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Bifurcations of new eigenvalues for the Benjamin–Ono equation

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A criterion for the emergence of new eigenvalues is found for the linear scattering problem associated with the Benjamin–Ono (BO) equation. This bifurcation occurs due to perturbations of nongeneric potentials which include the soliton solutions of the BO equation. The asymptotic approximation of an exponentially small new eigenvalue is derived. The method is based on the expansion of a localized function through a complete set of unperturbed eigenfunctions. Explicit expressions are obtained for the soliton potentials.
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I. INTRODUCTION

This paper studies the problem of soliton generation in the integro-differential Benjamin–Ono (BO) equation,

$$u_t + 2uu_x + Hu_{xx} = 0, \tag{1.1}$$

where

$$Hu = \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty} \frac{u(y)\,dy}{y-x}$$

and p.v. stands for the principal value of the integral. The evolution equation (1.1) describes the propagation of long internal waves in a deep fluid (see Ref. 1 for review and references). The algebraic and spectral properties of the BO equation were intensely studied in development of the bilinear and inverse scattering transform methods.[2,3] The latter method allows one to relate the number and parameters of solitons generated by a smooth localized initial perturbation $u(x) = u(x,0)$ to the number and location of eigenvalues of the discrete spectrum for the linear spectral problem associated with the potential $u(x)$.

Generations of new solitons (and associated eigenvalues) were studied for several $(1+1)$-dimensional evolution equations in the limit of small and, alternatively, large number of solitons.[4] For instance, the Korteweg–de Vries (KdV) equation associated with the Schrödinger operator possesses *no threshold* for the generation of solitons by a small initial disturbance[4] while the modified KdV (mKdV) equation *does* possess a certain threshold on the area of the potential,

$$\mathcal{M}[u] = \int_{-\infty}^{\infty} u\,dx. \tag{1.2}$$

This threshold is related to spectral properties of the Dirac operator.[5] When the potential has a large amplitude, the problem under consideration has a uniform quasiclassical solution (see Ref. 4 for the KdV equation) which was derived also within the rigorous analysis of the dispersionless limit of the evolution equations.[6]

Surprisely enough, the problem of soliton generation for the BO equation (1.1) has not been finally solved since some independent results have appeared to be contradictory. Suppose that the initial perturbation can be renormalized in dimensionless units, $u = U_0 U(x/L_0)$, where U_0 is a characteristic amplitude and L_0 is a characteristic length. Then, the number of solitons essentially depends on the sole parameter (the so-called Ursell parameter), $\sigma = U_0 L_0$. In the limit $\sigma \gg 1$, the potential generates a large number m of solitons, an approximation for m was found by Matsuno[7] and confirmed by Miloh *et al.*,[8]

$$m = \frac{1}{2\pi} \int_{u(x)\geq 0} u(x)\,dx. \tag{1.3}$$

Some early numerical and analytical data on the generation of solitons by an initial disturbance of an algebraic profile, $u = 2a/(1+x^2)$ were reported in Refs. 9 and 10. An important question, whether a small initial perturbation, i.e., that in the limit $\sigma \ll 1$ $(a \ll 1)$, can support propagation of at least a single soliton, remained open however. This problem was first addressed by Pelinovsky and Stepanyants[11] who considered linear approximation in the pulse propagation. The linear part of the BO equation (1.1) has a self-similar solution preserving the area (1.2),

$$u = \frac{1}{t^{1/2}}\, U\left(\frac{x}{t^{1/2}}\right),$$

i.e., the characteristic amplitude and length of an initial pulse evolves as $U_0 \sim t^{-1/2}$ and $L_0 \sim t^{1/2}$, so that the Ursell parameter σ is effectively constant in time. These arguments imply that an initially small wave disturbance with $\sigma \ll 1$ remains effectively linear for all times t and hence does not support propagation of a soliton since the latter realizes a balance of nonlinear and dispersive effects for the finite value of σ.

Another way to predict the existence of a threshold for the soliton generation (see Ref. 8) is to use the fact that the area (1.2) is invariant for soliton solutions of Eq. (1.1). A single soliton is

$$u = u_s(x - vt) = \frac{2v}{1 + v^2(x - vt - x_0)^2}, \tag{1.4}$$

where x_0, v are arbitrary parameters with the constraint $v > 0$. Using Eqs. (1.2) and (1.4), one concludes that $\mathcal{M}_s = \mathcal{M}[u_s] = 2\pi$, and thus the area does not depend on the amplitude of the soliton. For the mKdV equation, this property is related to the existence of threshold on the soliton generation, where perturbations with $\mathcal{M}[u] \leq \frac{1}{2}\mathcal{M}_s$ do not support a soliton.[5] Direct numerical simulations of the BO equation did not display formation of a soliton for an initial perturbation with $\mathcal{M} < \mathcal{M}_s$ (see Fig. 2 in Ref. 8). However, contrary to these preliminary predictions, we show that a soliton can still be generated in the BO equation (1.1) by a small initial perturbation without a threshold.

The problem at the center of the analysis is the linear spectral problem associated with the potential $u(x)$,

$$i\phi_x^+ + k(\phi^+ - \phi^-) = -u(x)\phi^+, \tag{1.5}$$

where k is a spectral parameter and $\phi^{\pm}(x)$ are the limit values, when x tends to the real axis, of analytical functions in upper/lower half-planes of x. The pioneer analysis of direct and inverse scattering problems for Eq. (1.5) was developed by Fokas and Ablowitz,[12] who showed that the continuous spectrum is located for real positive values of k while the discrete spectrum consists of isolated nondegenerate eigenvalues for real negative values of k. Various aspects of the spectral theory including in particular the complicated asymptotic behavior of the Jost functions (the eigenfunctions associated with the continuous spectrum) were addressed by many authors.[13–18]

Based on the ''improved'' eigenfunctions (i.e., those free of secular divergences at $k \to 0^+$), Coifman and Wickerhauser[15] proved that the scattering problem (1.5) has no bound states in a neighborhood of the origin, if $u(x) \to O(|x|^{-1-\mu})$, $\mu > 0$ (Theorem 7.1 of Ref. 15). Although this result seems to agree with qualitative arguments mentioned above, direct analysis of the problem (1.5) for the algebraic potential $u(x) = 2a/(1+x^2)$ did reveal the existence of a single negative eigenvalue for $0 < a < 1$.[10] The eigenvalue approaches zero very fast as $a \to 0^+$ and this fact makes the detection of the eigenvalue difficult for a small.

The reason for apparent discrepancy between the analytical and numerical results is caused by a very special structure of the Jost functions in the limit $k \to 0^+$. Theorem 7.1 of Ref. 15 is valid for *generic* potentials satisfying $n_0 \neq 0$, where

$$n_0 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} u(x) n(x) dx. \tag{1.6}$$

Here $n(x)$ is the limiting Jost function (see Eq. (2.13) below) which is logarithmically divergent as $k \to 0^+$. However, if the potential is *nongeneric*, i.e., it satisfies $n_0 = 0$, then the limiting Jost function is bounded in the limit $k \to 0^+$ and properties of the scattering problem are modified.[16,17] The zero potential $u(x) = 0$ as well as soliton solutions are particular examples of nongeneric potentials.

Recently, Kaup *et al.*[18] proved the completeness relation for the problem (1.5) and calculated variations of the scattering data under the action of a perturbation of the potential $u(x)$. These variational formulas are derived for generic potentials when the number of bound states is invariant under small perturbations of a potential.

In this paper we study the perturbation of nongeneric potentials where the number of bound states may change depending on the type of the perturbation. More precisely, we consider a potential $u(x)$ in the form $u^\epsilon = u(x) + \epsilon \Delta u(x)$, where $\epsilon \ll 1$ and $u(x)$ satisfies the constraint $n_0 = 0$. In the particular case $u(x) = 0$, this problem reduces to the problem of soliton generation by a small initial perturbation. We derive a criterion for a new eigenvalue to emerge from the edge of the continuous spectrum at $k = 0$. In particular, new eigenvalues may always appear due to perturbations of the zero background and the soliton solutions. We find that the asymptotic behavior of the new eigenvalue is *exponentially small* in terms of the parameter of the perturbation ϵ and obtain the corresponding bound state and the perturbed Jost functions of the continuous spectrum.

This result is to be compared to the classical spectral problem for the Schrödinger operator $-\Delta \psi + \epsilon V(x) \psi = \lambda \psi$, with $\epsilon \ll 1$.[19,20] As above, the problem is to find the condition on the potential so that there exists an eigenvalue λ that bifurcates from the edge of the continuous spectrum, and to compute, as $\epsilon \to 0$, its asymptotic value and the corresponding eigenfunction. In one space dimension, the bifurcation occurs when $\mathcal{M} = \int_{-\infty}^{\infty} V(x) dx < 0$ and one finds the eigenvalue $\lambda \sim -\epsilon^2 \mathcal{M}^2/4$.[19] The situation is more complex in two dimensions even with a radially symmetric potential. The criterion for the bifurcation to occur is $\mathcal{M} = \int_0^\infty V(r) r dr < 0$ and the eigenvalue is $\lambda \sim -\lambda_0 \exp(2/\epsilon \mathcal{M})$, where λ_0 can be calculated explicitly in terms of integrals of the potential.[21] Note that the behavior of the new eigenvalue for the BO spectral problem is very similar to the latter case.

