spectral problem (2.4). We introduce a fundamental matrix of solutions of (2.4),

$$\Psi = (\mu\, e^{ikz}, \chi\, e^{-ik\bar{z}}) \tag{2.5}$$

where the vectors $\mu = (\mu_1, \mu_2)^{\mathrm{T}} = \mu(z, k)$ and $\chi = (\chi_1, \chi_2)^{\mathrm{T}} = \chi(z, k)$ solve the problems,

$$\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \mu + ik \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mu = \mathcal{P}_k \mu \tag{2.6}$$

$$\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \chi - ik \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \chi = \mathcal{P}_k \chi \tag{2.7}$$

where

$$\mathcal{P}_k = \tfrac{1}{2}(P(z) - \sigma_3)(\partial_z - \partial_{\bar{z}} + ik). \tag{2.8}$$

The eigenvectors $\mu(z, k)$ and $\chi(z, k)$ satisfy the boundary conditions on the plane (x, y):

$$\lim_{|z| \to \infty} \mu(z, k) = \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \lim_{|z| \to \infty} \chi(z, k) = \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.9}$$

If $\mathbf{S}(z)$ is real-valued, the two fundamental solutions $\mu(z, k)$ and $\chi(z, k)$ of the problems (2.6) and (2.7) are related by means of the involution transformation:

$$\chi(z, \bar{k}) = \sigma \overline{\mu(z, k)} \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.10}$$

In the limit $|k| \to 0$, the eigenvector $\mu(z, k)$ becomes a constant vector in z, which is uniquely defined by the normalization condition (2.9) as

$$\lim_{|k| \to 0} \mu(z, k) = \mathbf{e}_1. \tag{2.11}$$

In the limit $|k| \to \infty$, the eigenvector $\mu(z, k)$ becomes a vector $\mu_\infty(z)$, which solves the algebraic problem,

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mu_\infty(z) = \frac{1}{2}(P(z) - \sigma_3)\mu_\infty(z). \tag{2.12}$$

Components of the vector $\mu_\infty(z)$ are related by means of the stereographical variable,

$$\frac{\mu_{2\infty}}{\mu_{1\infty}} = \frac{1 - S_3}{S_1 - iS_2} = \frac{S_1 + iS_2}{1 + S_3}. \tag{2.13}$$

The stereographical variable in (2.13) satisfies the same family of nonlinear problem as the DS-II equations (1.3) [S92]. We define the matrix $m_\infty = m_\infty(z)$ as

$$m_\infty(z) = (\mu_\infty(z), \sigma \overline{\mu_\infty(z)}) = \begin{pmatrix} \mu_{1\infty} & -\overline{\mu_{2\infty}} \\ \mu_{2\infty} & \overline{\mu_{1\infty}} \end{pmatrix} \tag{2.14}$$

such that $\det(m_\infty) = |\mu_{1\infty}|^2 + |\mu_{2\infty}|^2$. As a result,

$$P(z) = m_\infty(z)\sigma_3 m_\infty^{-1}(z) \tag{2.15}$$

or explicitly,

$$S_1 - \mathrm{i}S_2 = \frac{2\mu_{1\infty}\overline{\mu_{2\infty}}}{|\mu_{1\infty}|^2 + |\mu_{2\infty}|^2} \qquad S_3 = \frac{|\mu_{1\infty}|^2 - |\mu_{2\infty}|^2}{|\mu_{1\infty}|^2 + |\mu_{2\infty}|^2}. \tag{2.16}$$

The differential problem (2.6) can be transformed to the linear integral equation by means of the Green function,

$$\mu_1(z, k) = 1 + \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}z' \wedge \mathrm{d}\bar{z}'}{z' - z} (\mathcal{P}_k\boldsymbol{\mu})_1(z')$$

$$\mu_2(z, k) = \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}z' \wedge \mathrm{d}\bar{z}'}{\bar{z}' - \bar{z}} (\mathcal{P}_k\boldsymbol{\mu})_2(z')\, \mathrm{e}^{\mathrm{i}k(z'-z)+\mathrm{i}\bar{k}(\bar{z}'-\bar{z})}. \tag{2.17}$$

The same Fredholm integral equations (2.17) occur in the spectral problem for the DS-II equations [PS00b]. The integral equations (2.17) have a unique solution if the potential function $P(z)$ in $L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ [BC84]. If the potential function $P(z)$ in $L^2(\mathbb{R}^2)$ but not in $L^1(\mathbb{R}^2)$, the integral equations (2.17) may have homogeneous solutions, corresponding to bound states [FA84, APP89]. Bound states $\boldsymbol{\Phi}_j(z)$, are decaying solutions of the equation,

$$\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \boldsymbol{\Phi}_j + \mathrm{i}k_j \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\Phi}_j = \mathcal{P}_{k_j}\boldsymbol{\Phi}_j \tag{2.18}$$

subject to the boundary conditions,

$$\lim_{|z|\to\infty} z\boldsymbol{\Phi}_j(z) = \mathbf{e}_1. \tag{2.19}$$

Each eigenvalue k_j is double, since there exists a linearly independent eigenfunction $\boldsymbol{\Phi}'_j(z)$ solving the same equation (2.18) and satisfying the boundary conditions

$$\lim_{|z|\to\infty} z\boldsymbol{\Phi}'_j(z)\, \mathrm{e}^{\mathrm{i}k_j z + \mathrm{i}\overline{k_j z}} = \mathbf{e}_2. \tag{2.20}$$

The two linearly independent eigenfunctions are related as

$$\boldsymbol{\Phi}'_j(z) = \sigma\overline{\boldsymbol{\Phi}_j(z)}\, \mathrm{e}^{-\mathrm{i}(k_j z + \overline{k_j z})}. \tag{2.21}$$

We assume that there is a finite number (say n) of eigenvalues k_j on the plane $k \in \mathbb{C}$, such that $k_j \neq 0$, $j = 1, \ldots, n$. It follows from (2.17) that the eigenfunction $\boldsymbol{\mu}(z, k)$ satisfies the $\bar{\partial}$ problem in the domain $\mathcal{D} = \{k \in \mathbb{C} : k \neq k_j, j = 1, \ldots, n\}$, such that

$$\frac{\partial\boldsymbol{\mu}}{\partial\bar{k}} = b(k)\mathbf{N}_\mu(z, k) \tag{2.22}$$

where the spectral coefficient $b(k)$ is defined as

$$b(k) = \frac{1}{2\pi} \iint \mathrm{d}z \wedge \mathrm{d}\bar{z}(\mathcal{P}_k\boldsymbol{\mu})_2(z)\, \mathrm{e}^{\mathrm{i}(kz+\overline{kz})} \tag{2.23}$$

and the linearly independent eigenfunction $\mathbf{N}_\mu(z, k)$ solves the same integral equation (2.17) but with a different (oscillatory) behaviour at infinity:

$$N_{\mu 1}(z, k) = \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}z' \wedge \mathrm{d}\bar{z}'}{z' - z} (\mathcal{P}_k\mathbf{N}_\mu)_1(z')$$

$$N_{\mu 2}(z, k) = \mathrm{e}^{-\mathrm{i}(kz+\overline{kz})} + \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}z' \wedge \mathrm{d}\bar{z}'\, \mathrm{e}^{\mathrm{i}(k(z'-z)+\overline{k(z'-z)})}}{\bar{z}' - \bar{z}} (\mathcal{P}_k\mathbf{N}_\mu)_2(z'). \tag{2.24}$$

For real-valued $\mathbf{S}(z)$, there exists a local relation between two linearly independent eigenfunctions $\boldsymbol{\mu}(z, k)$ and $\mathbf{N}_\mu(z, k)$:

$$\mathbf{N}_\mu(z, k) = \sigma\overline{\boldsymbol{\mu}(z, k)}\, \mathrm{e}^{-\mathrm{i}(kz+\overline{kz})}. \tag{2.25}$$

When the eigenfunction $\boldsymbol{\mu}(z, k)$ approaches to the points $k = k_j$, $j = 1, \ldots, n$, where the bound state exists, it has a singular (pole) behaviour in the limit $k \to k_j$. We assume the following limiting behaviour of $\boldsymbol{\mu}(z, k)$:

$$\lim_{k \to k_j} \left(\boldsymbol{\mu}(z, k) - \frac{i}{k - k_j} \boldsymbol{\Phi}_j(z) \right) = (z + z_j) \boldsymbol{\Phi}_j(z) + c_j \boldsymbol{\Phi}'_j(z) \tag{2.26}$$

where z_j, c_j are arbitrary constants. We notice that this limiting behaviour introduces a constraint on the class of potentials as for the DS-II equations in [PS00b]. The constraint is always satisfied for lump potentials. No such constraints were introduced in the original paper [APP89], where a different limiting behaviour of $\boldsymbol{\mu}(z, k)$ was suggested as $k \to k_j$:

$$\lim_{k \to k_j} \frac{\partial}{\partial k} ((k - k_j) \boldsymbol{\mu}(z, k)) = (z + z_j) \boldsymbol{\Phi}_j(z) + c_j \boldsymbol{\Phi}'_j(z). \tag{2.27}$$

We neglect this discrepancy between (2.26) and (2.27), since the spectral decomposition and structural instability of points $k = k_j$ does not depend on the types of singular behaviour of $\boldsymbol{\mu}(z, k)$ as $k \to k_j$.

