Normal and anomalous scattering, formation and decay of bound states of two-dimensional solitons described by the Kadomtsev–Petviashvili equation

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Multisoliton solutions of the Kadomtsev–Petviashvili equation describing nonlinear wave processes in media with positive dispersion are analyzed using exact and approximate methods. The scattering of two-dimensional solitons is found to be accompanied by an infinite phase shift of their trajectories. It is shown that the stationary multistructures found earlier are degenerate states appearing during anomalous (slow) soliton scattering and disintegrating under the action of perturbations.

1. INTRODUCTION

The Kadomtsev–Petviashvili (KP) equation

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+\alpha u\cdot\frac{\partial u}{\partial x}-\beta\cdot\frac{\partial^3 u}{\partial x^3}\right)=-\frac{c_0}{2}\cdot\frac{\partial^2 u}{\partial y^2}, \tag{1.1}$$

which has been denied to describe quasiplane wave beams in weakly nonlinear dispersive media[1] has acquired the status of a prototype equation in the modern physics of nonlinear waves, thanks to a large number of special properties: the presence of symmetries and integrals of motion, broad classes of exact solutions, etc. The characteristics of solutions to this equation are essentially different for wave processes in media with positive $\beta>0$ (KP1) and negative $\beta<0$ (KP2) dispersion. Thus, even in the original work[1] it was established that the plane soliton solutions are stable with respect to wave front modulation in media with negative dispersion and are unstable in media with positive dispersion. At the same time, as was shown first numerically[2] and later analytically,[3,4] two-dimensional solitons localized in all directions and falling off as x^{-2}, y^{-2} (Fig. 1) may exist in media with positive dispersion. The stability of such solitons has been established.[5]

Investigation of the multisoliton formulas describing the scattering of two-dimensional solitons[3,4] reveals a striking property. Not only do two-dimensional solitons retain their shapes and initial parameters after collisions (amplitude, velocity, size), but their phase shift also turns out to be equal to zero. But it does not mean that the interaction of such solitons is as trivial as the interaction of pulses in linear nondispersive media. Detailed studies, the results of which are presented in this paper, show that the interaction even between two solitons by no means reduces to superposition of their fields, and can lead to unexpected effects which are characterized by an infinite rather than zero phase shift.

The structure of two-dimensional solitons allows one to suggest the existence of bound states of two and more solitons, i.e. multistructures, because their fields do not decrease monotonically in space but, instead, contain local minima (see Fig. 1). In fact, a set of bisolitons was discovered by means of numerical simulations, whose amplitudes depend on the distance between them.[6] Explicit analytical expressions both of steady-state bisolitons and of more complicated structures were found in Ref. 7, but the stability of these structures has not been studied yet. Here we shall show that the interpretation of bi- and multistructures as bound states of individual solitons is not a simple matter. Besides, the multistructures are themselves unstable.

At present there exist, besides the canonical KP model, similar two-dimensional equations which possess the above properties,[8,9] but are not integrable. Soliton interactions can be studied by means of an approximate approach based on soliton perturbation theory. In this theory the solitons are considered as classical particles obeying Newton's equations.[10] In the one-dimensional case this approach is effective and obvious enough,[11] but its generalization to two dimensions involves a number of unusual properties (thus, for example, the mass of interacting particles which correspond to two-dimensional solitons is anisotropic). As the perturbation theory is quite general and is used both for integrable and nonintegrable models, it is useful for studying two-dimensional solitons. Within the framework of the KP1 equation we succeed in analyzing the familiar solution and also new exact ones and in comparing them with results of the approximate approach. This allows one to estimate the efficiency and range of applicability of this approach.

2. INTERACTION OF TWO-DIMENSIONAL SOLITONS: EXACT SOLUTIONS

Below, we investigate the properties of solutions of the KP1 equation, reducing (1.1) to the dimensionless form

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+v\cdot\frac{\partial v}{\partial x}+\frac{\partial^3 v}{\partial x^3}\right)=\frac{\partial^2 v}{\partial y^2}. \tag{2.1}$$

The exact multisoliton solutions of this equation can be constructed by different methods (see, for example, Refs. 3, 4). For our purposes it is convenient to write them in the Hirota form:

$$v(x,y,t)=12\frac{\partial^2 \ln\varphi}{\partial x^2}, \tag{2.2}$$

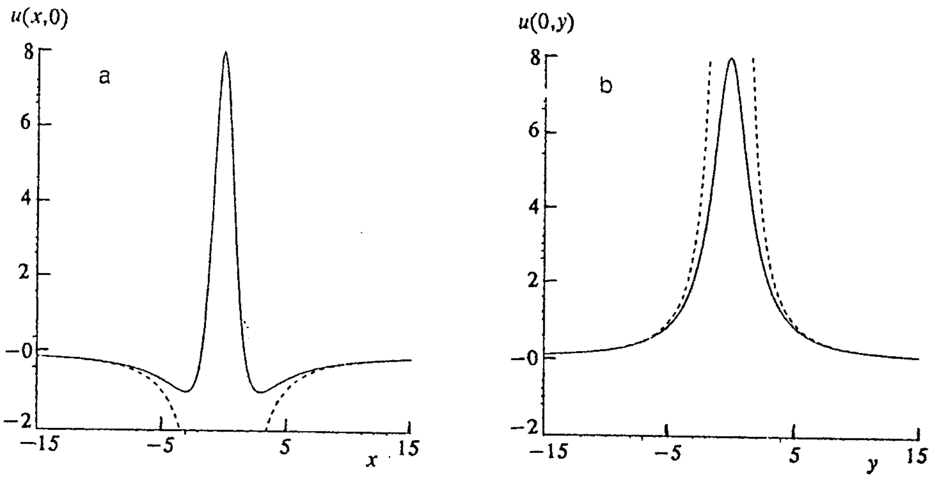

FIG. 1. The longitudinal (a) and transverse (b) cross-sections of a two-dimensional soliton (solid lines) and their linear asymptotic forms (dashed lines).