For the Schrödinger spectral problem, the analysis of the bifurcation can be performed by various methods such as a direct two-scale asymptotic expansion technique,[19] the analysis based on the positive definiteness of the self-adjoint operators,[20] the analysis involving the Green's function representation,[21] or the study of analytical behavior of transmission coefficients in the scattering problem or, their generalizations given by the Evans' function.[22]

The BO spectral problem has the form of an integro-differential equation which make direct applications of the above methods difficult. In this article, we develop a new approach based on the completeness of Jost functions of the unperturbed problem.

The plan of this article is as follows. In Sec. II we recall the main results on the spectral properties of Eq. (1.5). The regular perturbation theory is described in Sec. III. The bifurcation of a new negative eigenvalue from the origin cannot be obtained through a regular perturbation method and we develop a revised method in Sec. IV to analyze this bifurcation. We derive the leading order terms for the new eigenvalue and for the associated bound state at short and large distances, as well as for the variation of the continuous spectrum data. Explicit expressions are given in Sec. V for the soliton potentials. Section VI is devoted to discussions. Appendix A gives useful relations for eigenfunctions of the BO scattering problem and Appendix B presents new relations for the roots of the Laguerre polynomials which appear in the course of our analysis.

II. SPECTRAL PROPERTIES OF THE BO ASSOCIATED PROBLEM

In this section, we recall some basic results on the direct scattering problem (1.5) that will be used in the analysis. We assume that the potential $u=u(x)$ is real and has a sufficient decay to ensure the convergence of the area integral (1.2), i.e., $u\sim O(|x|^{-2-\mu})$ as $x\to\infty$, $\mu>0$. The details and proofs of these results can be found in Refs. 12 and 17.

A. Spectrum and scattering data

For all $k>0$, there are two solutions N and $\bar{N}$ (the Jost functions) to Eq. (1.5), defined by their boundary conditions as $x\to\infty$,

$$\bar{N}(x,k)\to 1, \quad N(x,k)\to e^{ikx}. \tag{2.1}$$

Using the projection operators

$$P^{\pm}(v)=\pm\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{v(y)dy}{y-(x\pm i0)}=\frac{1}{2}(v\mp iHv), \tag{2.2}$$

it is proved that $\bar{N}$ and N satisfy the integro-differential equations

$$i\bar{N}_x+k\bar{N}=-P^{+}(u\bar{N})+k, \tag{2.3a}$$

$$iN_x+kN=-P^{+}(uN). \tag{2.3b}$$

Equation (2.1) has also a discrete number of bound states $\{\Phi_j(x)\}_{j=1}^{m}$, where m is finite, associated to discrete eigenvalues $k_j<0$. They satisfy

$$i\Phi_{jx}+k_j\Phi_j=-P^{+}(u\Phi_j), \tag{2.3c}$$

with the boundary condition

$$\Phi_j(x)\to\frac{1}{x} \quad \text{as} \quad x\to\infty.$$

We denote ϕ^{+} the set of eigenfunctions $\phi^{+}=\{\bar{N}(\underline{x},k);N(x,k);\{\Phi_j(x)\}_{j=1}^{m}\}$. Alternatively, one can consider the set of eigenfunctions $\{M(x,k);\bar{M}(x,k);\{\Phi_j(x)\}_{j=1}^{m}\}$ which satisfy the same equations (2.3) but with the left-side boundary conditions

$$M(x,k)\to 1, \quad \bar{M}(x,k)\to e^{ikx}, \quad \Phi_j(x)\to\frac{1}{x}, \tag{2.4}$$

as $x\to-\infty$. These two sets are related by the scattering equations

$$M(x,k)-\bar{N}(x,k)=\beta(k)N(x,k), \quad \bar{M}(x,k)=\Gamma(k)N(x,k), \quad k>0. \tag{2.5}$$

In the case of a real potential $u(x)$, the scattering coefficients are given by[17]

$$\Gamma(k)=\exp\left[\frac{1}{2\pi i}\int_0^k\frac{|\beta(k')|^2}{k'}\,dk'\right] \tag{2.6a}$$

and

$$\beta(k)=i\int_{-\infty}^{\infty}u(x)N^{*}(x,k)dx. \tag{2.6b}$$

The scattering equation (2.5) relates the functions $M(x,k)$ and $\bar{N}(x,k)$ which are analytic in $\mathrm{Im}(k)>0$ and $\mathrm{Im}(k)<0$, respectively. It has the form of a Riemann–Hilbert problem.[3] In Ref. 12, Fokas and Ablowitz derived a solution of this Riemann–Hilbert problem for the function $\bar{N}(x,k)$ at $\mathrm{Im}(k)\leqslant 0$, in terms of $N(x,k)$ for real k,

$$\bar{N}(x,k)=1-i\sum_{j=1}^{m}\frac{\Phi_j(x)}{k-k_j}+\frac{1}{2\pi i}\int_0^\infty\frac{\beta(k')N(x,k')dk'}{k'-(k-i0)}. \tag{2.7}$$

Here simple poles correspond to the discrete spectrum located at m negative eigenvalues $k_j<0$ while the integral represents the contribution from the continuous spectrum located for $k>0$. The scattering data form a closed system when Eq. (2.7) is coupled with the relation,

$$\frac{\partial N(x,k)}{\partial k}=ixN(x,k)+\frac{\beta^*(k)}{2\pi ik}\bar{N}(x,k) \tag{2.8a}$$

and the limiting behavior of $\bar{N}(x,k)$ as $k\to k_j$ is to be used,

$$\lim_{k\to k_j}\left[\bar{N}(x,k)+\frac{i\Phi_j(x)}{k-k_j}\right]=(x+\gamma_j)\Phi_j(x), \tag{2.8b}$$

where γ_j are the phase constants. The spectral data $\{k_j,\gamma_j\}_{j=1}^m$ can be expressed through the eigenfunctions $\Phi_j(x)$ according to the expressions,

$$k_j=\frac{1}{2\pi i}\int_{-\infty}^{\infty}u(x)\Phi_j(x)dx \tag{2.9a}$$

and

$$\gamma_j=\frac{1}{2\pi k_j}\int_{-\infty}^{\infty}x|\Phi_j(x)|^2dx-\frac{i}{2k_j}. \tag{2.9b}$$

Equation (2.9a) follows from Eq. (2.3c) (see Refs. 12 and 17), while Eq. (2.9b) can be obtained from Eq. (A4a) of Appendix A.

It was recently found for other spectral problems of inverse scattering[23] that the coalescence of two simple poles may occur and, in this case, the proper solution to the Riemann–Hilbert problem must include also multiple poles corresponding to degenerate eigenvalues. However, this coalescence is not possible for the BO spectral problem (1.5) associated with the real potentials. Indeed, suppose that $k_1=k_2+\epsilon$, where $\epsilon\ll 1$ and expand a solution to Eqs. (2.7) and (2.8) into an asymptotic series,[23]

$$\begin{cases}\Phi_1=-\dfrac{1}{\epsilon}\Phi^{(-1)}(x)+\Phi_1^{(0)}(x)+O(\epsilon),\\[2mm] \Phi_2=\dfrac{1}{\epsilon}\Phi^{(-1)}(x)+\Phi_2^{(0)}(x)+O(\epsilon),\end{cases} \tag{2.10a}$$

$$\begin{cases}\gamma_1=\dfrac{i}{\epsilon}+\gamma^{(0)}+O(\epsilon),\\[2mm] \gamma^2=-\dfrac{i}{\epsilon}+\gamma^{(0)}+O(\epsilon).\end{cases} \tag{2.10b}$$

This expansion provides a double pole in the limit $\epsilon\to 0$ for the function $\bar{N}(x,k)$ and satisfy the limiting relations (2.8b). However, it contradicts the balance prescribed by Eq. (2.9b), the latter being valid for real potential $u(x)$. Therefore, multiple poles are not present in the representation (2.7) and the eigenvalues $k=k_j$ for $j=\overline{1,m}$ are all simple.

The spectral data $\{\beta(k)\},\{k_j,\gamma_j\}_{j=1}^m$ characterize the BO equation as a completely integrable system[18] and they can be calculated from the Jost functions $N(x,k)$ and the bound states $\Phi_j(x)$ by means of Eqs. (2.6) and (2.9). The functions $N(x,k)$ and $\Phi_j(x)$ satisfy a self-consistent system of integral and algebraic equations[17] which is a crucial step in inverse scattering. The potential $u(x,t)$ can be recovered for $t\geqslant 0$ by means of the formula,

$$u=\frac{1}{2\pi i}\int_0^\infty \beta(k')N(x,k')dk'+i\sum_{j=1}^m \Phi_j(x)+\text{c.c.} \tag{2.11}$$

if the time evolution of the spectral data is taken into account.[12,17] Inverse scattering results will not be used for the problem under consideration. Instead, we mention the well-known fact[12] that the continuous spectrum of the scattering problem (1.5) corresponds to the dispersive wave packets (radiation) induced by the initial potential $u(x)$ while the bound states of the discrete spectrum correspond to solitons (1.4) with the parameters $v=-2k_j$ and $x_0=-\text{Re}(\gamma_j)$. These parameters induced by the initial potential are constant in the time, while the solitons become separated due to different values of their velocities.