Solving the $\bar{\partial}$ problem (2.22) on plane $k \in \mathbb{C}$ subject to the boundary conditions $\lim_{|k| \to \infty} \boldsymbol{\mu}(z, k) = \boldsymbol{\mu}_\infty(z)$ and using the limiting conditions (2.26) at the poles $k = k_j$, $j = 1, \ldots, n$, we derive the expansion formula,

$$\boldsymbol{\mu}(z, k) = \boldsymbol{\mu}_\infty(z) + \sum_{j=1}^{n} \frac{i}{k - k_j} \boldsymbol{\Phi}_j(z) + \frac{1}{2\pi i} \iint \frac{dk' \wedge d\bar{k}'}{k' - k} b(k') \mathbf{N}_\mu(z, k'). \tag{2.28}$$

Using the limiting condition (2.11) as $k \to 0$, we express the limiting function $\boldsymbol{\mu}_\infty(z)$ explicitly as

$$\boldsymbol{\mu}_\infty(z) = \mathbf{e}_1 + \sum_{j=1}^{n} \frac{i}{k_j} \boldsymbol{\Phi}_j(z) - \frac{1}{2\pi i} \iint \frac{dk' \wedge d\bar{k}'}{k'} b(k') \mathbf{N}_\mu(z, k'). \tag{2.29}$$

As a result, we find an explicit expansion formula for $\boldsymbol{\mu}(z, k)$ as

$$\boldsymbol{\mu}(z, k) = \mathbf{e}_1 + \sum_{j=1}^{n} \frac{ik}{k_j(k - k_j)} \boldsymbol{\Phi}_j(z) + \frac{k}{2\pi i} \iint \frac{dk' \wedge d\bar{k}'}{k'(k' - k)} b(k') \mathbf{N}_\mu(z, k'). \tag{2.30}$$

Using the limiting relations as $k \to k_j$, we close the system (2.30) with a set of algebraic equations for $j = 1, \ldots, n$:

$$\left(z + z_j - \frac{i}{k_j} \right) \boldsymbol{\Phi}_j(z) + c_j \boldsymbol{\Phi}'_j(z) = \mathbf{e}_1 + \sum_{l \neq j}^{n} \frac{ik_j}{k_l(k_j - k_l)} \boldsymbol{\Phi}_l(z)$$

$$+ \frac{k_j}{2\pi i} \iint \frac{dk' \wedge d\bar{k}'}{k'(k' - k_j)} b(k') \mathbf{N}_\mu(z, k'). \tag{2.31}$$

We conclude this section with an explicit example of a single lump solution of the IS-II equations. Let $b(k) = 0$ and $n = 1$ in the equations above. Then the algebraic system (2.31) is decoupled from (2.30) and it has a unique solution:

$$\Phi_{1j}(z) = \frac{\bar{z} + \bar{z}_j + \frac{i}{k_j}}{\left| z + z_j - \frac{i}{k_j} \right|^2 + |c_j|^2} \qquad \Phi_{2j}(z) = \frac{-c_j\, e^{-i(k_j z + \overline{k_j z})}}{\left| z + z_j - \frac{i}{k_j} \right|^2 + |c_j|^2}. \tag{2.32}$$

The eigenfunction $\boldsymbol{\mu}(z, k)$ has a pole decomposition representation,

$$\boldsymbol{\mu}(z, k) = \mathbf{e}_1 + \frac{ik}{k_j(k - k_j)} \boldsymbol{\Phi}_j(z) \qquad \boldsymbol{\mu}_\infty(z) = \mathbf{e}_1 + \frac{i}{k_j} \boldsymbol{\Phi}_j(z). \tag{2.33}$$

The single lump solution for the vector $\mathbf{S}(z)$ takes the explicit form:

$$S_1 - iS_2 = \frac{2i\bar{c}_j \left[(z + z_j)\left(\bar{z} + \bar{z}_j + \frac{i}{\bar{k}_j}\right) + |c_j|^2 \right] e^{i(k_j z + \overline{k_j z})}}{\bar{k}_j (|z + z_j|^2 + |c_j|^2)\left(\left|z + z_j - \frac{i}{k_j}\right|^2 + |c_j|^2 \right)} \tag{2.34}$$

$$S_3 = 1 - \frac{2|c_j|^2}{|k_j|^2(|z + z_j|^2 + |c_j|^2)\left(\left|z + z_j - \frac{i}{k_j}\right|^2 + |c_j|^2 \right)}. \tag{2.35}$$

We note that the single lump solution of the IS-II equations is different from the single vortex solution found in [I84] with the Hirota's method. However, the single vortex solution in [I84] can be recovered from the single lump solution (2.32)–(2.35) in the singular limit $k_j \to 0$, with the appropriate shifts of z_j and $\bar{z}_j$. In our paper, we will be working only with the lump solutions which are derived in the inverse scattering transform for a general non-singular case $k_j \neq 0$, $j = 1, \ldots, n$.

3. Gauge transformation between the DS-II and IS-II equations

The DS-II equations (1.3) in complex coordinates (2.3) are related with the following linear operator:

$$\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \Psi_{\mathrm{DS}} = Q(z)\Psi_{\mathrm{DS}} \tag{3.1}$$

where

$$Q(z) = \begin{pmatrix} 0 & -q(z) \\ \bar{q}(z) & 0 \end{pmatrix}. \tag{3.2}$$

The fundamental matrix of the linear operator for the DS-II equations is $\Psi_{\mathrm{DS}} = (\mu_{\mathrm{DS}}\, e^{ikz}, \chi_{\mathrm{DS}}\, e^{-ik\bar{z}})$, where $\chi_{\mathrm{DS}}(z, \bar{k}) = \sigma\overline{\mu_{\mathrm{DS}}(z, k)}$ and $\mu_{\mathrm{DS}}(z, k)$ satisfies the problem:

$$\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \mu_{\mathrm{DS}} + ik \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mu_{\mathrm{DS}} = Q(z)\mu_{\mathrm{DS}}. \tag{3.3}$$

Linear spectral problems (2.4) and (3.1) for the IS-II and DS-II equations are related to the gauge transformation reported in [BC88, LS89].

Lemma 3.1. *Let $m_\infty(z)$ be defined by (2.14) such that $P(z) = m_\infty \sigma_3 m_\infty^{-1}$. Solutions of the linear system (2.4) are equivalent to solutions of the linear system (3.1) with the gauge transformation*

$$\Psi_{\mathrm{DS}}(z, k) = m_\infty^{-1}(z)\Psi(z, k) \tag{3.4}$$

such that

$$Q(z) = \left(\mathcal{D}_z m_\infty^{-1}\right) m_\infty \qquad \mathcal{D}_z = \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \tag{3.5}$$

Proof. We first show that if $\mathbf{m} = (\mu(z, k), \chi(z, k))$ with μ, χ satisfying (2.6), (2.7) and $\mathbf{m}_{\mathrm{DS}} = m_\infty^{-1}\mathbf{m} = (\mu_{\mathrm{DS}}, \chi_{\mathrm{DS}})$, then μ_{DS} solves the problem (3.3), where the matrix $Q(z)$ has zero diagonal entries, while $\chi_{\mathrm{DS}}(z, \bar{k}) = \sigma\overline{\mu_{\mathrm{DS}}(z, k)}$. For simplicity, we consider only the case $n = 0$. We rewrite (2.28) in the operator form,

$$\mathbf{m} = m_\infty + \mathcal{CT}(\mathbf{m}) \tag{3.6}$$

where $\mathcal{C}$ is the Cauchy integral operator,

$$\mathcal{C}(\mathbf{f}) = \frac{1}{2\pi i} \iint \frac{dk' \wedge d\bar{k}'}{k' - k} \mathbf{f}(k') \tag{3.7}$$

and $\mathcal{T}$ is the right multiplicative operator, such that

$$\mathcal{T}(\boldsymbol{f}) = \mathcal{T} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} (z, k) = \boldsymbol{f}(z, \bar{k}) \mathbf{S}_z(k). \tag{3.8}$$