where $\varphi = \det[(x+ip_k y - p_k^2 t+\theta_k)\cdot\delta_{k,l} + (1-\delta_{k,l})/(p_k-p_l)\cdot 2\sqrt{3}]$ and $\delta_{k,l}$ is the Kronecker symbol. Here $k,l = 1,2,3,\ldots,N\equiv 2M$, M is a natural number, the constants p_k and θ_k determine the velocity and the phase of each soliton, $p_{k+M}=\bar{p}_k$, and $\theta_{k+M}=\bar{\theta}_k$. It can easily be shown that the leading term of the polynomial $\varphi(x,y,t)$ is a product of the polynomials $\varphi_1(x,y,t)$ corresponding to individual solitons; for the variable $v(x,y,t)$ it means that when the solitons are far enough from each other, their fields form a linear superposition. This implies that individual solitons in the solution (2.2) are asymptotically free as $t\to\pm\infty$ and move along their unperturbed trajectories. We shall call the process of soliton interaction described by formula (2.2) *normal soliton scattering*. In Fig. 2 a typical set of trajectories of two solitons is shown. It was determined by the coordinates of the soliton maxima on the x,y plane in the solution (2.2). All the trajectories are divided into reflected and transmitted ones depending on the values of the impact parameter.

Until recently it was considered that the above solutions present a complete picture of multisoliton scattering which is even simpler than the soliton scattering in the framework of other integrable models, where, at a minimum there is a phase shift along the soliton trajectories.[12] The absence of any traces remaining after two-dimensional solitons interact, which is also characteristic of the interactions of linear pulses in media without dispersion, even misled some researchers to believe that such solitons "don't interact at all."[13]

But detailed analysis of the multisoliton formula shows that besides the ones described above there are other types of scattering which can be obtained from (2.2) using appropriate renormalization. For example, from this formula for $M=2$ one can obtained a new solution of the KP1 equation, in the limit $p_k\to p_l$ by changing the origin of time: $t\to t\pm 8\sqrt{3}/(p_k^2-p_l^2)^2$. In the reference frame which moves along the x axis with a unit velocity ($p_k=p_l=1$), this solution is as follows:

$$\varphi(\xi,y,t;a)=(\xi^2+y^2)^2\pm 8\sqrt{3}(\xi y^2-\xi^2 t+y^2 t)+12\xi^2+a\xi y+24y^2+48t^2\mp 24\sqrt{3}\xi\pm\sqrt{3}ay+72+a^2/16, \quad (2.3)$$

where $\xi=x-t$ and a is an arbitrary parameter. In the particular case $a=0$, a similar solution was discovered first in the work of Johnson and Thompson,[14] but its structure and the physical sense were not analyzed.

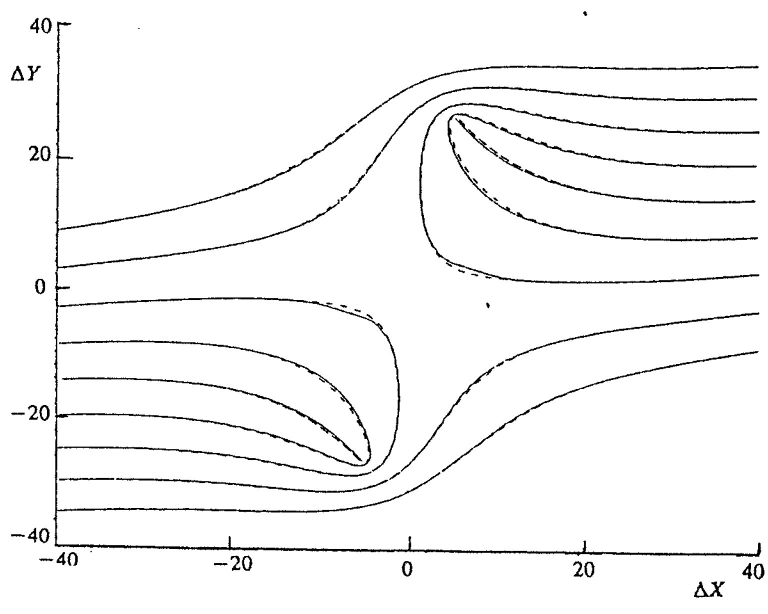

FIG. 2. The trajectories of the relative distances between soliton maxima in the exact solution (solid lines) and in the asymptotic approach (dashed lines) for asymptotic velocities: $\Delta V_x=0, 5$; $\Delta V_y=0, 1$.

For the upper signs, the set of solutions (2.3) describes the collision of two solitons moving along the x axis from $-\infty$ at $t\to-\infty$, with equal asymptotic velocities. The constant a plays the role of impact parameter in the scattering theory, since at $t=0$ it determines the distance between the solitons. As will be seen from the asymptotic approach developed below, two-dimensional solitons moving one after the other attract one another. Therefore, they move closer together even if the initial distance between them is much larger than their characteristic sizes. During this process the amplitude of the front soliton is decreasing while the amplitude of the back soliton is increasing. At the same time, they repel one another along the y axis. As a result, solitons get closer in x and move apart in y, their amplitudes equalizing again as the solitons diverge. As $t\to+\infty$, a pair of identical solitons forms that are infinitely far apart in y and propagate to parallel one another in the x direction. The rate of convergence and subsequent divergence changes in time as $\sim|t|^{-1/2}$.

This solution can be inverted by choosing the lower signs in formula (2.3). In this case two identical solitons moving in the x direction parallel to each other and at an infinite distance along y slowly approach, are rearranged and, at $t\to+\infty$, diverge in x, acquiring their initial amplitudes and velocities.

We shall call such processes, as well as more complicated ones where a larger number of solitons with identical amplitudes take part, anomalous soliton scattering because the distances between solitons change asymptotically slower than they would according to a linear law.