B. Asymptotic behavior of the Jost functions

The asymptotic behavior of the Jost functions and of the scattering data as $k\to 0^+$ can be studied by means of their Green's function representation.[16,17] For generic potentials, the Jost functions vanish as $k\to 0^+$ according to the scaling approximation,

$$N(x,k)\to\frac{n(x)}{1+n_0(\gamma+\ln(ik))}+O\left(\frac{k}{\ln k}\right), \tag{2.12}$$

where $\gamma\approx 0.577$ is the Euler's constant, n_0 is defined by Eq. (1.6) and $n(x)$ satisfies

$$in_x=-P^+(un). \tag{2.13a}$$

Equivalently, $n(x)$ is the unique solution to the integral equation

$$n(x)=1-i\int_x^\infty u(y)n(y)dy-\frac{1}{2\pi}\int_{-\infty}^\infty u(y)n(y)\ln(x-y-i0)dy. \tag{2.13b}$$

We notice that the representation (2.12) describes an improved approximation for the asymptotic behavior of the Jost function compared to the conventional expansion,[13]

$$N(x,k)\to\frac{n(x)}{n_0\ln k}+O\left(\frac{1}{(\ln k)^2}\right). \tag{2.14a}$$

Using Eqs. (2.6) and (2.14a) we find the uniform asymptotic representation for the scattering coefficients as $k\to 0^+$,

$$\beta(k)\to\frac{2\pi i}{\ln k}+O\left(\frac{1}{(\ln k)^2}\right),\quad \Gamma(k)=1+\frac{2\pi i}{\ln k}+O\left(\frac{1}{(\ln k)^2}\right). \tag{2.14b}$$

It follows from Eq. (2.13b) that the limiting function $n(x)$ is logarithmically secular and violates the boundary condition (2.1)

$$n(x)\to\begin{cases}1-n_0\ln x & \text{as } x\to\infty\\ 1-n_0\,\text{Ln}\,x & \text{as } x\to-\infty\end{cases}, \tag{2.15}$$

where $\text{Ln}\,x$ is the principal value of the logarithmic function. Thus, the limiting point $k=0$ does not belong to the continuous spectrum for generic potentials, i.e., $N(x,k)\to 0$ as $k\to 0^+$.

On the contrary, the condition $n_0=0$ defines a class of nongeneric potentials which include the zero background $u(x)=0$ and soliton solutions. In this case, the limiting function $n(x)$ is not secular [see Eq. (2.15)] and the limiting point $k=0$ *does* belong the continuous spectrum, i.e.,

$$N(x,k) \to n(x) + kn_1(x) + O(k^2 \ln k), \tag{2.16a}$$

and

$$\beta(k) \to k\beta_1 + O(k^2 \ln k)$$

$$\Gamma(k) \to 1 + \frac{|\beta_1|^2}{4\pi i} k^2 + O(k^3 \ln k) \tag{2.16b}$$

as $k \to 0^+$. In order to prove validity of the expansions (2.16), we use the relations (2.6b) and (2.8a) in the limit $k \to 0^+$. At leading order, $\bar{N}(x,k) \sim n(x)$ as $k \to 0^+$.[17] As a result, we find from Eq. (2.8a) the following representation for $n_1(x)$:

$$n_1(x) = i\left(x - \frac{\beta_1^*}{2\pi}\right) n(x), \tag{2.17a}$$

and from Eq. (2.6b) the explicit formula for β_1,

$$\beta_1 = \text{p.v.} \int_{-\infty}^{\infty} xu(x) n^*(x) dx. \tag{2.17b}$$

Any perturbation to the nongeneric potential changes the asymptotic behavior as $k \to 0^+$ and may lead to the occurrence of a new bound state for $k < 0$. This bifurcation is studied in Sec. IV.

C. Number of bound states

For both generic and nongeneric potentials, the total area (1.2) is related to the number of bound states m by

$$\mathcal{M}[u] = 2\pi m - \frac{1}{2\pi} \int_0^{\infty} \frac{|\beta(k)|^2}{k} dk. \tag{2.18}$$

To obtain this formula (see Ref. 17), one writes Eq. (2.7) at $k = 0$, multiplies it by u and integrates over **R**. The integral $\int_{-\infty}^{\infty} u(x) \bar{N}(x,0) dx = 0$ for both generic and nongeneric potentials. The other terms are evaluated using Eqs. (2.6b) and (2.9a).

Another way to see that

$$\frac{1}{2\pi}\left(\mathcal{M}[u] + \frac{1}{2\pi} \int_0^{\infty} \frac{|\beta(k)|^2}{k} dk\right)$$

is an integer is to take the limit $k \to \infty$, $x \to -\infty$ in the second equation (2.5) and to use Eqs. (2.4), (2.6a) as well as the asymptotic representation of $N(x,k)$ for $k \to \infty$,[17]

$$N(x,k) \sim \exp\left[ikx - i\int_x^{\infty} u(x) dx\right] \quad \text{for real } x. \tag{2.19}$$

In agreement with the distribution (1.3) for large m, the positive regions of the potential $u(x)$ give birth to bound states (solitons) while the negative regions are sources of continuous spectrum (radiation).

III. REGULAR PERTURBATION THEORY

Suppose that the scattering problem (1.5) for $u = u(x)$ has m bound states $\Phi_j(x)$ for $k = k_j < 0$, $j = \overline{1,m}$ and let $N(x,k)$ be the Jost functions for $k > 0$. In Ref. 18, Kaup *et al.* proved the orthogonality conditions

$$\int_{-\infty}^{\infty} N^*(x,k') N(x,k) dx = 2\pi\delta(k - k'), \tag{3.1a}$$

$$\int_{-\infty}^{\infty} N^*(x,k)\Phi_j(x)dx = 0, \tag{3.1b}$$

$$\int_{-\infty}^{\infty} \Phi_l^*(x)\Phi_j(x)dx = -2\pi k_j \delta_{jl} \tag{3.1c}$$

as well as the completeness relation,

$$\int_0^{\infty} N^*(y,k)N(x,k)dk - \sum_{j=1}^{m} \frac{\Phi_j^*(y)\Phi_j(x)}{k_j} = \frac{-i}{y-(x+i0)}. \tag{3.2}$$

Any function $\Phi^+(x)$ such that $\Phi^+ \sim O(|x|^{-1-\mu})$ as $x\to\infty,\ \mu\geqslant 0$, can be expanded over the basis $\{N(x,k);\{\Phi_j(x)\}_{j=1}^m\}$ as follows

$$\Phi^+(x) = \int_0^{\infty} \frac{\alpha(k)N(x,k)}{k-\kappa}\, dk + \sum_{j=1}^{m} \frac{\alpha_j \Phi_j(x)}{k_j - \kappa}, \tag{3.3}$$

where $\alpha(k),\ \alpha_j$ are coefficients of the expansion and the arbitrary parameter κ is introduced for convenience in the following analysis and will be specified in the next subsection. The coefficients $\alpha(k)$ and α_j can be expressed through the function $\Phi^+(x)$ using the orthogonality relations (3.1).

A classical problem of perturbation theory is to study variations of the spectral data subject to a small variation of the potential. We assume that the perturbed potential is expressed as $u^{\epsilon}(x) = u(x) + \epsilon\Delta u(x)$, where $\epsilon \ll 1$, and the functions $u(x)$ and $\Delta u(x)$ decay at infinity as $u(x),\Delta u(x) \sim O(|x|^{-2-\mu})$ with $\mu \geqslant 0$. Here and in subsequent sections, we study the variations of the spectral data for perturbed generic and nongeneric potentials.