Here the scattering matrix $\mathbf{S}_z(k)$ is

$$\mathbf{S}_z(k) = \begin{pmatrix} 0 & e^{i(\bar{k}z + k\bar{z})} \overline{b(\bar{k})} \\ e^{-i(kz + \bar{k}\bar{z})} b(k) & 0 \end{pmatrix} \tag{3.9}$$

and we have used (2.10) and (2.25). Multiplying (3.6) with m_∞^{-1}, we find that

$$m_\infty^{-1} \mathbf{m} = I + \mathcal{C} m_\infty^{-1} \mathcal{T}(\mathbf{m}) = I + \mathcal{C}\mathcal{T}\left(m_\infty^{-1}\mathbf{m}\right) \tag{3.10}$$

where I is the 2-by-2 identity matrix. We apply the operator,

$$\mathcal{D}_k \boldsymbol{f} = \mathcal{D}_z \boldsymbol{f} + k(\text{ad}\mathcal{J} \boldsymbol{f}) = \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \boldsymbol{f} - \frac{i}{2} k[\sigma_3, \boldsymbol{f}] \tag{3.11}$$

to both sides of (3.10) and use the commutativity property,

$$[\mathcal{D}_k, \mathcal{T}]\boldsymbol{f} = 0. \tag{3.12}$$

As a result, we obtain

$$\mathcal{D}_k\left(m_\infty^{-1}\mathbf{m}\right) = \mathcal{D}_k \mathcal{C}\mathcal{T}\left(m_\infty^{-1}\mathbf{m}\right) = \mathcal{C}\mathcal{T}\mathcal{D}_k\left(m_\infty^{-1}\mathbf{m}\right) + Q(z)$$

where

$$Q(z) = [\mathcal{D}_k, \mathcal{C}]\mathcal{T}\left(m_\infty^{-1}\mathbf{m}\right) = -\frac{1}{2\pi i} \text{ad}\mathcal{J} \iint dk \wedge d\bar{k} \mathcal{T}\left(m_\infty^{-1}\mathbf{m}\right)(z, k)$$

$$= \frac{1}{2\pi} \iint dk \wedge d\bar{k} \begin{pmatrix} 0 & \left(\mathcal{T}\left(m_\infty^{-1}\mathbf{m}\right)\right)_{12}(z, k) \\ \left(\mathcal{T}\left(m_\infty^{-1}\mathbf{m}\right)\right)_{21}(z, \bar{k}) & 0 \end{pmatrix}. \tag{3.13}$$

We use the relation (3.10) and the commutativity relation,

$$[Q(z), \mathcal{T}]\mathbf{f} = 0 \tag{3.14}$$

and finish the first part of the proof,

$$\mathcal{D}_k\left(m_\infty^{-1}\mathbf{m}\right) = (I - \mathcal{C}\mathcal{D}_k)^{-1} Q(z) = Q(z)(I - \mathcal{C}\mathcal{D}_k)^{-1} I = Q(z) m_\infty^{-1}\mathbf{m}. \tag{3.15}$$

In the second part of the proof, we show that if Ψ solves (2.4) with $P(z) = m_\infty \sigma_3 m_\infty^{-1}$ and $\Psi = m_\infty \Psi_{\text{DS}}$, then Ψ_{DS} solves (3.1), where $Q(z)$ is related to m_∞ by means of (3.5). We rewrite (2.4) in the form

$$\mathcal{D}_z\left(m_\infty \Psi_{\text{DS}}\right) = \tfrac{1}{2}\left(m_\infty \sigma_3 m_\infty^{-1} - \sigma_3\right)\left(\partial_z - \partial_{\bar{z}}\right) m_\infty \Psi_{\text{DS}}. \tag{3.16}$$

Straightforward differentiations transform (3.16) to the form

$$\begin{pmatrix} \partial_{\bar{z}}\mu_{1\infty} & -\partial_{\bar{z}}\overline{\mu_{2\infty}} \\ \partial_z\mu_{2\infty} & \partial_z\overline{\mu_{1\infty}} \end{pmatrix} \Psi_{\text{DS}} + m_\infty \mathcal{D}_z \Psi_{\text{DS}}$$

$$= \frac{1}{\det(m_\infty)} \begin{pmatrix} -|\mu_{2\infty}|^2 & \mu_{1\infty}\overline{\mu_{2\infty}} \\ \overline{\mu_{1\infty}}\mu_{2\infty} & |\mu_{2\infty}|^2 \end{pmatrix} \left((\partial_z - \partial_{\bar{z}}) m_\infty\right) \Psi_{\text{DS}} \tag{3.17}$$

which further transforms to the form

$$m_\infty \mathcal{D}_z \Psi_{\text{DS}} = m_\infty Q(z) \Psi_{\text{DS}} \tag{3.18}$$

where

$$Q(z) = \frac{1}{\det(m_\infty)} \begin{pmatrix} -\overline{\mu_{1\infty}}\partial_{\bar{z}}\mu_{1\infty} - \overline{\mu_{2\infty}}\partial_{\bar{z}}\mu_{2\infty} & \overline{\mu_{1\infty}}\partial_{\bar{z}}\overline{\mu_{2\infty}} - \overline{\mu_{2\infty}}\partial_{\bar{z}}\overline{\mu_{1\infty}} \\ \mu_{2\infty}\partial_z\mu_{1\infty} - \mu_{1\infty}\partial_z\mu_{2\infty} & -\mu_{2\infty}\partial_z\overline{\mu_{2\infty}} - \mu_{1\infty}\partial_z\overline{\mu_{1\infty}} \end{pmatrix}. \tag{3.19}$$

Explicit differentiations in (3.5) shows that representations (3.5) and (3.19) are the same. $\square$

Corollary 3.2. *Components of the transformation matrix $m_\infty(z)$ in (2.14) satisfy the constraints:*

$$q(z) = \frac{1}{\det(m_\infty)} (\overline{\mu_{2\infty}}\partial_{\bar{z}}\overline{\mu_{1\infty}} - \overline{\mu_{1\infty}}\partial_{\bar{z}}\overline{\mu_{2\infty}}) \tag{3.20}$$

$$0 = \mu_{1\infty}\partial_z\overline{\mu_{1\infty}} + \mu_{2\infty}\partial_z\overline{\mu_{2\infty}}. \tag{3.21}$$

We continue the explicit example (2.32)–(2.35) and compute the gauge transformation between the single lump solutions of the IS-II and DS-II equations. It follows from (2.33) that

$$\mu_{1\infty} = 1 + \frac{i\left(\bar{z} + \bar{z}_j + \frac{i}{k_j}\right)}{k_j\left(\left|z + z_j - \frac{i}{k_j}\right|^2 + |c_j|^2\right)} \qquad \mu_{2\infty} = \frac{-ic_j\,e^{-i(k_jz + \overline{k_j}\bar{z})}}{k_j\left(\left|z + z_j - \frac{i}{k_j}\right|^2 + |c_j|^2\right)} \tag{3.22}$$

such that

$$\det(m_\infty) = \frac{|z + z_j|^2 + |c_j|^2}{\left|z + z_j - \frac{i}{k_j}\right|^2 + |c_j|^2}. \tag{3.23}$$

Constraints (3.20) and (3.21) are satisfied with

$$q = \frac{\bar{c}_j\,e^{i(k_jz + \overline{k_j}\bar{z})}}{|z + z_j|^2 + |c_j|^2}. \tag{3.24}$$

Therefore, the gauge transformation (3.5) generates a single lump solution of the DS-II equations (1.3) from a single lump solution of the IS-II equations (1.1). The eigenfunctions of the Lax operator for the DS-II equations are generated by the gauge transformation $\mu_{\mathrm{DS}} = m_\infty^{-1}\mu$ to the form

$$\mu_{\mathrm{DS}}(z, k) = \mathbf{e}_1 + \frac{i}{k - k_j}\Phi_{\mathrm{DS}}(z) \tag{3.25}$$

and

$$\Phi_{\mathrm{DS1}}(z) = \frac{\bar{z} + \bar{z}_j}{|z + z_j|^2 + |c_j|^2} \qquad \Phi_{\mathrm{DS2}}(z) = \frac{-c_j\,e^{-i(k_jz + \overline{k_j}\bar{z})}}{|z + z_j|^2 + |c_j|^2}. \tag{3.26}$$

4. The adjoint linear problem

Let $\mathcal{L}$ be a linear operator that maps $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$. We define the adjoint operator $\mathcal{L}^a$ by the inner product for two vector functions $f, g \in L^2(\mathbb{R}^2)$:

$$\langle\mathbf{f}|\mathcal{L}\mathbf{g}\rangle = \langle\mathcal{L}^+\mathbf{f}|\mathbf{g}\rangle \qquad \mathcal{L}^+ = \overline{\mathcal{L}^a} \tag{4.1}$$

where the standard inner product is

$$\langle\mathbf{f}|\mathbf{g}\rangle = \iint (\bar{f}_1 g_1 + \bar{f}_2 g_2)\,dz \wedge d\bar{z}. \tag{4.2}$$

If $\mathcal{L}\Psi = \mathbf{0}$ is a linear problem with an eigenfunction $\Psi \in L^2(\mathbb{R}^2)$, then $\mathcal{L}^a\Psi^a = \mathbf{0}$ is the adjoint problem with an adjoint eigenfunction $\Psi^a \in L^2(\mathbb{R}^2)$, which takes the form:

$$\begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix}\Psi^a = -\frac{1}{2}(\partial_z - \partial_{\bar{z}})(P^+(z) - \sigma_3)\Psi^a \tag{4.3}$$

where $P^+(z) = P(z)$. The fundamental matrix of adjoint eigenfunctions for the IS-II equations is

$$\Psi^a(z) = (\chi^a(z, k)\, e^{ik\bar{z}}, \mu^a(z, k)\, e^{-ikz}) \tag{4.4}$$

where $\chi^a(z, k)$ and $\mu^a(z, k)$ solve the adjoint problems to (2.6) and (2.7), respectively:

$$\begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} \chi^a + ik \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \chi^a = -\mathcal{P}_k^a \chi^a \tag{4.5}$$

$$\begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} \mu^a - ik \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mu^a = -\mathcal{P}_k^a \mu^a \tag{4.6}$$

where

$$\mathcal{P}_k^a = \tfrac{1}{2}(\partial_z - \partial_{\bar{z}} - ik)(P^+(z) - \sigma_3). \tag{4.7}$$