3. INTERACTION OF TWO-DIMENSIONAL SOLITONS: ASYMPTOTIC APPROACH

The exact solutions presented in Sec. 2, as well as many others where solitons are basic elements, can be accurately reproduced by using the asymptotic approach.[10,11] On the one hand, this method has a number of restrictions, since it is based on the expansion of unknown solutions in powers of a small parameter. On the other hand, it has great advantages since its range of applicability also includes nonintegrable models in the theory of nonlinear waves. Application of this approach to integrable models, for example, to the KP1 equation, as in our case, allows one to delimit its range of applicability by comparison with exact solutions and, besides, to understand better the physical meaning of the exact solutions.

The asymptotic approach is based on two main assumptions:[10]

1) The solitary waves are spaced rather far apart so that the correction to the field of each soliton is a superposition of "tails" of other solitons. These "tails" can be found by solving the linearized initial equations.

2) The relative soliton velocities are much smaller than their average speeds.

The first assumption allows one to represent a solution to the KP1 equation for M interacting solitons as a formal asymptotic series:

$$v(x,y,t)=\sum_{k=1}^{M} v_k(x,y,t,\mathbf{V}_k,\mathbf{R}_k)+\sum_{m=1}^{\infty}\varepsilon^m v^{(m)}(x,y,t), \tag{3.1}$$

where v_k are the fields of the individual unperturbed solitons, $\mathbf{V}_k$ are their velocities, taken to be of the same order of magnitude so that $\varepsilon=|\mathbf{V}_k-\mathbf{V}_l|/|\mathbf{V}_k+\mathbf{V}_l|$ is a small parameter, and $v^{(m)}(x,y,t)$ are the m-th order corrections caused by the soliton interaction.

Owing to the second assumption the initial equation can be solved systematically by assuming that the parameters of individual solitons vary in a quasistationary manner: $\mathbf{V}_k(\varepsilon t)$, $\mathbf{R}_k(\varepsilon t)$. Consequently, in the vicinity of each kth soliton one must solve a linear equation for $v^{(m)}$ that depends on time implicitly, and whose solvability conditions yield the equations of motion of solitons as classical particles:[10]

$$(\mathbf{R}_k\cdot\nabla_{V_k})\mathbf{P}_k=-\sum_{\substack{l=1\\ l\neq k}}^{M}\nabla_{R_k}W_l(\mathbf{R}_k-\mathbf{R}_l), \tag{3.2}$$

where $\mathbf{P}_k$ is the momentum of the unperturbed soliton with components

$$(P_k)_x=48\pi\sqrt{(V_k)_x^2-(V_k)_y^2/4},$$

$$(P_k)_y=-2(V_k)_y\cdot(P_k)_x,$$

and the pair-interaction potential $W_k(x,y)$ is related to the profile of a single soliton $v_k(x,y)$ by

$$W_k(x,y)=(P_k)_x\cdot v_k(x,y),\quad v_k(x,y)=12\frac{\partial^2\ln\varphi_k(X,Y)}{\partial x^2},$$

$$\varphi_k(X,Y)=3/b_k^2+(X+a_kY)^2+b_k^2\cdot Y^2,$$

$$X=x-(a_k^2+b_k^2)t-X_k,\quad Y=y+2a_kt-Y_k.$$

Here a_k and b_k determine the soliton velocity components: $(V_k)_x=a_k^2+b_k^2$, $(V_k)_y=-2a_k$, and X_k, Y_k are phase constants.

The assumptions 1) and 2) leading to the system (3.2) to the first order in ε yield a completely integrable system of particles in the complex plane $z=x+iy$. This system is referred to as the Calogero—Moser system. Expressing the soliton coordinates in the form $Z_k=X_k+iY_k$ we write the Hamiltonian of this system as

$$H=\sum_{k=1}^{M}(\dot{Z}_k)^2-\sum_{1\leqslant k<1\leqslant M}\sum\frac{2}{(Z_k-Z_1)^2}. \tag{3.3}$$

As is well known,[12] the coordinates $Z_k(t)$ are the eigenvalues of the matrix $\hat{B}$ with elements

$$B_{kl}(t)=(Z_{k0}+\dot{Z}_{k0}t)\cdot\delta_{kl}+\frac{1-\delta_{kl}}{Z_{k0}-Z_{l0}}\cdot t,$$

$$k,l=1,2,\ldots,M, \tag{3.4}$$

where Z_{k0} and $\dot{Z}_{k0}$ are the initial particle locations and velocities.

Thus, the 2D soliton interaction in the framework of the KP1 equation, which is solved exactly by formula

(2.2), is described asymptotically by the Calogero–Moser system with implicit solution (3.4). (It is interesting to note that the last solution coincides with the polynomial $\varphi(x,y)$ of degree M, corresponding to the rational solution of the KP1 equation and determining the y dependence of the poles in this solution.[16] As $\varepsilon\to 0$ the solution (3.4) asymptotically approaches the exact solution (2.2) in the region $|Z_k-Z_l|\gg 1$.

3.1. Normal scattering of two-dimensional solitons treated by the asymptotic method

Using momentum conservation we relate the reference frame to the center of mass of the solitons so that the conditions

$$\sum_{k=1}^{M} Z_k=0 \quad \text{and} \quad \sum_{k=1}^{M} \dot{Z}_k=0 \tag{3.5}$$

hold.