A. Variations of data of the discrete spectrum

Suppose that $\Phi^+(x)$ solves Eq. (2.3c) with $u^{\epsilon} = u(x) + \epsilon\Delta u(x)$ and κ is the corresponding eigenvalue. The linear eigenvalue problem (2.3c) can be reduced to a set of integral equations for the coefficients $\alpha(k)$ and α_j in the form

$$\alpha(k) = \frac{\epsilon}{2\pi}\left[\int_0^{\infty} \frac{K(k,k')\alpha(k')}{k'-\kappa}\, dk' + \sum_{j=1}^{m} \frac{K_j(k)\alpha_j}{k_j - \kappa}\right], \tag{3.4a}$$

$$\alpha_j = -\frac{\epsilon}{2\pi k_j}\left[\int_0^{\infty} \frac{K_j^*(k')\alpha(k')}{k'-\kappa}\, dk' + \sum_{l=1}^{m} \frac{K_{jl}\alpha_l}{k_l - \kappa}\right], \tag{3.4b}$$

where

$$K(k,k') = \int_{-\infty}^{\infty} \Delta u(x) N^*(x,k) N(x,k') dx,$$

$$K_j(k) = \int_{-\infty}^{\infty} \Delta u(x) N^*(x,k)\Phi_j(x) dx,$$

$$K_{jl} = \int_{-\infty}^{\infty} \Delta u(x) \Phi_j^*(x)\Phi_l(x) dx.$$

Consider a particular eigenvalue $\kappa = k_j^{\epsilon} < 0$ and the corresponding bound state $\Phi^+ = \Phi_j^{\epsilon}(x)$. The analysis of Eq. (3.4b) shows that the asymptotic balance is satisfied by the following Taylor expansion:

$$k_j^{\epsilon} = k_j + \epsilon\Delta k_j + \epsilon^2 \Delta_2 k_j + O(\epsilon^3) \tag{3.5a}$$

and

$$\alpha_j=(k_j-k_j^\epsilon)(1+\epsilon\alpha_j^{(1)}+O(\epsilon^2)), \tag{3.5b}$$

$$\alpha_l=-\frac{\epsilon K_{lj}}{2\pi k_l}+O(\epsilon^2), \quad l\neq j, \tag{3.5c}$$

$$\alpha(k)=\frac{\epsilon K_j(k)}{2\pi}+O(\epsilon^2). \tag{3.5d}$$

The correction Δk_j is obtained from Eq. (3.4b) as

$$\Delta k_j=\frac{1}{2\pi k_j}K_{jj}. \tag{3.6}$$

We then proceed with the first-order correction term in $\Phi_j^\epsilon(x)$ which is given through Eqs. (3.3) and (3.4) as $\Phi_j^\epsilon=\Phi_j+\epsilon\Delta\Phi_j+O(\epsilon^2)$, where

$$\Delta\Phi_j=\alpha_j^{(1)}\Phi_j+\frac{1}{2\pi}\int_0^\infty\frac{K_j(k)N(x,k)}{k-k_j}\,dk-\sum_{l\neq j}\frac{K_{lj}\Phi_l(x)}{2\pi k_l(k_l-k_j)}. \tag{3.7}$$

The constant $\alpha_j^{(1)}$ can be found from the boundary condition (2.3) by removing the $O(x^{-1})$ term as $x\to\infty$. Using the boundary conditions (A3) of Appendix A this leads to the expression,

$$\alpha_j^{(1)}=\sum_{l\neq j}\frac{K_{lj}}{2\pi k_l(k_l-k_j)}-\frac{K_j(0)}{2\pi i k_j}-\frac{1}{4\pi^2}\int_0^\infty\frac{\beta^*(k)K_j(k)}{k(k-k_j)}\,dk. \tag{3.8a}$$

This formula is equivalent to that given in Ref. 18,

$$\alpha_j^{(1)}=-\frac{1}{2\pi i k_j}\int_{-\infty}^\infty\Delta u(x)(x+\gamma_j)|\Phi_j(x)|^2dx. \tag{3.8b}$$

In order to prove this, we rewrite Eq. (3.8b) as

$$\alpha_j^{(1)}=-\frac{1}{2\pi i k_j}\left(\int_{-\infty}^\infty P^+(\Delta u\Phi_j)dx+\int_{-\infty}^\infty(x+\gamma_j)\Phi_j^*P^+(\Delta u\Phi_j)dx\right). \tag{3.9}$$

Using the completeness formula (3.2) we represent $P^+(\Delta u\Phi_j)$ as

$$P^+(\Delta u\Phi_j)=\frac{1}{2\pi}\int_0^\infty K_j(k)N(x,k)dk-\sum_{l=1}^m\frac{K_{lj}\Phi_l(x)}{2\pi k_l}. \tag{3.10}$$

Substituting Eq. (3.10) into Eq. (3.9) and using Eqs. (A2) and (A4) of Appendix A we recover Eq. (3.8a).

Applying the same method, we expand the relations (2.9) into the asymptotic series for k_j^ϵ [see Eq. (3.5a)] and for γ_j^ϵ, $\gamma_j^\epsilon=\gamma_j+\epsilon\Delta\gamma_j+O(\epsilon^2)$. The first-order perturbation terms are thus expressed as

$$\Delta k_j=\frac{1}{2\pi i}\int_{-\infty}^\infty(\Delta u\Phi_j+u\Delta\Phi_j)dx, \tag{3.11a}$$

$$\Delta\gamma_j=\frac{1}{2\pi k_j}\int_{-\infty}^\infty(x+\gamma_j)(\Phi_j\Delta\Phi_j^*+\Phi_j^*\Delta\Phi_j)dx. \tag{3.11b}$$

Substitution of Eqs. (3.7) and (3.10) into (3.11a) reproduces the result (3.6), while substitution of Eqs. (3.7), (3.8a), and (A4) into Eq. (3.11b) leads to the expression,

$$\Delta \gamma_j = -\frac{1}{2\pi k_j^2}\int_{-\infty}^{\infty}\Delta u(x)(x+\gamma_j)|\Phi_j(x)|^2 dx - \frac{i}{2\pi k_j}\sum_{l\neq j}\frac{K_{lj}-K_{jl}}{(k_l-k_j)^2}$$
$$+\frac{i}{4\pi^2 k_j}\int_0^{\infty}\frac{\beta^*(k)K_j(k)-\beta(k)K_j^*(k)}{(k-k_j)^2}\,dk. \tag{3.12}$$

The formulas (3.6)–(3.8) and (3.12) coincide with the variational derivatives found in Ref. 18 with a different approach. In addition, we find from Eq. (3.4b) the second-order correction to the eigenvalue,

$$\Delta_2 k_j = \frac{1}{2\pi k_j}\left(\int_0^{\infty}\frac{|K_j(k)|^2}{k-k_j}\,dk - \sum_{l\neq j}\frac{|K_{jl}|^2}{2\pi k_l(k_l-k_j)}\right). \tag{3.13}$$

These results are valid both for generic and nongeneric potentials, i.e., those with the asymptotic behavior (2.14) and (2.16), respectively. The variations of data of the discrete spectrum for the BO spectral problem are similar to standard form of perturbation theory in quantum mechanics.[19]

B. Variations of data of the continuous spectrum

The perturbation theory for the Jost function can be constructed with the help of a modified integral representation,

$$N^{\epsilon}(x,\kappa) = N(x,\kappa) + \int_0^{\infty}\frac{\alpha(k,\kappa)N(x,k)}{k-(\kappa-i0)}\,dk + \sum_{j=1}^{m}\frac{\alpha_j(\kappa)\Phi_j(x)}{k_j-\kappa}, \tag{3.14}$$

where $N^{\epsilon}(x,\kappa)$ solves Eq. (2.3b) for $u^{\epsilon}=u(x)+\epsilon\Delta u(x)$ and $k=\kappa$. According to the boundary conditions (2.1), (2.4) and the scattering problem (2.5), Eq. (3.14) reduces in the limit $x\to-\infty$ to the following relation:

$$\frac{1}{\Gamma^{\epsilon}(k)} - \frac{1}{\Gamma(k)} = -\frac{2\pi i\,\alpha(k,k)}{\Gamma(k)}. \tag{3.15}$$

The coefficients $\alpha(k,\kappa)$ and $\alpha_j(\kappa)$ satisfy a system of integral equations modified compared to Eqs. (3.4),

$$\alpha(k,\kappa) = \frac{\epsilon}{2\pi}\left[K(k,\kappa) + \int_0^{\infty}\frac{K(k,k')\alpha(k',\kappa)}{k'-(\kappa-i0)}\,dk' + \sum_{j=1}^{m}\frac{K_j(k)\alpha_j(\kappa)}{k_j-\kappa}\right], \tag{3.16a}$$

$$\alpha_j(\kappa) = -\frac{\epsilon}{2\pi k_j}\left[K_j^*(\kappa) + \int_0^{\infty}\frac{K_j^*(k')\alpha(k',\kappa)}{k'-(\kappa-i0)}\,dk' + \sum_{l=1}^{m}\frac{K_{jl}\alpha_l(\kappa)}{k_l-\kappa}\right]. \tag{3.16b}$$

As $\epsilon\to 0$, these integral equations can be solved by a Taylor expansion,

$$\alpha(k,\kappa) = \frac{\epsilon}{2\pi}K(k,\kappa) + O(\epsilon^2), \tag{3.17a}$$

$$\alpha_j(\kappa) = -\frac{\epsilon}{2\pi k_j}K_j^*(\kappa) + O(\epsilon^2). \tag{3.17b}$$