The eigenvectors $\chi^a(z, k)$ and $\mu^a(z, k)$ satisfy the boundary conditions on the plane (x, y):

$$\lim_{|z|\to\infty} \chi^a(z, k) = \mathbf{e}_1 \qquad \lim_{|z|\to\infty} \mu^a(z, k) = \mathbf{e}_2. \tag{4.8}$$

If $\mathbf{S}(z)$ is real-valued, the two fundamental solutions $\chi^a(z, k)$ and $\mu^a(z, k)$ of the problems (4.5) and (4.6) are related by means of the involution transformation:

$$\chi^a(z, \bar{k}) = -\sigma \overline{\mu^a(z, k)}. \tag{4.9}$$

In the limit $|k| \to \infty$, the eigenvector $\mu^a(z, k)$ becomes a vector $\mu_\infty^a(z)$, which solves the algebraic problem,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mu_\infty^a(z) = -\frac{1}{2}(P(z) - \sigma_3)\, \mu_\infty^a(z). \tag{4.10}$$

Components of the vector $\mu_\infty^a(z)$ are related by means of the adjoint stereographical variable,

$$\frac{\mu_{2\infty}^a}{\mu_{1\infty}^a} = -\frac{1 + S_3}{S_1 - iS_2} = -\frac{S_1 + iS_2}{1 - S_3}. \tag{4.11}$$

We define the adjoint matrix $m_\infty^a = m_\infty^a(z)$ as

$$m_\infty^a(z) = \left(-\sigma \overline{\mu_\infty^a(z)}, \mu_\infty^a(z)\right) = \begin{pmatrix} \overline{\mu_{2\infty}^a} & \mu_{1\infty}^a \\ -\overline{\mu_{1\infty}^a} & \mu_{2\infty}^a \end{pmatrix} \tag{4.12}$$

such that $\det\left(m_\infty^a\right) = \left|\mu_{1\infty}^a\right|^2 + \left|\mu_{2\infty}^a\right|^2$. As a result,

$$P^+(z) = m_\infty^a(z)\sigma_3 \left(m_\infty^a\right)^{-1}(z) \tag{4.13}$$

or explicitly,

$$S_1 - iS_2 = \frac{-2\mu_{1\infty}^a \overline{\mu_{2\infty}^a}}{\left|\mu_{1\infty}^a\right|^2 + \left|\mu_{2\infty}^a\right|^2} \qquad S_3 = \frac{\left|\mu_{2\infty}^a\right|^2 - \left|\mu_{1\infty}^a\right|^2}{\left|\mu_{1\infty}^a\right|^2 + \left|\mu_{2\infty}^a\right|^2}. \tag{4.14}$$

The differential problem (4.6) can be transformed to the linear integral equation by means of the Green function,

$$\mu_1^a(z, k) = -\frac{1}{2\pi i} \iint \frac{dz' \wedge d\bar{z}'}{\bar{z}' - \bar{z}} \left(\mathcal{P}_k^a \mu^a\right)_1(z')\, e^{-ik(z'-z) - i\bar{k}(\bar{z}'-\bar{z})}$$

$$\mu_2^a(z, k) = 1 - \frac{1}{2\pi i} \iint \frac{dz' \wedge d\bar{z}'}{z' - z} \left(\mathcal{P}_k^a \mu^a\right)_2(z'). \tag{4.15}$$

If the potential function $P(z)$ in $L^2(\mathbb{R}^2)$ but not in $L^1(\mathbb{R}^2)$, the integral equations (4.15) may have homogeneous solutions, corresponding to bound states. The bound states $\Phi_j^a(z)$ are decaying solutions of the equation,

$$\begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} \Phi_j^a - \mathrm{i}k_j^a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Phi_j^a = -\mathcal{P}_{k_j}^a \Phi_j^a \tag{4.16}$$

subject to the boundary conditions,

$$\lim_{|z| \to \infty} z\Phi_j^a(z) = \mathbf{e}_2. \tag{4.17}$$

Each eigenvalue k_j^a of the adjoint problem is double, since there exists a linearly independent eigenfunction $\Phi_j^{a\prime}(z)$ solving the same equation (4.16) and related to $\Phi_j^a(z)$ by means of the involution:

$$\Phi_j^{a\prime}(z) = -\sigma \overline{\Phi_j^a(z)}\, \mathrm{e}^{\mathrm{i}(k_j^a z + \overline{k_j^a} z)}. \tag{4.18}$$

It follows from (4.15) that the eigenfunction $\mu^a(z, k)$ satisfies the $\bar{\partial}$ problem in the domain $\mathcal{D}^a = \{k \in \mathbb{C} : k \neq k_j^a, j = 1, \ldots, n\}$, such that

$$\frac{\partial \mu^a}{\partial \bar{k}} = -b^a(k) \mathbf{N}_\mu^a(z, k) \tag{4.19}$$

where the spectral coefficient $b^a(k)$ is defined as

$$b^a(k) = \frac{1}{2\pi} \iint \mathrm{d}z \wedge \mathrm{d}\bar{z}\, \left(\mathcal{P}_k^a \mu^a\right)_1 (z)\, \mathrm{e}^{-\mathrm{i}(kz + \overline{k}\bar{z})} \tag{4.20}$$

and the linearly independent eigenfunction $\mathbf{N}_\mu^a(z, k)$ solves the same integral equation (4.15) but with a different (oscillatory) behaviour at infinity,

$$\begin{aligned}
N_{\mu 1}^a(z, k) &= \mathrm{e}^{\mathrm{i}(kz + \overline{k}\bar{z})} - \frac{1}{2\pi \mathrm{i}} \iint \frac{\mathrm{d}z' \wedge \mathrm{d}\bar{z}'}{\bar{z}' - \bar{z}} \left(\mathcal{P}_k^a \mathbf{N}_\mu^a\right)_1(z')\, \mathrm{e}^{-\mathrm{i}(k(z'-z) + \overline{k}(\overline{z'-z}))} \\
N_{\mu 2}^a(z, k) &= -\frac{1}{2\pi \mathrm{i}} \iint \frac{\mathrm{d}z' \wedge \mathrm{d}\bar{z}'}{z' - z} \left(\mathcal{P}_k^a \mathbf{N}_\mu^a\right)_2 (z').
\end{aligned} \tag{4.21}$$

For real-valued $\mathbf{S}(z)$, there exists a local relation between two linearly independent eigenfunctions $\mu^a(z, k)$ and $\mathbf{N}_\mu^a(z, k)$:

$$\mathbf{N}_\mu^a(z, k) = -\sigma \overline{\mu^a(z, k)}\, \mathrm{e}^{\mathrm{i}(kz + \overline{k}\bar{z})}. \tag{4.22}$$

When the eigenfunction $\mu(z, k)$ approaches to the points $k = k_j^a$, $j = 1, \ldots, n$, where the bound states exist, it has a singular (pole) behaviour in the limit $k \to k_j^a$. We assume the following limiting behaviour of $\mu^a(z, k)$:

$$\lim_{k \to k_j^a} \left(\mu^a(z, k) + \frac{\mathrm{i}}{k - k_j^a} \Phi_j^a(z) \right) = \left(z + z_j^a\right) \Phi_j^a(z) + c_j^a \Phi_j^{a\prime}(z) \tag{4.23}$$

where z_j^a, c_j^a are arbitrary constants.

Solving the $\bar{\partial}$ problem (4.19) on plane $k \in \mathbb{C}$ subject to the boundary conditions $\lim_{|k| \to \infty} \mu^a(z, k) = \mu_\infty^a(z)$ and using the limiting conditions (4.23) at the poles $k = k_j^a$, $j = 1, \ldots, n$, we derive the expansion formula,

$$\mu^a(z, k) = \mu_\infty^a(z) - \sum_{j=1}^n \frac{\mathrm{i}}{k - k_j^a} \Phi_j^a(z) - \frac{1}{2\pi \mathrm{i}} \iint \frac{\mathrm{d}k' \wedge \mathrm{d}\bar{k}'}{k' - k} b^a(k') \mathbf{N}_\mu^a(z, k'). \tag{4.24}$$

With the use of the limiting relations as $k \to k_j^a$, we close the system (4.24) with a set of algebraic equations for $j = 1, \ldots, n$:

$$
\left(z + z_j^a \right) \Phi_j^a(z) + c_j^a \Phi_j^{a\prime}(z) = \mu_\infty^a(z) - \sum_{l \neq j}^n \frac{i}{k_j^a - k_l^a} \Phi_l^a(z)
$$

$$
- \frac{1}{2\pi i} \iint \frac{dk' \wedge d\bar{k}'}{k' - k_j^a} b^a(k') \mathbf{N}_\mu^a(z, k'). \tag{4.25}
$$

The system of equations (4.24) and (4.25) is incomplete, since $\mu_\infty^a(z)$ is unknown. The adjoint problem (4.6) does not exhibit any simple solution at $k = 0$, in contrast to the direct problem (2.6). We complete the system (4.24)–(4.25) by deriving the relation between m_∞^a and m_∞. The relation appears as the commutativity condition between the gauge and adjoint transformations.