Then the solution (3.4) for two solitons ($M=2$) is described by the complex second-order curves

$$1+2HZ_1^2=[H(t+it_0)]^2, \tag{3.6}$$

where $H=H_r+iH_i\neq 0$ is the complex Hamiltonian of two particles in the system (3.3). Analysis of these curves shows that the soliton coordinates increase linearly in time at $t\to\pm\infty$ according to the formulas

$$X_{1,2}=\pm\frac{\sqrt{|H|+H_i}}{2}\cdot t\mp\frac{H_it_0}{2\sqrt{|H|+H_r}},$$
$$Y_{1,2}=\pm\frac{\sqrt{|H|+H_i}}{2}\cdot t\pm\frac{H_it_0}{2\sqrt{|H|-H_i}}. \tag{3.7}$$

Thus, (3.6), like the exact solution (2.2), describes a normal collision of two solitons with nonzero asymptotic velocity which move along the unperturbed trajectories after the interaction.

In Fig. 3a the relative distances ΔX, ΔY between the Moser particles in the lower-half plane (all trajectories are mirror-symmetric in the upper-half plane) are shown by dashed lines versus the impact parameter (initial y displacement of the solitons, which is determined by the parameter t_0. Here the solitons are copropagating with the parameters $\Delta V_y=0$, $\Delta V_x=0,5$; $H_r>0$, $H_i=0$. In terms of the exact solution (2.2) the trajectories of the individual soliton maxima are shown in the same figure by solid lines. As is seen from Fig. 3a, there is not only qualitative but also a good quantitative agreement between these two types of trajectories.

In absolute coordinates the trajectories of the Moser particles differ from those of the soliton maxima, which are more complicated and not mirror-symmetrical relative to the x-axis (see Fig. 3b). But this difference is substantial only in the region of strong overlapping of the soliton fields ($|\Delta X|$, $|\Delta Y|\leqslant 10$), where the approximate theory is obviously incorrect.

It is interesting to note that for a two-soliton collision with zero impact parameter the copropagating motion is transformed to the transverse motion after some time interval. In the particle system with the Hamiltonian (3.3) this process is described by the solution (3.6) ($H_i=0$, $t_0=0$), which consists of two branches corresponding to motions along the x and y axes and joined at $t=\pm 1/H_r$ (see dashed segments in Fig. 3a).

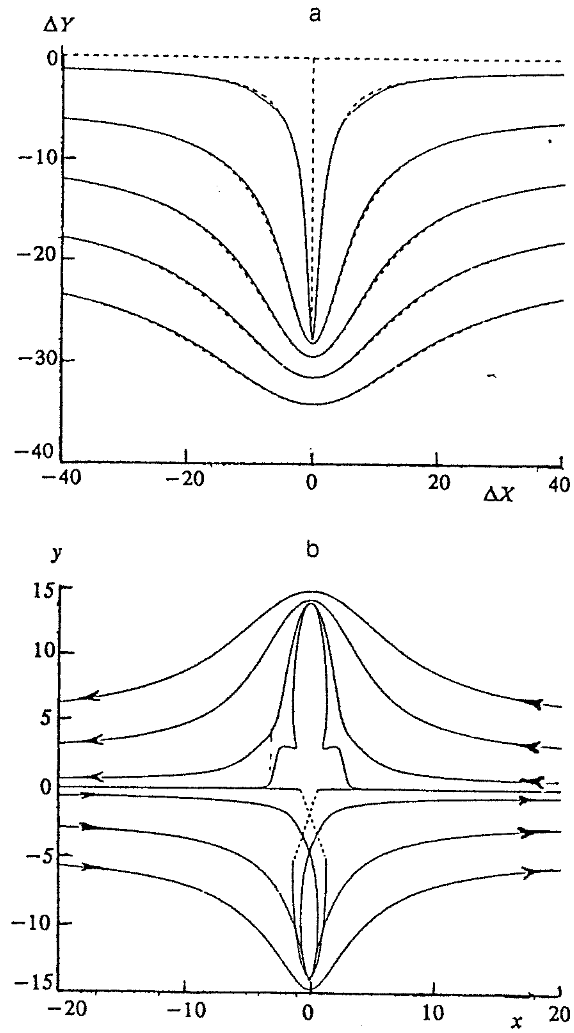

FIG. 3. a) The same as Fig. 2, but for $\Delta V_x=0,5$; $\Delta V_y=0$. b) The trajectories of two solitons in the exact solution.

But for the exact solution, the wave field shows a more complicated picture at the time of collision. Under the action of the local minimum in the profile of the rear fast soliton the front soliton splits symmetrically along x into two pulses which propagate perpendicular to the initial motion and turn into solitons. This process was first illustrated computationally by Freeman.[17] At nonzero but small impact parameter this gives rise to loops which are described by temporal local maxima (Fig. 3b) rather than by real solitons.

To demonstrate the efficiency and accuracy of the approximate method we compare the largest distance between two solitons during copropagating ($H_r>0$) motion with the relative asymptotic velocity ΔV_x (Fig. 4) for the exact solution (solid line) and for the approximate one (dashed line) where $|\Delta Y|=2/\Delta V_x$. As is seen from this figure, the agreement between the two approaches worsens

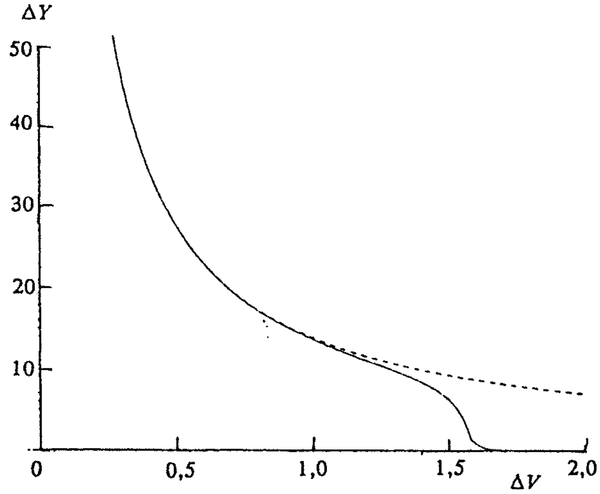

FIG. 4. Plot of the largest distance between two solitons in copropagating motion vs the relative asymptotic velocity ΔV_x.