Using this explicit asymptotic behavior, the spectral coefficient can also be expanded in the Taylor series, $\Gamma^{\epsilon}(k)=\Gamma(k)+\epsilon\Delta\Gamma(k)+O(\epsilon^2)$, where

$$\frac{\Delta\Gamma(k)}{\Gamma(k)} = iK(k,k). \tag{3.18}$$

Since $\lim_{k\to\infty} K(k,k)=\int_{-\infty}^{\infty}\Delta u(x)dx \equiv \Delta\mathcal{M}$, it follows from Eqs. (2.6a) and (3.18) that

$$\Delta \mathcal{M} = -\frac{1}{2\pi} \Delta \int_0^\infty \frac{|\beta(k)|^2}{k} dk. \tag{3.19}$$

This relation is nothing but the variation of Eq. (2.18) which proves that the number m of solitons is invariant under a small change of the potential, i.e., $\Delta m = 0$. However, this result as well as formulas (3.17)–(3.19) of the regular perturbation theory are valid only for generic potentials. When the potential is nongeneric, i.e., it possesses the asymptotic representation (2.16), the regular perturbation theory fails in the limit $k \to 0^+$. Indeed, using Eqs. (2.16) we find that

$$\Delta \Gamma(0) = i \int_{-\infty}^\infty \Delta u(x) |n(x)|^2 dx \neq 0. \tag{3.20}$$

This result contradicts the asymptotic behavior (2.14b), which provides $\Delta\Gamma(k) \to 0$ as $k \to 0^+$. For comparison, the relation (3.18) for generic potentials exhibits the limit behavior, $\Delta\Gamma(k) \sim O((\ln k)^{-2})$ as $k \to 0^+$ which is in agreement with Eq. (2.14b). The reason for failure of the regular perturbation theory is related to the fact that the integral in Eq. (3.14) becomes singular as $\kappa \to 0^+$. Thus, the perturbation theory must be revised for perturbations of the nongeneric potentials. This revision is closely related to analysis of bifurcation of a new eigenvalue in a neighborhood of $k = 0$.

IV. REVISED PERTURBATION THEORY

Since multiple degenerate eigenvalues are not permitted in the BO spectral problem, as shown in Sec. II A, the only possible bifurcation may be a formation of a new bound state from the delocalized limiting eigenfunction of the continuous spectrum. In this section we show that this bifurcation can happen due to perturbations of the nongeneric potentials. Using the integral representation (3.3) and (3.4) we find the asymptotic approximations for the new eigenvalue and the corresponding eigenfunction.

A. Asymptotic series for a new eigenvalue

Suppose that $\kappa = k_{m+1} = -p(\epsilon)$, where k_{m+1} is a new eigenvalue detaching from the continuum as $\epsilon \to 0$ and $p(\epsilon) > 0$. The integral in Eq. (3.4a) is singular as $\kappa \to 0^-$ unless $K(k,0) = 0$. The latter condition is always true for the generic potentials since $N(x,k)$ vanishes in the limit $k \to 0^+$. However, for nongeneric potentials, since $N(x,0) \sim n(x)$ [see Eq. (2.16a)], $K(k,0)$ is generally nonzero. Thus, the singularity in Eq. (3.4a) is of resonance pole type and accounts for a nontrivial solution of integral equations which corresponds to a new bound state $\Phi^+ = \Phi_{m+1}(x)$.

In order to construct an asymptotic solution to Eqs. (3.4) at $\kappa = -p(\epsilon)$ we evaluate the singular contribution explicitly,

$$\alpha(k) = -\frac{\epsilon}{2\pi} \ln\left(\frac{p}{1+p}\right) K(k,0)\alpha(0) + \frac{\epsilon}{2\pi}\left[\int_0^1 \frac{K(k,k')\alpha(k') - K(k,0)\alpha(0)}{k'+p} dk' + \int_1^\infty \frac{K(k,k')\alpha(k')}{k'+p} dk' + \sum_{j=1}^m \frac{K_j(k)\alpha_j}{k_j + p}\right]. \tag{4.1}$$

[The other equation (3.4b) can be transformed in a similar manner.] We notice that the integrals in Eq. (4.1) are now free of singular terms in the limit $p(\epsilon) \to 0^+$. [Still a Taylor expansion with respect to $p(\epsilon)$ would lead to new (weak) singular terms in Eq. (4.1). This modification is required only for calculation of exponentially small corrections of the asymptotic series (4.2) and will be neglected henceforth.] The asymptotic balance of the singular term occurs for $\epsilon \ln p \sim O(1)$ as $\epsilon \to 0$. Thus, the new eigenvalue is exponentially small in terms of ϵ. Under the balance above, the asymptotic series solving Eq. (4.1) can be presented in the form,

$$\ln p = \frac{1}{\epsilon} p_{-1} + \sum_{l=0}^\infty \epsilon^l p_l + O(\epsilon^{-1} p(\epsilon)), \tag{4.2a}$$

$$\alpha(k)=\alpha_0(k)+\sum_{l=1}^{\infty}\epsilon^l\alpha_l(k)+O(p(\epsilon)), \tag{4.2b}$$

$$\alpha_j=\alpha_j^{(0)}+\sum_{l=1}^{\infty}\epsilon^l\alpha_j^{(l)}+O(p(\epsilon)). \tag{4.2c}$$

We require $p_{-1}<0$ to ensure the eigenvalue $p=p(\epsilon)$ to be small in the limit $\epsilon\to 0$. All terms of the Taylor series for $\alpha(k)$ and α_j can be found recursively in an explicit form from Eq. (4.1) by neglecting the exponentially small residue terms, i.e., the terms containing the eigenvalue p. We confine our analysis only to the leading-order and first-order approximations in these asymptotic series. The leading order term is

$$\alpha_0(k)=-\frac{p_{-1}}{2\pi}K(k,0)\alpha_0(0). \tag{4.3}$$

The coefficient p_{-1} follows from this equation in the limit $k\to 0$,

$$p_{-1}=-\frac{2\pi}{K(0,0)}. \tag{4.4}$$

Since the restriction $p_{-1}<0$ was assumed in Eq. (4.2a), the following criterion,

$$K(0,0)=\int_{-\infty}^{\infty}\Delta u(x)|n(x)|^2dx>0, \tag{4.5}$$

should be satisfied for the bifurcation of the new eigenvalue $k=k_{m+1}=-p(\epsilon)$ to occur.

In order to find the pre-exponential factor p_0 in the asymptotic expansion for $p=p(\epsilon)$, we derive from Eqs. (3.4b) and (4.1) the following expressions for $\alpha_j^{(0)}$ and $\alpha_1(k)$:

$$\alpha_j^{(0)}=-\frac{K_j^*(0)}{k_jK(0,0)}\alpha_0(0) \tag{4.6a}$$

and

$$\begin{aligned}\alpha_1(k)=&\frac{K(k,0)}{K(0,0)}\alpha_1(0)-\frac{p_0}{2\pi}K(k,0)\alpha_0(0)\\&+\frac{\alpha_0(0)}{2\pi K(0,0)}\left[\int_0^1\frac{K(k,k')K(k',0)-K(k,0)K(0,0)}{k'}dk'\right.\\&\left.+\int_1^{\infty}\frac{K(k,k')K(k',0)}{k'}dk'-\sum_{j=1}^{m}\frac{K_j(k)K_j^*(0)}{k_j^2}\right].\end{aligned} \tag{4.6b}$$

Then, the coefficient p_0 follows from Eq. (4.6b) in the limit $k\to 0$,

$$\begin{aligned}p_0=&\frac{1}{[K(0,0)]^2}\int\int_{-\infty}^{\infty}\Delta u(x)\Delta u(y)n^*(x)n(y)dxdy\left[\int_0^1\frac{N^*(y,k)N(x,k)-n^*(y)n(x)}{k}dk\right.\\&\left.+\int_1^{\infty}\frac{N^*(y,k)N(x,k)}{k}dk-\sum_{j=1}^{n}\frac{\Phi_j^*(y)\Phi_j(x)}{k_j^2}\right].\end{aligned} \tag{4.7}$$

This formula can be simplified if the nongeneric potential reduces to a pure m-soliton solution (see examples in Sec. V). We formulate the main result of our analysis in the following statement.

Proposition 1: Suppose that the potential $u(x)$ *satisfies the constraint* $n_0=0$ *and the perturbation* $\Delta u(x)$ *satisfies the criterion* $K(0,0)>0$. *Then, the potential* $u^{\epsilon}=u(x)+\epsilon\Delta u(x)$ *supports an eigenvalue* $k=-p(\epsilon)$ *in a neighborhood of* $k=0$ *for* $\epsilon>0$, *where*

$$p(\epsilon)=c\ \exp(\epsilon^{-1}p_{-1})[1+O(\epsilon)] \tag{4.8}$$

and $c=e^{p_0}$. The coefficients p_{-1}, p_0 are given by Eqs. (4.4) and (4.7). No eigenvalue exists in a neighbourhood of $k=0$ for $\epsilon<0$.