5. Commutativity between the gauge and adjoint transformations

The adjoint problem for the DS-II equations is

$$
\begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} \Psi_{DS}^a = -Q^+(z) \Psi_{DS}^a \tag{5.1}
$$

where $Q^+(z) = -Q(z)$. The fundamental matrix of the adjoint operator for the DS-II equations is $\Psi_{DS}^a = \left(\chi_{DS}^a\, e^{ik\bar{z}}, \mu_{DS}^a\, e^{-ikz} \right)$, where $\chi_{DS}^a(z, \bar{k}) = -\sigma \overline{\mu_{DS}^a(z, k)}$. Adjoint spectral problems (4.3) and (5.1) for the IS-II and DS-II equations are related with the adjoint gauge transformation.

Lemma 5.1. *Let $m_\infty^a(z)$ be defined by (4.12) such that $P^+(z) = m_\infty^a \sigma_3 \left(m_\infty^a \right)^{-1}$. Solutions of the linear system (4.3) are equivalent to solutions of the linear system (5.1) with the gauge transformation,*

$$
\Psi_{DS}^a(z, k) = \left(m_\infty^a \right)^{-1}(z) \Psi^a(z, k) \tag{5.2}
$$

such that

$$
Q^+(z) = \left(m_\infty^a \right)^{-1} \left(D_{\bar{z}} \left(m_\infty^a \right)^{\mathrm{T}} \right)^{\mathrm{T}} \qquad \mathcal{D}_{\bar{z}} = \begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix}. \tag{5.3}
$$

Proof. We show by direct differentiation that if Ψ^a satisfies (4.3) and $\Psi^a = m_\infty^a \Psi_{DS}^a$, then Ψ_{DS}^a satisfies (5.1) with $Q^+(z)$ given in (5.3). Using (4.13), we notice that

$$
- \frac{1}{2} (P^+(z) - \sigma_3) m_\infty^a = \begin{pmatrix} 0 & \mu_{1\infty}^a \\ \mu_{1\infty}^a & 0 \end{pmatrix}. \tag{5.4}
$$

Substitution $\Psi^a = m_\infty^a \Psi_{DS}^a$ reduces (4.3) to the form:

$$
\mathcal{D}_{\bar{z}} m_\infty^a \Psi_{DS}^a = (\partial_z - \partial_{\bar{z}}) \begin{pmatrix} 0 & \mu_{1\infty}^a \\ \mu_{1\infty}^a & 0 \end{pmatrix} \Psi_{DS}^a. \tag{5.5}
$$

Differentiation expands (5.5) to the form

$$
m_\infty^a \mathcal{D}_{\bar{z}} \Psi_{DS}^a = - \begin{pmatrix} \partial_z \overline{\mu_{2\infty}^a} & \partial_{\bar{z}} \mu_{1\infty}^a \\ -\partial_z \mu_{1\infty}^a & \partial_{\bar{z}} \mu_{2\infty}^a \end{pmatrix} \Psi_{DS}^a. \tag{5.6}
$$

Inverting m_∞^a and matching with (5.1), we express $Q^+(z)$ in terms of m_∞^a as

$$Q^+(z) = \frac{1}{\det(m_\infty^a)} \begin{pmatrix} \mu_{2\infty}^a \partial_z \overline{\mu_{2\infty}^a} + \mu_{1\infty}^a \partial_z \overline{\mu_{1\infty}^a} & \mu_{2\infty}^a \partial_{\bar{z}} \mu_{1\infty}^a - \mu_{1\infty}^a \partial_{\bar{z}} \mu_{2\infty}^a \\ \mu_{1\infty}^a \partial_z \overline{\mu_{2\infty}^a} - \mu_{2\infty}^a \partial_z \overline{\mu_{1\infty}^a} & \mu_{1\infty}^a \partial_{\bar{z}} \mu_{1\infty}^a + \mu_{2\infty}^a \partial_{\bar{z}} \mu_{2\infty}^a \end{pmatrix}. \tag{5.7}$$

Direct differentiation shows equivalence of representations (5.3) and (5.7).   □

Corollary 5.2. *Components of the transformation matrix $m_\infty^a(z)$ in (4.12) satisfy the constraints:*

$$q(z) = \frac{1}{\mu_{2\infty}^a} \partial_{\bar{z}} \mu_{1\infty}^a \tag{5.8}$$

$$0 = \mu_{1\infty}^a \partial_z \overline{\mu_{1\infty}^a} + \mu_{2\infty}^a \partial_z \overline{\mu_{2\infty}^a}. \tag{5.9}$$

if $Q^+ = -Q$.

We plot the diagram of the gauge and adjoint transformations for the IS-II and DS-II equations:

$$\begin{array}{ccc} \text{IS-II (2.4) for } P(z) & \rightarrow & \text{DS-II (3.1) for } Q(z) \\ \downarrow & & \vdots \\ \text{adjoint IS-II (4.3) for } P^+(z) & \rightarrow & \text{adjoint DS-II (5.1) for } Q^+(z). \end{array} \tag{5.10}$$

The gauge transformation from IS-II to DS-II is given by lemma 3.1. The adjoint transformation from IS-II to adjoint IS-II is described in section 4 for $P^+ = P$. The gauge transformation from adjoint IS-II to adjoint DS-II is given by lemma 5.1. It remains to show that the system (5.1) is indeed the adjoint system of (3.1) with $Q^+ = -Q$. This would serve as a commutativity condition between the gauge and adjoint transformations. We prove the commutativity condition and close the diagram (5.10) with an important relation between $m_\infty^a(z)$ and $m_\infty(z)$.

Theorem 5.3. *The gauge transformation and the adjoint transformation are commutative if and only if*

$$m_\infty^a = \left(m_\infty^{-1} \right)^+ = \frac{m_\infty}{\det(m_\infty)} \tag{5.11}$$

such that

$$\mu_{1\infty}^a = -\frac{\overline{\mu_{2\infty}}}{|\mu_{1\infty}|^2 + |\mu_{2\infty}|^2} \qquad \mu_{2\infty}^a = \frac{\overline{\mu_{1\infty}}}{|\mu_{1\infty}|^2 + |\mu_{2\infty}|^2} \tag{5.12}$$

and $\det(m_\infty^a) \det(m_\infty) = 1$.

Proof. A relation between m_∞ and m_∞^a follows from comparison of representations (2.15) and (4.13):

$$P^+(z) = \left(m_\infty^{-1} \right)^+ \sigma_3 m_\infty^+ = m_\infty^a \sigma_3 \left(m_\infty^a \right)^{-1}. \tag{5.13}$$

Another relation follows from comparison of representations (3.5) and (5.3):

$$Q^+ = m_\infty^+ \left(D_{\bar{z}} \left(m_\infty^{-1} \right)^{+T} \right)^T = \left(m_\infty^a \right)^{-1} \left(D_{\bar{z}} \left(m_\infty^a \right)^T \right)^T. \tag{5.14}$$

The constraint (5.11) is a sufficient condition for the relations (5.13) and (5.14). We prove that this is also a necessary condition for commutability of the gauge and adjoint transformations. By comparing (2.13) and (4.11), we derive a relation between μ_∞ and μ_∞^a:

$$\mu_{1\infty} \overline{\mu_{1\infty}^a} + \mu_{2\infty} \overline{\mu_{2\infty}^a} = 0. \tag{5.15}$$

Using combinations of constraints (3.20), (3.21), (5.8) and (5.9), combined with the relation (5.15), we also derive that

$$\mu_{1\infty}^a \mu_{2\infty} = \overline{\mu_{1\infty}^a}\,\overline{\mu_{2\infty}} \qquad \mu_{2\infty}^a \mu_{1\infty} = \overline{\mu_{2\infty}^a}\,\overline{\mu_{1\infty}}. \tag{5.16}$$