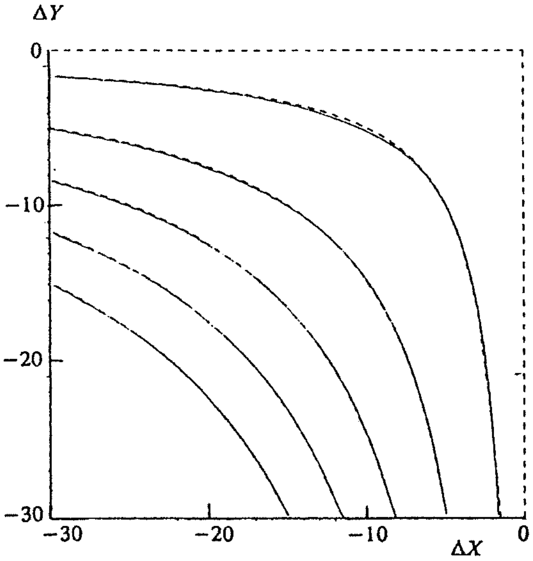

FIG. 5. As in Fig. 2, but for $\Delta V_x=0$, $\Delta V_y=0$.

in the region of large asymptotic velocities at $\Delta V_x \simeq 1$ where the second assumption of the asymptotic method does not hold. Nevertheless, qualitative (and even satisfactory quantitative) agreement of the curves is observed in the region $\Delta V_x \geqslant 1$ too.

3.2. Anomalous scattering of two-dimensional solitons using the asymptotic method

As follows from Eq. (3.6), the distance between two solitons during copropagating motion increases without restriction when their relative asymptotic velocity (i.e., their energy H) tends to zero (Fig. 4). But the solution (3.6) does not hold in this case. Only by changing the origin of time by replacing $t \rightarrow t \pm 1/H$ one can obtain the solution describing the interaction of solitons with identical asymptotic velocities:

$$Z_1^2 = Z_2^2 = \pm (t + it_0). \tag{3.8}$$

The trajectories of particle motion within this solution (Fig. 5, dashed curves) are hyperbolas. Note that in the limit $t \rightarrow +\infty$ all trajectories of this set converge to the vertical axis. This means that after interaction the solitons move with identical velocities parallel to the x axis separated by an infinite distance. Comparison with the exact solution (2.3) shows that the trajectories of particles and soliton maxima do not, actually, differ for any values of t_0 except very small ones (see Fig. 5). Thus, at $H=0$ the solution (3.6) describes irreversible interaction of two particles, which gives rise to infinite phase shift of their trajectories.

4. BOUND STATES OF SOLITONS AND THEIR DECAY

4.1. Manifolds of stationary solutions

As has been noted above, the presence of local minima of the field in a single soliton structure (see Fig. 1) suggests the existence of bound states of two and more solitons. Such structures were in fact first discovered numerically[6] and later analytically.[7] But their interpretation in terms of interacting solitons is not trivial. First of all, we note that for two solitons the first order of perturbation theory yields the Calogero–Moser two-particle system, which has no stationary solutions. In principle, higher-order corrections can give rise to bound states, but these corrections are rather difficult to calculate and have not been obtained until now.

Another way to construct steady-state formations within the Calogero–Moser system is to take into account a larger number of particles. The problem of determining stationary solutions for the Hamiltonian field (3.3) was studied in the context of finding rational solutions for the Korteweg–de Vries (KdV) equation.[18,19] It was proved that such solutions exist for a certain number of particles (solitons, in our case): $M=N(N+1)/2=1, 3, 6, 10,\ldots$. Each rational solution is invariant to the action of N commuting flows of "higher"-order equations for the KdV-hierarchy and is stationary for all the others. From this it follows that N generalized coordinates exist for each manifold. For our case it means that the coordinates of the equilibrium configuration of M solitons depend on N complex parameters.

The simplest equilibrium configuration exists for three ($N=2$) particles. It is written as follows:

$$Z_k = Z_0 + q \cdot \exp(2\pi k i/3), \quad k=1, 2, 3, \tag{4.1}$$

where the complex parameter q specifies the distance between solitons and rotation angle of the entire configuration on the x,y plane. Thus, Moser particles are located at equal distances from the center of three rays intersecting at an angle of 120°.

We found a set of rational solutions[7] describing steady-state multisoliton structures, and proved that the degree P of the polynomial, related to solitons the number of M by $P=2M$, is $P=N(N+1)$. Hence each equilibrium state in the Calogero—Moser system corresponds to an exact

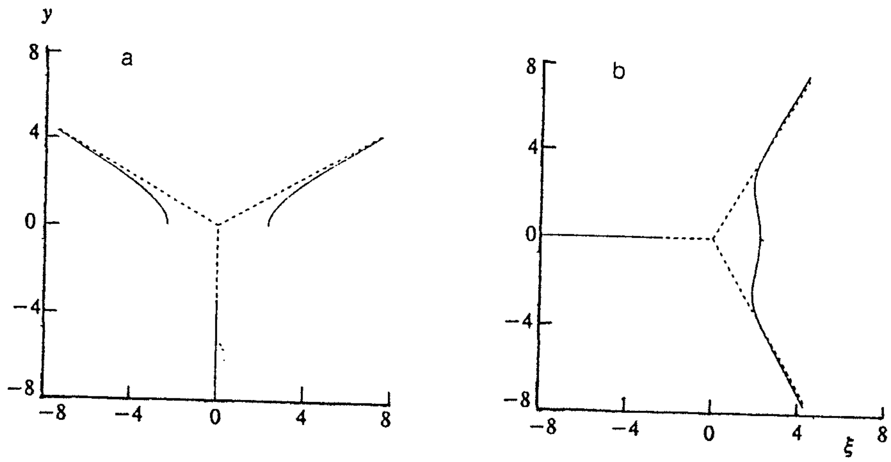

FIG. 6. The locations of three solitons which form a bound state in the approximate (4.1) (dashed traces) and the exact (4.2) (solid traces) solutions for two configurations relative to the mean velocity of motion.