We notice that if $K(0,0)=0$, the asymptotic approximation of $\ln p$ has generally the $O(\epsilon^{-2})$ term and the asymptotic expressions (4.2)–(4.8) must be reconsidered. This special situation occurs if the perturbation $\Delta u(x)$ still conserves the constraint $n_0=0$ for the limiting Jost function $n(x)$ extended to the first-order of the perturbation theory. Alternatively, if $K(0,0)<0$, the bifurcation analyzed above occurs for $\epsilon<0$.

B. The bound state corresponding to the new eigenvalue

Proposition 2: The eigenfunction $\Phi_{m+1}(x)$ corresponding to the eigenvalue $k=k_{m+1}=-p(\epsilon)$ has the asymptotic form

$$\Phi_{m+1}(x)=\frac{\alpha_0(0)}{K(0,0)}\int_0^\infty \frac{K(k,0)N(x,k)}{k+p(\epsilon)}\,dk+O(\epsilon). \tag{4.9}$$

The spectral data γ_{m+1} defined in Eq. (2.9b) is given by

$$\gamma_{m+1}=-\frac{\int_{-\infty}^{\infty}x\Delta u(x)|n(x)|^2dx}{\int_{-\infty}^{\infty}\Delta u(x)|n(x)|^2dx}+\frac{\beta_1+\beta_1^*}{4\pi}(1+\ln p)+\frac{i}{2p}, \tag{4.10}$$

where β_1 is defined in Eq. (2.17b).

The eigenfunction $\Phi_{m+1}(x)$ follows from Eqs. (3.3) and (4.2b). At the leading order, this produces the expression (4.9), where the integral has a singular contribution as $\epsilon\to 0$.

The behavior of this eigenfunction is different along different scales of the x variations. In the outer region, i.e., in the limit $x\to\infty$, the function $\Phi_{m+1}(x)$ is localized. Using the boundary conditions (A3) and the asymptotic representation (2.16), we find the limiting behavior,

$$\Phi_{m+1}(x)\big|_{\text{outer}}\to -\frac{\alpha_0(0)}{x}\left[\frac{1}{ip}+\frac{\beta_1^*\ \ln p}{2\pi}\right]+O(\epsilon), \tag{4.11a}$$

where β_1 is given by Eq. (2.17b). The second term in the brackets is exponentially small compared to the first one, and, therefore, we define $\alpha_0(0)$ as $\alpha_0(0)=-ip$ according to the boundary condition $\Phi_{m+1}(x)\to x^{-1}$ as $x\to\infty$.

In the inner region, where $x\sim O(1)$, the amplitude of the eigenfunction $\Phi_{m+1}(x)$ is exponentially small in terms of ϵ. Using the integral representation (4.9) we find the inner asymptotic expansion for $\Phi_{m+1}(x)$ as follows:

$$\Phi_{m+1}(x)\big|_{\text{inner}}\to ip(\ln p)n(x)+O(p). \tag{4.11b}$$

In order to compute the spectral data γ_{m+1} we use Eqs. (2.9b), (4.9), and (A5) from Appendix A. In addition, we find from Eqs. (2.16a) and (2.17a) that

$$\lim_{k\to 0}\left(\frac{\partial K(k,0)}{\partial k}\right)=-i\int_{-\infty}^{\infty}x\Delta u(x)|n(x)|^2dx-\frac{\beta_1}{2\pi i}K(0,0),$$

where β_1 is given by Eq. (2.17b). The result of these computations is expressed in Eq. (4.10).

C. Revised variations of data of the continuous spectrum

Here we revise the regular perturbation theory for Jost functions. The regular asymptotic expansion is described in Sec. III C, where it was shown that the contradiction (3.20) appears in the limit $k\to 0^+$ for perturbations of nongeneric potentials. In order to calculate this limit correctly, we notice that the integral in Eq. (3.16a) is singular in the limit $\kappa\to 0^+$ if $K(k,0)\neq 0$ and $\alpha(0,\kappa)\neq 0$. Therefore, we cannot use the Taylor expansion (3.17) to solve Eqs. (3.16). Instead, we evaluate the second term of the r.h.s. of Eq. (3.16a) in the form,

$$\int_0^\infty \frac{K(k,k')\alpha(k',\kappa)}{k'-(\kappa-i0)}\,dk' = -\pi i K(k,\kappa)\alpha(\kappa,\kappa) + K(k,\kappa)\alpha(\kappa,\kappa)\ln\left(\frac{1-\kappa}{\kappa}\right) + \text{p.v.}\int_0^1 \frac{K(k,k')\alpha(k',\kappa)-K(k,\kappa)\alpha(\kappa,\kappa)}{k'-\kappa}\,dk' + \int_1^\infty \frac{K(k,k')\alpha(k',\kappa)}{k'-\kappa}\,dk'.$$

Now we impose a scaling transformation,

$$\alpha(k,\kappa) = \frac{1}{\ln\,\kappa - \ln\,p + \pi i}\,\beta(k,\kappa), \tag{4.12a}$$

where $\ln p$ is defined by solving Eq. (4.1) and $\beta(k,\kappa)$ satisfies the integral equation,

$$\beta(k,\kappa) = \frac{\epsilon}{2\pi}K(k,\kappa)[-\ln\,p + (\ln\,\kappa + \pi i)(1-\beta(\kappa,\kappa))] + \frac{\epsilon}{2\pi}\left[\text{p.v.}\int_0^1 \frac{K(k,k')\beta(k',\kappa)-K(k,\kappa)\beta(\kappa,\kappa)}{k'-\kappa}\,dk' + \int_1^\infty \frac{K(k,k')\beta(k',\kappa)}{k'-\kappa}\,dk' + \sum_{j=1}^m \frac{K_j(k)\beta_j(\kappa)}{k_j-\kappa}\right]. \tag{4.13}$$

Here $\beta_j(\kappa)$ are defined through $\alpha_j(\kappa)$ by the same transformation as Eq. (4.12a). In the asymptotic limit $\kappa \sim p(\epsilon) \ll 1$ we use the approximation $K(k,\kappa) = K(k,0) + O(\kappa)$ and $\beta(\kappa,\kappa) = \beta(0,\kappa) + O(\kappa)$ and compare the integral equation (4.1) with Eq. (4.13). Neglecting the terms of $O(p(\epsilon))$ and $O(\kappa)$ in the integral equations, we derive the following simple result:

$$\beta(k,\kappa) = \frac{\alpha(k)}{\alpha(0)} + O(\kappa, p(\epsilon)), \tag{4.12b}$$

where $\alpha(k)$ is represented by the asymptotic series (4.2b). The remainder term of $O(\kappa)$ follows from the analysis of Eq. (4.13) in the limit $\epsilon \to 0$. Using Eqs. (3.14) and (4.12) we verify that the Jost function $N^\epsilon(x,k)$ has the correct asymptotic behavior as $k \to 0^+$,

$$N^\epsilon(x,k) = \frac{-\epsilon^{-1}p_{-1}}{\ln\,k - \ln\,p + \pi i}\,n(x) + O\left(\frac{k}{\ln\,k}\right). \tag{4.14}$$

This formula agrees with the representation (2.12). Furthermore, comparing the denominators of Eqs. (2.12) and (4.14), we find the relation between the asymptotic expansion of the parameters n_0^ϵ [see Eq. (1.6)] and $p(\epsilon)$ for the new eigenvalue,

$$\frac{1}{n_0^\epsilon} = -\ln\,p(\epsilon) + \frac{i\pi}{2} - \gamma + O(p(\epsilon)). \tag{4.15}$$

At the leading order, this formula implies that

$$n_0^\epsilon = \frac{\epsilon}{2\pi}K(0,0) + O(\epsilon^2). \tag{4.16}$$

Thus, the criterion for bifurcation of a new eigenvalue ($K(0,0) > 0$) can also be formulated as $n_0^\epsilon > 0$. Furthermore, we use Eqs. (3.15) and (4.12) to find the spectral data $\Gamma^\epsilon(k)$ as $k \sim p(\epsilon) \ll 1$. Since $\Gamma(k) = 1 + O(k^2)$ for nongeneric potentials [see Eq. (2.16b)], we have

Proposition 3: When $k \sim p(\epsilon) \ll 1$, the spectral data $\Gamma^\epsilon(k)$ are given by

$$\Gamma^{\epsilon}(k)=1+\frac{2\pi i}{\ln k-\ln p+\pi i}+O\left(\frac{k}{\ln k},\epsilon\right). \tag{4.17}$$

It is important to notice that the perturbation to the nongeneric potential produces an $O(1)$ effect which occurs however only for $k\sim p(\epsilon)\ll 1$. In the limit $k\ll p(\epsilon)$ the asymptotic solutions (4.14), (4.17) reproduce the behavior of scattering data for the generic potentials [see Eqs. (2.14)]. In the opposite limit $k\gg p(\epsilon)$ the asymptotic solution (4.12) matches with the Taylor expansion (3.17a), while the Jost function (4.14) approaches to the limiting function $n(x)$. Thus, the revised perturbation theory provides a uniform asymptotic solution describing a structural transformation of asymptotic behavior of the Jost function and the scattering coefficients as $k\to 0^+$ induced due to a perturbation of the nongeneric potential.