The relations (5.15) and (5.16) are satisfied if and only if

$$\mu_{1\infty}^a = -c(z)\overline{\mu_{2\infty}} \qquad \mu_{2\infty}^a = c(z)\overline{\mu_{1\infty}} \tag{5.17}$$

where $c \in \mathbb{R}$. Using constraints (3.20), (5.8), combined with the relation (5.17), we find that

$$c(z) = \frac{c_0}{\det(m_\infty)} \tag{5.18}$$

where c_0 is constant. It follows from the asymptotic limits,

$$\lim_{|z|\to\infty} m_\infty(z) = I \qquad \lim_{|z|\to\infty} m_\infty^a(z) = I \tag{5.19}$$

that $c_0 = 1$ and the formula (5.11) is thus proved. $\qquad\square$

We can develop an alternative proof of theorem 5.3, without using lemma 5.1 and corollary 5.2. The diagram of the gauge and adjoint transformations for the IS-II and DS-II equations can be closed as

$$\begin{array}{ccc} \text{IS-II (2.4) for } P(z) & \rightarrow & \text{DS-II (3.1) for } Q(z) \\ \downarrow & & \downarrow \\ \text{adjoint IS-II (4.3) for } P^+(z) & \cdots & \text{adjoint DS-II (5.1) for } Q^+(z). \end{array} \tag{5.20}$$

Here we will use the adjoint system for DS-II equations (5.1) with $Q^+ = -Q$ and show that the system is related to the adjoint system for IS-II equations (4.3) with $P^+ = P$ by means of the gauge transformation:

$$\Psi^a(z,k) = \frac{m_\infty(z)}{\det(m_\infty(z))}\Psi_{\text{DS}}^a(z,k). \tag{5.21}$$

This relation is equivalent to the gauge transformation between the adjoint IS-II and DS-II equations in lemma 5.1, where $m_\infty^a(z)$ is given by (5.11).

The left-hand side of (4.3) is written explicitly in the form,

$$D_{\bar{z}}\left(\frac{m_\infty}{\det m_\infty}\begin{pmatrix}\Psi_{1,\text{DS}}^a\\\Psi_{2,\text{DS}}^a\end{pmatrix}\right)$$

$$= \frac{1}{\det m_\infty}\begin{pmatrix}(\partial_z\mu_{1\infty})\Psi_{1,\text{DS}}^a + \mu_{1\infty}\partial_z\Psi_{1,\text{DS}}^a - (\partial_z\bar{\mu}_{2\infty})\Psi_{2,\text{DS}}^a - \bar{\mu}_{2\infty}\partial_z\Psi_{2,\text{DS}}^a\\(\partial_{\bar{z}}\mu_{2\infty})\Psi_{1,\text{DS}}^a + \mu_{2\infty}\partial_{\bar{z}}\Psi_{1,\text{DS}}^a + (\partial_{\bar{z}}\bar{\mu}_{1\infty})\Psi_{2,\text{DS}}^a + \bar{\mu}_{1\infty}\partial_{\bar{z}}\Psi_{2,\text{DS}}^a\end{pmatrix}$$

$$- \frac{1}{(\det m_\infty)^2}\begin{pmatrix}(\mu_{1\infty}\Psi_{1,\text{DS}}^a - \bar{\mu}_{2\infty}\Psi_{2,\text{DS}}^a)[(\partial_z\mu_{1\infty})\bar{\mu}_{1\infty} + (\partial_z\mu_{2\infty})\bar{\mu}_{2\infty}]\\(\mu_{2\infty}\Psi_{1,\text{DS}}^a + \bar{\mu}_{1\infty}\Psi_{2,\text{DS}}^a)[\mu_{1\infty}\partial_{\bar{z}}\bar{\mu}_{1\infty} + \mu_{2\infty}\partial_{\bar{z}}\bar{\mu}_{2\infty}]\end{pmatrix}. \tag{5.22}$$

We apply the gauge transformation $Q = \mathcal{D}_z(m_\infty^{-1})m_\infty$ to the adjoint DS-II system (5.1) and find that

$$D_{\bar{z}}\left(\frac{m_\infty}{\det(m_\infty)}\begin{pmatrix}\Psi_{1,\text{DS}}^a\\\Psi_{2,\text{DS}}^a\end{pmatrix}\right) = \frac{1}{\det m_\infty}\begin{pmatrix}(\partial_z\mu_{1\infty})\Psi_{1,\text{DS}}^a - (\partial_z\bar{\mu}_{2\infty})\Psi_{2,\text{DS}}^a - \bar{\mu}_{2\infty}\partial_z\Psi_{2,\text{DS}}^a\\(\partial_{\bar{z}}\mu_{2\infty})\Psi_{1,\text{DS}}^a + \mu_{2\infty}(\partial_{\bar{z}}\Psi_{1,\text{DS}}^a) + (\partial_{\bar{z}}\bar{\mu}_{1\infty})\Psi_{2,\text{DS}}^a\end{pmatrix}$$

$$+ \frac{1}{(\det m_\infty)^2}\begin{pmatrix}\mu_{1\infty}(\bar{\mu}_{1\infty}\partial_{\bar{z}}\bar{\mu}_{2\infty} - \bar{\mu}_{1\infty}\partial_{\bar{z}}\bar{\mu}_{2\infty})\Psi_{1,\text{DS}}^a\\(\bar{\mu}_{1\infty}\mu_{1\infty} - \mu_{1\infty}\partial_z\mu_{2\infty})\Psi_{1,\text{DS}}^a\end{pmatrix}$$

$$+ \frac{1}{(\det m_\infty)^2}\begin{pmatrix}-|\mu_{1\infty}|^2(\partial_z\mu_{1\infty})\Psi_{1,\text{DS}}^a + \bar{\mu}_{1\infty}\bar{\mu}_{2\infty}(\partial_z\mu_{1\infty})\Psi_{2,\text{DS}}^a\\-|\mu_{1\infty}|^2(\partial_{\bar{z}}\bar{\mu}_{1\infty})\Psi_{2,\text{DS}}^a - \mu_{2\infty}\mu_{1\infty}(\partial_{\bar{z}}\bar{\mu}_{1\infty})\Psi_{1,\text{DS}}^a\end{pmatrix}$$

$$+ \frac{1}{(\det m_\infty)^2}\begin{pmatrix}-\mu_{1\infty}\bar{\mu}_{2\infty}(\partial_z\mu_{2\infty})\Psi_{1,\text{DS}}^a + \bar{\mu}_{2\infty}^2(\partial_z\mu_{2\infty})\Phi_{21}^a\\-\mu_{2\infty}\bar{\mu}_{1\infty}(\partial_{\bar{z}}\bar{\mu}_{2\infty})\Psi_{2,\text{DS}}^a - \mu_{2\infty}^2(\partial_{\bar{z}}\bar{\mu}_{2\infty})\Psi_{1,\text{DS}}^a\end{pmatrix}. \tag{5.23}$$

On the other hand, the right-hand side of (4.3) can be rewritten as

$$
-\frac{1}{2} (\partial_z - \partial_{\bar{z}}) \, (P(z) - \sigma_3) \left(\frac{m_\infty}{\det(m_\infty)} \begin{pmatrix} \Psi^a_{1,\mathrm{DS}} \\ \Psi^a_{2,\mathrm{DS}} \end{pmatrix} \right)
$$

$$
= \frac{1}{\det m_\infty} \begin{pmatrix} (\partial_{\bar{z}} \bar{\mu}_{2\infty}) \Psi^a_{2,\mathrm{DS}} + \bar{\mu}_{2\infty} \partial_{\bar{z}} \Psi^a_{2,\mathrm{DS}} - (\partial_z \bar{\mu}_{2\infty}) \Psi^a_{2,\mathrm{DS}} - \bar{\mu}_{2\infty} \partial_z \Psi^a_{2,\mathrm{DS}} \\ (\partial_{\bar{z}} \mu_{2\infty}) \Psi^a_{1,\mathrm{DS}} + \mu_{2\infty} \partial_{\bar{z}} \Psi^a_{1,\mathrm{DS}} - (\partial_z \mu_{2\infty}) \Psi^a_{1,\mathrm{DS}} - \mu_{2\infty} \partial_z \Psi^a_{1,\mathrm{DS}} \end{pmatrix}
$$

$$
- \frac{1}{(\det m_\infty)^2} \begin{pmatrix} \bar{\mu}_{2\infty} \Psi^a_{2,\mathrm{DS}} \, (\mu_{1\infty} \partial_{\bar{z}} \bar{\mu}_{1\infty} + \mu_{2\infty} \partial_{\bar{z}} \bar{\mu}_{2\infty} - (\partial_z \mu_{1\infty}) \bar{\mu}_{1\infty} - (\partial_z \mu_{2\infty}) \bar{\mu}_{2\infty}) \\ \mu_{2\infty} \Psi^a_{1,\mathrm{DS}} \, (\mu_{1\infty} \partial_{\bar{z}} \bar{\mu}_{1\infty} + \mu_{2\infty} \partial_{\bar{z}} \bar{\mu}_{2\infty} - (\partial_z \mu_{1\infty}) \bar{\mu}_{1\infty} - (\partial_z \mu_{2\infty}) \bar{\mu}_{2\infty}) \end{pmatrix}.
$$

$$(5.24)$$

We prove that the right-hand sides of (5.23) and (5.24) are equal by showing that the following two expressions are equal