steady-state multisoliton solution of the KP1 equation. For three solitons this solution can be written explicitly in the Hirota form:

$$\varphi_{st}(\xi,y,t;a,b) = 1875 + (a^2+b^2)/2 - b\xi - 5ay - 125\xi^2 + 475y^2 + b\xi^3 - 3ay\xi^2 - 3b\xi y^2 + ay^3 + 25\xi^4 + 90\xi^2y^2 + 17y^4 + (\xi^2+y^2)^3, \tag{4.2}$$

where $\xi = x - t$. The soliton configuration for the states symmetrical about the ξ axis are shown in Fig. 6a versus the parameter a at $b=0$. A similar dependence on the parameter b at $a=0$ is shown in Fig. 6b. In the latter case the solitons are arranged symmetrically about the y axis. The dashed lines denote the approximate solution (4.1) which, as is clearly seen, agrees well with the exact solution for $|Z_k| \geqslant 5$. As an illustration the function $v(\xi,y)$ for $a=1500$, $b=0$ is shown in Fig. 7a.

For $q=0$ three particles of the approximate solution (4.1) merge into one particle, but, as has been mentioned already, the asymptotic theory is incorrect for small distances between solitons. It follows from the exact solution (4.2) in that the limit $a\to 0$, $b\to 0$ the three-soliton solution degenerates into a two-humped structure (bisoliton) (Fig. 7b). It can be shown[7] that there is only one solution which is symmetrical about both the ξ and y axes.

At the same time, a one-parameter set of two-humped steady-state solutions which are similar to the bisoliton displayed in Fig. 7b was simulated in Ref. 6. As we see now, these structures are actually the two-soliton part of the solution (4.2), which is located far from the third soliton (see Fig. 7a). (The numerical simulation was carried out using an iterative scheme only for the first quarter of the x,y plane and then this solution was completed by reflecting about the coordinate axes.)

Note also, that the explicit formulas of steady-state multistructures presented in Ref. 7 describe only the degenerate case of a configuration that is symmetrical in ξ,y when N solitons are arranged on the common straight line and the other $M-N$ solitons are hidden in a complicated multi-humped relief between them.

In the Appendix we prove that the energy and momentum of all steady-state multistructures are independent of their location relative to each other and are equal to the energy and momentum of M individual solitons propagating with the same velocity. In the presence of perturbations, as we shall show in the next section, equilibrium

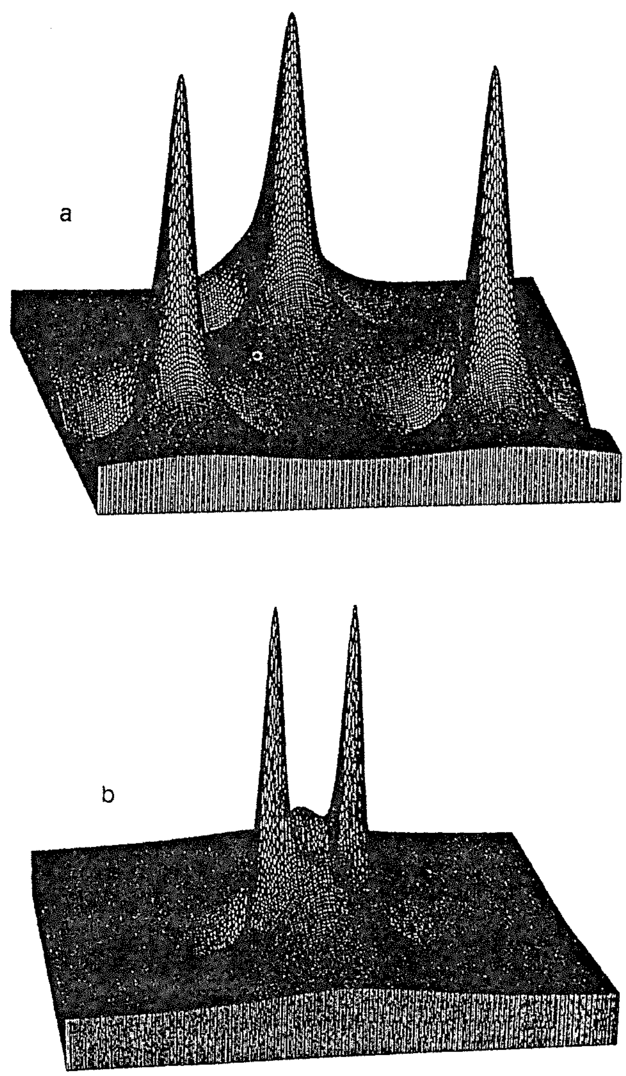

FIG. 7. The typical pattern of steady-state three-soliton structure (a); the steady-state bisoliton which is formed as a result of degeneracy of the three-soliton solution (b).

states of solitons are destroyed and the multistructure decays into $M>N$ individual solitons. Apparently, this phenomenon can be explained as the birth of $M-N$ solitons from the binding energy of the unstable N-soliton structures.

4.2. Instability of the steady-state structures.

One can prove that there are no exponential modes in the system linearized about the stationary solutions. However, it is clear from the solution (3.4) that all trajectories outside the manifolds tend to infinity, growing secularly with time. Therefore, there are no periodic breather motions, which agrees with the Krichever theorem[20] about the algebraic time dependence of the algebraic poles of the solutions of the KP1 equation. This implies that all equilibrium states corresponding to multisoliton structures are unstable against small perturbation.

Employing analysis of the solution (3.4) for $M=3$ we shall show that the decay time of an equilibrium state largely depends on the type of perturbation.