Finally, we notice that the uniform asymptotic solution (4.12) prescribes a ''hidden'' pole at $k=p(\epsilon)$. This resonant pole occurs due to strong excitation of radiation in the BO equation (1.1) which compensates the unbalance of the area integral (1.2) induced due to formation of a new soliton. Indeed, if the small perturbation to the potential leads to a new soliton (new eigenvalue), then the following relation must be satisfied for Eq. (2.18) at the leading order,

$$\lim_{\epsilon\to 0^+}\frac{1}{2\pi}\int_0^{\infty}\frac{|\beta^{\epsilon}(k)|^2-|\beta(k)|^2}{k}\,dk=2\pi. \tag{4.18}$$

We check this limit by using the asymptotic representation for $\beta^{\epsilon}(k)$ following from Eqs. (2.6b), (2.12), and (4.15),

$$\beta^{\epsilon}(k)=\frac{2\pi i}{\ln k-\ln p-\pi i}+O\left(\frac{k}{\ln k},\epsilon\right). \tag{4.19}$$

The dominant pole-type term is the leading-order term in the difference between $\beta^{\epsilon}(k)$ and $\beta(k)$, so that we find

$$\lim_{\epsilon\to 0^+}\frac{1}{2\pi}\int_0^{\infty}\frac{|\beta^{\epsilon}(k)|^2-|\beta(k)|^2}{k}\,dk=2\pi\int_0^{\infty}\frac{dk}{k((\ln k-\ln p)^2+\pi^2)}=2\pi.$$

V. EXPLICIT EXAMPLES

In this section we apply our general results to the important class of the pure m-soliton potential $u(x)$. In particular, these explicit formulas solve the problem of soliton generation by a small initial disturbance, i.e., when $u(x)=0$. Explicit expressions are also computed for an algebraic potential $u(x)=2a/(1+x^2)$.

A. Asymptotic approximations for *m*-soliton potentials

For m-soliton potentials, the scattering data are $\beta(k)=0$ and $\Gamma(k)=1$ for all k, and the Jost function $N(x,k)$ has the simple explicit form,

$$N(x,k)=n(x)e^{ikx},\quad n(x)=\exp\left[-i\int_x^{\infty}u_0(x)dx\right]. \tag{5.1}$$

Thus, the asymptotic expression (2.19) for large k is valid for any k in the case of m-soliton potentials and provides a function which is analytic in $\mathrm{Im}(x)\geqslant 0$. To prove the latter statement, we rewrite the m-soliton solution through its pole expansions,[2]

$$u(x)=\sum_{j=1}^{m}\left[\frac{i}{x-x_j}-\frac{i}{x-x_j^*}\right], \tag{5.2}$$

where x_j are complex poles of the solutions with $\mathrm{Im}(x_j)<0$. Then, we find the explicit formula for $n(x)$,

$$n(x)=\prod_{j=1}^{m}\frac{x-x_j^*}{x-x_j} \tag{5.3}$$

which is analytical for Im(x)⩾0. It follows from Eqs. (1.6) and (5.1) that the constraint $n_0=0$ is satisfied identically for any m-soliton solution. As a result, we find from Eq. (4.5) that the variation of m-soliton solutions may generate an additional eigenvalue, if

$$\Delta\mathcal{M}=\int_{-\infty}^{\infty}\Delta u(x)dx>0, \tag{5.4}$$

where $\Delta\mathcal{M}$ is the variation of the area integral (1.2). [Note that the area integral for the m-soliton solution is $\mathcal{M}=2\pi m$.] The asymptotic approximation of the eigenvalue $k=-p(\epsilon)$ follows from Eqs. (4.4), (4.7), and (4.8). Using the fact that $|n(x)|^2=1$ [see Eq. (5.1)] and the table integral (see p. 432 in Ref. 24),

$$\int_0^1\frac{e^{ik(x-y)}-1}{k}\,dk+\int_1^{\infty}\frac{e^{ik(x-y)}}{k}\,dk=-\gamma-\ln|x-y|+\frac{i\pi}{2}\,\mathrm{sign}(x-y) \tag{5.5}$$

we find the parameters p_{-1} and p_0 explicitly

$$p_{-1}=-\frac{2\pi}{\Delta\mathcal{M}} \tag{5.6a}$$

and

$$p_0=-\gamma-\frac{\int\int_{-\infty}^{\infty}\Delta u(x)\Delta u(y)\ln|x-y|dxdy}{(\Delta\mathcal{M})^2}-\sum_{j=1}^{m}\frac{|\int_{-\infty}^{\infty}\Delta u(x)n^*(x)\Phi_j(x)dx|^2}{k_j^2(\Delta\mathcal{M})^2}. \tag{5.6b}$$

The first integral term in Eq. (5.6b) represents the second-order correction to the new eigenvalue from the continuous spectrum while the second integral term is the correction induced due to the presence of other bound states. We notice that the latter contribution is additive which is a consequence of the Darboux transformation of linear eigenvalue problems with bound states.[25] In the particular case $m=0$, when all other bound states are absent, the expression (5.6) give an approximation for a single eigenvalue supported by a small perturbation to the zero background $u(x)=0$ with a positive area integral (1.2).

B. Algebraic potential

The single-humped algebraic potential $u(x)=2a/(1+x^2)$ was considered as an initial condition for numerical simulations of the BO equation.[8,9] In the case $a=1$, this potential is exactly the soliton solution (1.4) for $t=0$, $x_0=0$, and $v=1$ [Due to the scaling invariance of the BO equation the parameter v can be always scaled by unity.] Kodama *et al.*[10] proved that this potential generates an m-soliton solution for $a=a_m=m$ with eigenvalues k_j given by zeros of the Laguerre polynomials $L_m(-2k)$ (see Appendix B). They also proved that, if $a_m<a\leqslant a_{m+1}$, there are $m+1$ bound states. Since the exact solutions of Eq. (1.5) for the algebraic potential are not known, the eigenvalues were calculated numerically. Here we find the explicit analytical approximations of the new small eigenvalue which emerges for $a=a_m+\epsilon$, where $0<\epsilon\ll 1$.

Since the function $n(x)$ is analytical for Im(x)⩾0 we calculate $P^+[un]$ for the algebraic potential $u=2a/(1+x^2)$ as follows:

$$P^+[un]=un+\frac{ian(i)}{x-i}.$$

In addition, we find from Eq. (1.6) that

$$n_0=\frac{1}{2\pi}\int_{-\infty}^{\infty}u(x)n(x)dx=an(i).$$

Then, the limiting function $n(x)$ satisfies a simple differential equation following from Eq. (2.13a),

$$in_x+un=-\frac{in_0}{x-i}.$$

This equation has an explicit solution

$$n(x)=-n_0\left(\frac{x-i}{x+i}\right)^a(I_{a+1}(x)-\alpha), \tag{5.7}$$

where

$$I_{a+1}(x)=\int\frac{(x+i)^a}{(x-i)^{a+1}}\,dx$$

and the constant α is introduced to cancel the nonanalyticity of $n(x)$ at the branching point $x=i$. Using the recurrent relation,

$$I_{a+1}(x)=I_a(x)-\frac{1}{a}\left(\frac{x+i}{x-i}\right)^a \tag{5.8}$$

the function $n(x)$ can be represented for $m\leqslant a<(m+1)$ as

$$n(x)=n_0\left[\sum_{k=0}^{m}\frac{1}{a-k}\left(\frac{x-i}{x+i}\right)^k-\left(\frac{x-i}{x+i}\right)^a(I_{a-m}(x)-I_{a-m}(i))\right]. \tag{5.9}$$

The integral $I_a(x)$ for $0\leqslant a<1$ can be evaluated in two exceptional cases: for $a=1/2$ (see Ref. 16), when $I_{1/2}=\ln(x+\sqrt{1+x^2})$, and for $a=0$, when $I_0=\ln(x+i)$. Using the last formula and comparing the asymptotical behavior of $n(x)$ as $x\to\infty$ given by Eqs. (2.15) and (5.9), we find the asymptotic representation for n_0 as $a=m+\epsilon$,

$$\frac{1}{n_0}=\frac{1}{\epsilon}+\ln 2+\frac{i\pi}{2}+\sum_{k=1}^{m}\frac{1}{k}+O(\epsilon). \tag{5.10}$$

This exact computation of n_0 enables us to find from Eq. (4.15) the asymptotic approximation for a new eigenvalue $k_{m+1}=-p(\epsilon)$ supported by a perturbed algebraic potential,

$$p(\epsilon)=\tfrac{1}{2}\exp(-\epsilon^{-1}-\gamma-q_m+O(\epsilon))\quad q_m=\sum_{k=1}^{m}\frac{1}{k}. \tag{5.11}$$

In Appendix B we derive the formula (5.11) with the help of Eqs. (5.6). This (equivalent) technique leads to a new relation for zeros of Laguerre polynomials.