$$
\frac{1}{\det m_\infty} \begin{pmatrix} (\partial_z \mu_{1\infty}) \Psi^a_{1,\mathrm{DS}} + \mu_{1\infty} \frac{(\bar{\mu}_{1\infty} \bar{\mu}_{2\infty} - \bar{\mu}_{2\infty} \partial_{\bar{z}} \bar{\mu}_{1\infty}) \Psi^a_{2,\mathrm{DS}}}{\det m_\infty} \\ (\partial_{\bar{z}} \bar{\mu}_{1\infty}) \Psi^a_{2,\mathrm{DS}} + \bar{\mu}_{1\infty} \frac{(\mu_{2\infty} \partial_z \mu_{1\infty} - \mu_{1\infty} \partial_z \mu_{2\infty}) \Psi^a_{1,\mathrm{DS}}}{\det m_\infty} \end{pmatrix}
$$

$$
+ \frac{1}{(\det m_\infty)^2} \begin{pmatrix} -|\mu_{1\infty}|^2 (\partial_z \mu_{1\infty}) \Psi^a_{1,\mathrm{DS}} - \mu_{1\infty} \bar{\mu}_{2\infty} (\partial_z \mu_{2\infty}) \Psi^a_{1,\mathrm{DS}} \\ -|\mu_{1\infty}|^2 (\partial_{\bar{z}} \bar{\mu}_{1\infty}) \Psi^a_{2,\mathrm{DS}} - \mu_{2\infty} \bar{\mu}_{1\infty} (\partial_{\bar{z}} \bar{\mu}_{2\infty}) \Psi^a_{2,\mathrm{DS}} \end{pmatrix}
$$

$$(5.25)$$

and

$$
\frac{1}{\det m_\infty} \begin{pmatrix} (\partial_{\bar{z}} \bar{\mu}_{2\infty}) \Psi^a_{2,\mathrm{DS}} + \bar{\mu}_{2\infty} \frac{(\mu_{2\infty} \partial_z \mu_{1\infty} - \mu_{1\infty} \partial_z \mu_{2\infty}) \Psi^a_{1,\mathrm{DS}}}{\det m_\infty} \\ -(\partial_z \mu_{2\infty}) \Psi^a_{1,\mathrm{DS}} - \mu_{2\infty} \frac{(\bar{\mu}_{1\infty} \partial_{\bar{z}} \bar{\mu}_{2\infty} - \bar{\mu}_{2\infty} \partial_{\bar{z}} \bar{\mu}_{1\infty}) \Psi^a_{2,\mathrm{DS}}}{\det m_\infty} \end{pmatrix}
$$

$$
- \frac{1}{(\det m_\infty)^2} \begin{pmatrix} |\mu_{2\infty}|^2 (\partial_{\bar{z}} \bar{\mu}_{2\infty}) \Psi^a_{2,\mathrm{DS}} + \mu_{1\infty} \bar{\mu}_{2\infty} (\partial_{\bar{z}} \bar{\mu}_{1\infty}) \Psi^a_{2,\mathrm{DS}} \\ -|\mu_{2\infty}|^2 (\partial_z \mu_{2\infty}) \Psi^a_{1,\mathrm{DS}} - \bar{\mu}_{1\infty} \mu_{2\infty} (\partial_z \mu_{1\infty}) \Psi^a_{1,\mathrm{DS}} \end{pmatrix}.
$$

$$(5.26)$$

Therefore, the gauge and adjoint transformations are commutative under the condition (5.11).

Lemma 5.4. *Spectral data of the adjoint problem (4.3) are related to the spectral data of the direct problem (2.4) as*

$$
b^a(k) = \overline{b(k)} \qquad k^a_j = k_j \qquad z^a_j = z_j \qquad c^a_j = -\bar{c}_j. \tag{5.27}
$$

Proof. Applying the gauge transformations of lemmas 3.1 and 5.1 to spectral representations (2.28) and (4.24) for $\mu(z, k)$ and $\mu^a(z, k)$, we obtain the spectral representations for $\mu_{\mathrm{DS}}(z, k)$ and $\mu^a_{\mathrm{DS}}(z, k)$:

$$
\mu_{\mathrm{DS}}(z, k) = \mathbf{e}_1 + \sum_{j=1}^n \frac{\mathrm{i}}{k - k_j} (m_\infty)^{-1} \Phi_j(z) + \frac{1}{2\pi \mathrm{i}} \iint \frac{\mathrm{d}k' \wedge \mathrm{d}\bar{k}'}{k' - k} b(k') (m_\infty)^{-1} \mathbf{N}_\mu(z, k')
$$

$$(5.28)$$

$$
\mu^a_{\mathrm{DS}}(z, k) = \mathbf{e}_2 - \sum_{j=1}^n \frac{\mathrm{i}}{k - k^a_j} (m^a_\infty)^{-1} \Phi^a_j(z) - \frac{1}{2\pi \mathrm{i}} \iint \frac{\mathrm{d}k' \wedge \mathrm{d}\bar{k}'}{k' - k} b^a(k') (m^a_\infty)^{-1} \mathbf{N}^a_\mu(z, k').
$$

$$(5.29)$$

Similarly, applying gauge transformations to the limiting formulae (2.26) and (4.23), we obtain

$$
\lim_{k \to k_j} \left(\mu_{\mathrm{DS}}(z, k) - \frac{\mathrm{i}}{k - k_j} (m_\infty)^{-1} \Phi_j(z) \right) = (z + z_j) (m_\infty)^{-1} \Phi_j(z) + c_j (m_\infty)^{-1} \Phi'_j(z)
$$

$$(5.30)$$

$$\lim_{k \to k_j^a} \left(\boldsymbol{\mu}_{\mathrm{DS}}^a(z, k) + \frac{\mathrm{i}}{k - k_j^a} \left(m_\infty^a \right)^{-1} \boldsymbol{\Phi}_j^a(z) \right) = \left(z + z_j^a \right) \left(m_\infty^a \right)^{-1} \boldsymbol{\Phi}_j^a(z) + c_j^a \left(m_\infty^a \right)^{-1} \boldsymbol{\Phi}_j^{a\prime}(z).$$

(5.31)

Since eigenfunctions of the DS-II and IS-II equations are related by the gauge transformation as

$$(m_\infty)^{-1} \mathbf{N}_\mu = \mathbf{N}_{\mu,\mathrm{DS}} \qquad \left(m_\infty^a \right)^{-1} \mathbf{N}_\mu^a = \mathbf{N}_{\mu,\mathrm{DS}}^a$$

$$(m_\infty)^{-1} \boldsymbol{\Phi}_j = \boldsymbol{\Phi}_{j,\mathrm{DS}} \qquad \left(m_\infty^a \right)^{-1} \boldsymbol{\Phi}_j^a = \boldsymbol{\Phi}_{j,\mathrm{DS}}^a \qquad (5.32)$$

$$(m_\infty)^{-1} \boldsymbol{\Phi}_j' = \boldsymbol{\Phi}_{j,\mathrm{DS}}' \qquad \left(m_\infty^a \right)^{-1} \boldsymbol{\Phi}_j^{\prime a} = \boldsymbol{\Phi}_{j,\mathrm{DS}}^{\prime a}$$

we find the representations (5.32) and (5.28) and limiting formulae (5.29) and (5.31) for the DS-II equations with the same spectral data $\{b(k), k_j, c_j, z_j\}$ and $\{b^a(k), k_j^a, c_j^a, z_j^a\}$. The symmetry relations (5.27) were proved for the DS-II equations in [PS00b]. □

The relation (5.11) between $m_\infty^a(z)$ and $m_\infty(z)$ closes the system of equations (4.24) and (4.25) for eigenfunctions of the adjoint problem. For the explicit example of the single lump solution of the IS-II equations, we find from (3.22), (3.23) and (5.11) that

$$\mu_{1\infty}^a = \frac{-\mathrm{i}\overline{c_j}\, \mathrm{e}^{\mathrm{i}(k_j z + k_j^- z)}}{\bar{k}_j(|z + z_j|^2 + |c_j|^2)} \qquad \mu_{2\infty}^a = 1 - \frac{\mathrm{i}(\bar{z} + \bar{z}_j)}{k_j(|z + z_j|^2 + |c_j|^2)}. \qquad (5.33)$$

The algebraic system (4.25) for $b(k) = 0$ and $n = 1$ becomes closed with (5.11) and (5.27) and it has an exact solution:

$$\Phi_{1j}^a(z) = \frac{\bar{c}_j\, \mathrm{e}^{\mathrm{i}(k_j z + \overline{k_j} z)} \left[|z + z_j|^2 + |c_j|^2 + \frac{\mathrm{i}}{k_j}(z + z_j - \bar{z} - \bar{z}_j) \right]}{(|z + z_j|^2 + |c_j|^2)^2} \qquad (5.34)$$

$$\Phi_{2j}^a(z) = \frac{(\bar{z} + \bar{z}_j)(|z + z_j|^2 + |c_j|^2) - \frac{\mathrm{i}}{k_j}((\bar{z} + \bar{z}_j)^2 + |c_j|^2)}{(|z + z_j|^2 + |c_j|^2)^2}. \qquad (5.35)$$

Using the gauge transformation $\boldsymbol{\mu}_{\mathrm{DS}}^a = \left(m_\infty^a \right)^{-1} \boldsymbol{\mu}^a$, we find the eigenfunction of the adjoint problem for DS-II equations from (5.27) and (5.29):

$$\boldsymbol{\mu}_{\mathrm{DS}}^a(z, k) = \mathbf{e}_2 - \frac{\mathrm{i}}{k - k_j} \boldsymbol{\Phi}_{j,\mathrm{DS}}^a(z) \qquad (5.36)$$

where the bound state of the adjoint problem $\boldsymbol{\Phi}_{j,\mathrm{DS}}^a(z)$ is

$$\Phi_{1j,\mathrm{DS}}^a(z) = \frac{\bar{c}_j\, \mathrm{e}^{\mathrm{i}(k_j z + \overline{k_j} z)}}{|z + z_j|^2 + |c_j|^2} \qquad \Phi_{2j,\mathrm{DS}}^a(z) = \frac{\bar{z} + \bar{z}_j}{|z + z_j|^2 + |c_j|^2}. \qquad (5.37)$$

6. Applications

We use the main results of our analysis to study the spectral decomposition and surface representation theorems for the Ishimori-II equations.