Using (3.5) we rewrite (3.4) in an explicit form:

$$Z_k^3+P(t)Z_k-Q(t)=0, \tag{4.3}$$

where $P(t)=p_0+p_1t+p_2t^2$ and $Q(t)=q_0+q_1t+q_2t^2+q_3t^3$ are polynomials with coefficients p_i, q_i which are expressed in terms of the initial coordinates and velocities of the particles.

As follows from the analysis of the asymptotic form of the solution for Z_k at $t\to\pm\infty$, the solution of Eq. (4.3) describes the normal scattering of three solitons in two cases: either for $q_3\neq0$ or for $q_3=0$ and $p_2\neq0$ (in the second case, one soliton has the asymptotic velocity of the center of mass).

When the parameters p_2 and q_3 vanish simultaneously, a set of algebraic equations arises for the initial data, whose solvability leads to solutions describing the anomalous soliton scattering: $Z_k\sim t^s$, where $s<1$, at $t\to\pm\infty$. Comparison of (4.3) with the polynomials $\varphi(x,y)$ corresponding to three-pole rational solutions of KP1 equation shows that in this case there exist only two groups of solutions $Z_k(t)$, determined implicitly by cubic equations:

$$Z_k^3+(p\pm3t)Z_k-q=0, \tag{4.4}$$

$$Z_k^3-3r^2Z_k-q^3-3rt=0. \tag{4.5}$$

In the first case we have $s=1/2$, and the solitons asymptotically approach the coordinate axes. Therefore, like (3.8), the solution (4.4) describes the anomalous interaction of three solitons, which results in particles scattering at an angle $\pi/2$ with the direction of initial motion.

In the particular case $r=0$ the other solution (4.5) has a branch of the equilibrium three-particle states (4.1), where the rotation angle is determined by the phase of the complex parameter q. If $r\neq0$, the particles move from this configuration with the power $s=1/3$, tending asymptotically to another branch of the stationary configuration with rotation angle depending on the phase of the parameter r. Thus, the presence of rays where the potential of three-particle interaction has an extremum slows down the soliton scattering still more but does not lead to a phase shift after interaction.

This analysis of particle dynamics in the approximate Calogero–Moser system is corroborated by the corresponding exact solutions. The process of three-soliton anomalous scattering which behaves as $t^{1/3}$ is determined by four parameters and has the following form:

$$\begin{aligned}\varphi(\xi,y,t;a,b,c,d)=&\varphi_{\mathrm{st}}(\xi,y,a,b)+(150c^2+3c^4+1200d+342d^2+48d^3+3d^4+48c^2d+6c^2d^2)-12(bc+ad)t\\&+144(c^2+d^2)t^2+(ac^2-2bc-2ad-2bcd-ad^2)\xi+24[2c^2+c^2d+(1+d)^2]t\xi\\&+(20c^2+c^4+4c^2d+2c^2d^2-140d-12d^2+4d^3+d^4)\xi^2-24dt\xi^3+2(c^2-2d-d^2)\xi^4\\&+(4ac-bc^2-2acd-4bd+bd^2)y+(120c+24c^3+24cd^2)ty+(300c+12c^3+96cd+12cd^2)\xi y\\&+72ct\xi^2y+4(5c+2cd)\xi^3y+(8c^2+c^4-8c^2d+2c^2d^2+184d+24d^2-8d^3+d^4)y^2\\&+72dt\xi y^2+36d\xi^2y^2-24cty^3+4(-7c+2cd)\xi y^3-2(c^2+4d-d^2)y^4.\end{aligned} \tag{4.6}$$

Here again we have set $\xi=x-t$, the pair of parameters a,b determines the steady-state three-soliton solution $\varphi_{\mathrm{st}}(\xi,y,a,b)$, and the other pair c,d determines directions to which the scattering solitons tend asymptotically. The typical picture of anomalous scattering is presented in Fig. 8a for $a=1500$, $b=0$, and $c=0$ for parameter d varying (the dark circles denote the soliton positions in the equilibrium state at $d=0$).

As is noted above, the complication of soliton dynamics in comparison with the asymptotic solution for Moser particles occurs only when the distances between solitons are small ($a,b,c,d\simeq0$), when instead of triple soliton merging at the intermediate scattering stage a two-humped structure forms similar to that shown in Fig. 7b. Because of the anisotropic properties of the wave fields this structure may decay in different ways, depending on the type of perturbation. For deformation in the direction parallel to the mean motion ($a=0$, $c=0$), one observes the splitting of one hump of the multistructure into two solitons and their subsequent scattering (Fig. 8b). For a perturbation

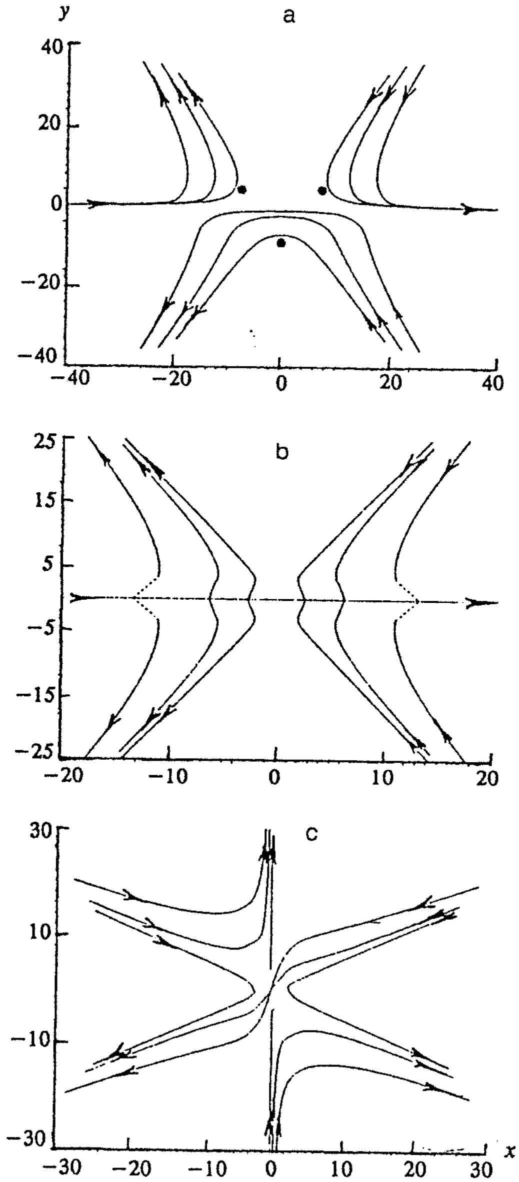

FIG. 8. The processes of anomalous decay of the bound three-soliton state described by the solution (4.6).