VI. CONCLUSION

We have shown that small variations of the nongeneric potentials support a new bound state in the BO scattering problem if the generic perturbation leads to a positive value of the governing parameter n_0. For the soliton potentials, the latter condition is equivalent to a positive contribution to the area integral (1.2) induced by the perturbation. The self-consistent asymptotic expansions are constructed and studied not only for the new eigenvalue and an associated bound state but also for the perturbed Jost functions of the BO scattering problem.

Our results on bifurcation of new eigenvalues in the spectral problem (1.5) solve the classical problem of soliton generation by a small initial perturbation within the BO equation. The exponential behavior of the new eigenvalue explains why our results do not fit into the classification of Ref. 11. Although the BO equation (1.1) realizes a marginal case in the criterion on the soliton generation given in Ref. 11, the generation still takes place but is accompanied by a strong radiation. The energy of an initial pulse measured by the integral $P=\int_{-\infty}^{\infty}u^2dx$ all goes to the linear dispersive waves except for an exponentially small addition to the algebraic soliton (1.4). As an indication of this smallness, the new soliton was not identified in the numerical simulations of Ref. 8.

It is interesting to compare the no-threshold soliton generation in the BO equation (1.1) with that in the KdV equation, $u_t+6uu_x+u_{xxx}=0$. In the KdV equation,[4] a small initial pulse with the mass invariant $M=\epsilon\Delta M$, where $\Delta M=\int_{-\infty}^{\infty}u(x,0)dx$, also leads to generation of a single soliton with the mass $M_s=2\epsilon\Delta M$ and emission of linear radiative waves with the mass $M_{\rm rad}=M-M_{\rm sol}=-\epsilon\Delta M$, i.e., all integral quantities remain small in this evolution process. In the BO equation, an initial pulse with the mass $M=\epsilon\Delta M$ generates a soliton with finite mass $M_{\rm sol}=2\pi$ and strong radiation with finite mass $M_{\rm rad}=M-M_{\rm sol}=-2\pi+\epsilon\Delta M$ [see formulas (2.18) and (4.18)].

Similar bifurcations of new eigenvalues (solitons) may also occur when the perturbation of the potential is given not by an initial condition but by a perturbation to the underlying equation itself. The perturbed BO equation was considered recently by Matsuno and Kaup[26,27] who studied variations of the soliton solutions induced by the perturbation. The formation of new solitons has not been discussed yet.

Finally, since the BO equation is believed to be a pivot to multidimensional evolution equations such as the KP1 and DS2 equations (see Ref. 3), another interesting problem is to construct the bifurcation theory for new eigenvalues (solitons) in the multidimensional evolution equation integrable by the inverse scattering method.

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APPENDIX A: USEFUL RELATIONS FOR PERTURBATION THEORY

Taking the limits (2.8b) into Eq. (2.7) we derive a system of algebraic equations for $\Phi_j(x)$, $j=\overline{1,m}$,

$$(x+\gamma_j)\Phi_j(x)=1-i\sum_{l\neq j}\frac{\Phi_l(x)}{k_j-k_l}+\frac{1}{2\pi i}\int_0^{\infty}\frac{\beta(k)N(x,k)}{k-k_j}\,dk. \tag{A1}$$

Furthermore, using Eqs. (2.3b), (2.3c), (2.6b), (2.9a) as well as the formula

$$\lim_{x\to+\infty}\text{p.v.}\,\frac{e^{\pm ikx}}{k}=\pm i\pi\delta(k),$$

we find the following relations:

$$\int_{-\infty}^{\infty}\Phi_j dx=-\pi i, \tag{A2a}$$

$$\int_{-\infty}^{\infty}N(x,k)dx=2\pi\delta(k)-\frac{i\beta^*(k)}{2k}. \tag{A2b}$$

Equivalently, these formulas also follow from the boundary conditions for $N(x,k)$ and $\Phi_j(x)$ as $x\to\infty$ [cf. Eq. (2.1)],

$$N(x,k)\to e^{ikx}+\frac{\beta^*(k)}{2\pi kx}+O(x^{-2}), \tag{A3a}$$

$$\Phi_j(x)\to\frac{1}{x}+O(x^{-2}). \tag{A3b}$$

We derive from Eqs. (A1), (A2), and orthogonality conditions (3.1) the following important relations:

$$\int_{-\infty}^{\infty}(x+\gamma_j)|\Phi_j(x)|^2dx=\pi i, \tag{A4a}$$

$$\int_{-\infty}^{\infty}(x+\gamma_j)\Phi_l^*(x)\Phi_j(x)dx=\frac{\pi i(k_j+k_l)}{k_j-k_l},\quad j\neq l, \tag{A4b}$$

$$\int_{-\infty}^{\infty}(x+\gamma_j)N^*(x,k)\Phi_j(x)dx=2\pi\delta(k)+\frac{i\beta(k)(k_j+k)}{2k(k_j-k)}. \tag{A4c}$$

In addition, we use Eqs. (2.7), (2.8a), (3.1a), and (A2b) in order to compute an equivalent formula for the Jost functions $N(x,k)$. As a result of straightforward calculations, we derive the relation,

$$\int_{-\infty}^{\infty}xN^*(x,k')N(x,k)dx=-2\pi i\frac{d\delta}{dk}(k-k')+\frac{\beta^*(k)}{2k}\delta(k')+\frac{\beta^*(k')}{2k'}\delta(k)+\frac{i\beta(k')\beta^*(k)(k+k')}{4\pi kk'(k-k')}. \tag{A5}$$

APPENDIX B: DERIVATION OF THE ASYMPTOTIC FORMULA (5.11)

Here we check the asymptotic result (5.11) by virtue of formulas (5.6). First, the potential $u(x)=2a/(1+x^2)$ can be decomposed for $a=m+\epsilon$ as

$$u(x)=\frac{2m}{1+x^2}\ \text{and}\ \Delta u(x)=\frac{2}{1+x^2}. \tag{B1}$$

Then, we find from Eqs. (5.4) and (5.6a) that $\Delta M=2\pi$ and $p_{-1}=-1$ in agreement with Eq. (5.11). Then, the first integral term in Eq. (5.6b) can be evaluated explicitly[24] as

$$\int\int_{-\infty}^{\infty}\Delta u(x)\Delta u(y)\ln|x-y|dxdy=-8\int_0^{\pi}t\ \ln(\sin t)dt=4\pi^2\ \ln 2 \tag{B2}$$

which gives the factor $\frac{1}{2}$ in Eq. (5.11).

Next, in order to evaluate the contribution from the other bound states, we find from Eq. (5.3) the limiting eigenfunction $n(x)$ for the algebraic potential $u(x)$ given by Eq. (B1),

$$n(x)=\frac{(x-i)^m}{(x+i)^m}. \tag{B3}$$

Using Eq. (2.9a), we have

$$k_j=\frac{1}{2\pi i}\int_{-\infty}^{\infty}u(x)\Phi_j(x)dx=-im\Phi_j(i).$$

Similarly to Eq. (5.7), the bound states $\Phi_j(x)$ can also be expressed through an integral representation (see Eq. (7.23) in Ref. 10),

$$\Phi_j(x)=-ik_j\left(\frac{x-i}{x+i}\right)^m e^{ik_jx}\int_{-\infty}^{x}\frac{(y+i)^m}{(y-i)^{m+1}}\,e^{-ik_jy}dy.$$

This integral can be computed through the finite sum,

$$\Phi_j(x)=ik_j\sum_{k=1}^{m}\sum_{l=1}^{k}\frac{(l-1)!m!}{(k!)^2(m-k)!}\frac{(2i)^l(2k_j)^{k-l}}{(x+i)^m(x-i)^{l-m}}-ik_j\left(\frac{x-i}{x+i}\right)^m e^{ik_jx}\int_{-\infty}^{x}\frac{e^{ik_jy}}{y-i}\,dy\;L_m(-2k_j), \tag{B4}$$

where $L_m(x)$ is the Laguerre polynomial[24]

$$L_m(x)=\sum_{k=0}^{m}\frac{m!}{(k!)^2(m-k)!}\,(-x)^k. \tag{B5}$$

The eigenvalues k_j satisfy $L_m(-2k_j)=0$. Therefore, the integrals $\int_{-\infty}^{\infty}\Delta u(x)n^*(x)\Phi_j(x)dx$ in Eq. (5.6b) can be evaluated from (B1), (B3), and (B4). As a result, we find from Eqs. (5.6) and (5.11) the following representation for q_m:

$$q_m=\sum_{j=1}^{m}\left(\sum_{k=1}^{m}\sum_{l=1}^{k}\frac{(-1)^l(l-1)!m!}{(k!)^2(m-k)!}\,(2k_j)^{k-l}\right)^2. \tag{B6}$$

According to Eq. (5.11), the sum q_m is equal to $q_m=\Sigma_{k=1}^{m}(1/k)$ which gives a new relation for zeros of the Laguerre polynomials. Indeed, we have checked the first terms following from Eqs. (B5) and (B6): $q_1=1$, $q_2=3/2$, $q_3=11/6$. These results agree with Eq. (5.11).

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