The spectral decomposition theorem for the DS-II equations was proved in [PS00b]. Structural instability of lump solutions of the DS-II equations was studied by means of perturbation series expansions over a complete set of eigenfunctions of the linear problem (3.1). Using orthogonality and completeness of eigenfunctions of the linear problem for the DS-II equations and the gauge transformation between DS-II and IS-II equations, we construct a basis of the spectral decomposition for the IS-II equations. Similar to [PS00b], the

perturbation series expansions can be used to prove structural instability of lump solutions of the IS-II equations.

Theorem 6.1. *An arbitrary scalar function $f(z) \in L^2(\mathbb{R}^2)$ can be decomposed through the set of eigenfunctions $S = \{[m_\infty^{-1}\mathbf{N}_\mu]_2(z, k), [m_\infty^{-1}\Phi_j]_2, j = 1, \ldots, n\}$ as follows*

$$f(z) = \iint dk \wedge d\bar{k} \alpha(k) [m_\infty^{-1}\mathbf{N}_\mu]_2(z, k) + \sum_{j=1}^{n} \alpha_j [m_\infty^{-1}\Phi_j]_2(z) \qquad (6.1)$$

where

$$\alpha(k) = -\frac{1}{4\pi^2} \left(\left[(m_\infty^a)^{-1}\mathbf{N}_\mu^a \right]_1 (k), f \right) \qquad \alpha_j = \frac{1}{2\pi i} \left(\left[(m_\infty^a)^{-1}\Phi_j^a \right]_1, f \right) \qquad (6.2)$$

and the inner product for scalar functions is defined by

$$(g(k'), f(k)) = \iint g(z, k') f(z, k) \, dz \wedge d\bar{z}. \qquad (6.3)$$

Proof. The proof follows from propositions 3.3–3.5 of [PS00b].   □

Orthogonality and completeness relations for eigenfunctions of the IS-II equations can be formulated explicitly as follows.

Lemma 6.2. *Eigenfunctions $[m_\infty^{-1}\mathbf{N}_\mu]_2(z, k)$ and $[m_\infty^{-1}\Phi_j]_2$ are orthogonal with respect to the eigenfunctions $[(m_\infty^a)^{-1}\mathbf{N}_\mu^a]_1(z, k)$ and $[(m_\infty^a)^{-1}\Phi_j^a]_1$ according to the orthogonality relations:*

$$\left(\left[(m_\infty^a)^{-1}\mathbf{N}_\mu^a \right]_1 (k'), [m_\infty^{-1}\mathbf{N}_\mu]_2(k) \right) = -2\pi^2 i \delta(k' - k) \qquad (6.4)$$

$$\left(\left[(m_\infty^a)^{-1}\mathbf{N}_\mu^a \right]_1 (k'), [m_\infty^{-1}\Phi_j]_2 \right) = 0 \qquad (6.5)$$

$$\left(\left[(m_\infty^a)^{-1}\Phi_j^a \right]_1, [m_\infty^{-1}\mathbf{N}_\mu]_2(k) \right) = 0 \qquad (6.6)$$

$$\left(\left[(m_\infty^a)^{-1}\Phi_l^a \right]_1, [m_\infty^{-1}\Phi_j]_2 \right) = 2\pi i \delta_{j,l}. \qquad (6.7)$$

Corollary 6.3. *The orthogonality relations for eigenfunctions of the IS-II equations take the form of weighted inner vector products defined by (4.2)*

$$\left(\left[(m_\infty^a)^{-1}\mathbf{g}^a \right]_1, [m_\infty^{-1}\mathbf{f}]_2(k) \right) = \langle \sigma \overline{g_1^a} \mathbf{e}_1 | f_2 \mathbf{e}_2 \rangle - \tfrac{1}{2} \langle \sigma \overline{\mathbf{g}^a} | (P - \sigma_3) \mathbf{f} \rangle. \qquad (6.8)$$

Remark 6.4. The orthogonality relations (6.4)–(6.7) for the IS-II equations can be proved directly from the linear systems (2.4) and (4.3), with the use of weighted inner products in the right-hand side of (6.8).

Lemma 6.5. *Eigenfunctions $[m_\infty^{-1}\mathbf{N}_\mu]_2(z, k)$ and $[m_\infty^{-1}\Phi_j]_2$ are complete with respect to the adjoint eigenfunctions $[(m_\infty^a)^{-1}\mathbf{N}_\mu^a]_1(z, k)$ and $[(m_\infty^a)^{-1}\Phi_j^a]_1$ according to the completeness relation:*

$$\delta(z' - z) = -\frac{1}{2\pi^2 i} \iint dk \wedge d\bar{k} \left[(m_\infty^a)^{-1}\mathbf{N}_\mu^a \right]_1 (z', k) [m_\infty^{-1}\mathbf{N}_\mu]_2(z, k)$$

$$- \frac{1}{\pi} \sum_{j=1}^{n} \left[(m_\infty^a)^{-1}\Phi_j^a \right]_1 (z') [m_\infty^{-1}\Phi_j]_2(z). \qquad (6.9)$$

The linear problem for the DS-II equations and its adjoint was used in [K96, K00] to parametrize surfaces and integrable evolutions of surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$. Their results generalize the classical Weierstrass–Enneper representation for conformal parametrizations of minimal surfaces in $\mathbb{R}^3$, or stationary Lagrangian surfaces in $\mathbb{R}^4$ [HR02]. Similarly, we formulate the surface representation theorem by using the gauge transformation between the DS-II and IS-II equations.

Theorem 6.6. *Define the generalized Weierstrass formulae*

$$X^1 + iX^2 = \int_\Gamma (-\phi_1\phi_2\, dz' + \psi_1\psi_2\, d\bar{z}')$$

$$X^1 - iX^2 = \int_\Gamma (\bar{\psi}_1\bar{\psi}_2\, dz' - \bar{\phi}_1\bar{\phi}_2\, d\bar{z}')$$

$$X^3 + iX^4 = \int_\Gamma (\bar{\psi}_2\phi_1\, dz' + \psi_1\bar{\phi}_2\, d\bar{z}')$$

$$X^3 - iX^4 = \int_\Gamma (\bar{\psi}_1\phi_2\, dz' + \psi_2\bar{\phi}_1\, d\bar{z}')$$

where

$$(\phi_1, \psi_1)^{\mathrm{T}} = m_\infty^{-1}\boldsymbol{\mu}(z, k)\, e^{ikz} \qquad (\psi_2, -\phi_2)^{\mathrm{T}} = \left(m_\infty^a\right)^{-1}\boldsymbol{\mu}^a(z, k)\, e^{-ikz} \qquad (6.10)$$

and Γ is a contour in $\mathbb{C}$. Then, the Weierstrass formulae define a conformal immersion of a surface into $\mathbb{R}^4 : X^i(z, \bar{z}) : \mathbb{C} \mapsto \mathbb{R}^4$. The induced metric of the surface is

$$ds^2 = u_1 u_2\, dz\, d\bar{z} \qquad (6.11)$$

where $u_\alpha = |\phi_\alpha|^2 + |\psi_\alpha|^2$, $\alpha = 1, 2$, and the Gaussian and mean curvatures are respectively

$$K = -\frac{2}{u_1 u_2}\frac{\partial^2}{\partial z\partial\bar{z}}\log(u_1 u_2) \qquad H^2 = \frac{4}{u_1 u_2}\frac{|\bar{\mu}_{1\infty}\partial_{\bar{z}}\bar{\mu}_{2\infty} - \bar{\mu}_{2\infty}\partial_{\bar{z}}\bar{\mu}_{1\infty}|^2}{(|\mu_{1\infty}|^2 + |\mu_{2\infty}|^2)^2}. \qquad (6.12)$$

Proof. Proof follows from theorem 3 and proposition 1 in [K00]. □

Acknowledgments

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