transverse to the motion ($b=0$, $d=0$), the third soliton is born from the local maximum (see Fig. 7b) which bounds two solitons (Fig. 8c). These processes correspond to two nonlinear modes destroying the structure and determining its instability under small perturbations.

In Appendix we show that nonstationary processes described by Eq. (4.6) for c, $d\neq 0$ have energy and momentum identical to the steady-state multistructure. Hence, the normal or anomalous decay behavior is determined by the additional energy introduced by perturbance.

5. CONCLUSION

The analysis of soliton dynamics in terms of the KP1 equation, carried out using an asymptotic approach, has shown that when the asymptotic velocity vanishes, the soliton interaction is, again a scattering process but it proceeds more slowly than the normal one and leads to an infinite phase shift of their trajectories. Besides, for some number of solitons there exist equilibrium states corresponding to bound states of individual solitons. Evidently, all solutions of this type correspond to multiple poles of wave function in the "k-space" of the inverse transform method,[21] which have not been considered before. But finding a complete set of exact solutions in place of the general solutions (2.2) which are invalid in this case is beyond the scope of our consideration.

We would like to point out the good agreement of results obtained using the asymptotic technique and exact solutions. It gives us hope that application of this technique to nonintegrable models will be effective. To reach this end, we only need to know the linear asymptotic behavior of solitary waves that can be found by solving the linearized initial problem. However, in two-dimensional problems it is not easy to construct the Green's function for self-adjoint operators, and the successive determination of higher corrections in asymptotic series (3.1) is difficult. As a result, the behavior of the soliton interaction at small distances cannot be considered using this approach now.

APPENDIX

We shall show by direct calculation that the momentum and the energy of the wave field consisting of M interacting two-dimensional solitons are determined by the asymptotic values of each soliton velocity even if they form a bound steady state. For the equation (2.1) the momentum $U(x,t)$ of the field and the energy (Hamiltonian) are written as follows:

$$P_x=\frac{1}{2}\int v^2 dx\,dy, \tag{A1}$$

$$H=\frac{1}{2}\int\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\int\frac{\partial v}{\partial y}dx\right)^2-\frac{v^3}{3}\right]dx\,dy. \tag{A2}$$

Since the M-soliton solution is a rational $2M$-pole solution of the KP1 equation, in accordance with the general Krichever theorem[20] it can be written in the form

$$v(x,y,t)=-12\sum_{k=1}^{2M}\frac{1}{(x-x_k(y,t))^2}. \tag{A3}$$

It is not difficult to see by direct substitution of the formula (A.3) into the KP1 equation (2.1) that the motion of the poles $X_k(y,t)$ is determined by the pair of commuting equations

$$\frac{\partial^2 X_k}{\partial y^2}+\sum_{m\neq k}\frac{24}{(X_m-X_k)^3}=0, \tag{A4a}$$

$$\frac{\partial X_k}{\partial t}+\left(\frac{\partial X_k}{\partial y}\right)^2+\sum_{m\neq k}\frac{12}{(X_m-X_k)^2}=0. \tag{A4b}$$

Using the representation (A.3), Eq. (A.4a), and the Cauchy integral formula we can evaluate the integral (A.1):

$$P_x=72\sum_{k,m=1}^{2M}\int\int_{-\infty}^{+\infty}\frac{dx\,dy}{(x-X_k)^2(x-X_m)^2}$$

$$=12\sum_{k=1}^{2M}\int\int_{-\infty}^{+\infty}\frac{dx\,dy}{x-X_k}\sum_{m\neq k}\frac{24}{(X_m-X_k)^3}$$

$$=-12\sum_{k=1}^{2M}\int\int_{-\infty}^{+\infty}\frac{dx\,dy}{x-X_k}\cdot\frac{\partial^2 X_k}{\partial y^2}$$

$$=-24\pi i\sum_{k=1}^{2M}\left(\sigma_k\cdot\frac{\partial X_k}{\partial y}\right)\Bigg|_{y=-\infty}^{y=+\infty},$$

where $\sigma_k=1$ ($\sigma_k=0$), if $X_k(y,t)$ lies in the upper half (lower half) of the complex plane.

It follows from (A.4b) that the asymptotic behavior of the poles at $y\to\pm\infty$ is

$$X_k\simeq(V_k)_x\cdot t+\frac{(V_k)_y}{2}\cdot[y-(V_k)_y\cdot t]$$
$$\pm i[y-(V_k)_y\cdot t]\sqrt{(V_k)_x-(V_k)_y^2/4}.$$

Therefore

$$P_x=\sum_{k=1}^{M}48\pi\cdot\sqrt{(V_k)_x-(V_k)_y^2/4}=\sum_{k=1}^{M}(P_k)_x, \quad \text{(A5)}$$

where P_k is the momentum of the free kth soliton with velocity $\mathbf{V}_k$.

Similarly, the integrals in (A.2) can be calculated. As a result we have:

$$H=\sum_{k=1}^{M}16\pi\cdot((V_k)_y^2-(V_k)_x)\cdot\sqrt{(V_k)_x-(V_k)_y^2/4}$$

$$=\sum_{k=1}^{M}H_k. \quad \text{(A6)}$$

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