



Instability of H^1 -stable peakons in the Camassa–Holm equation

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Abstract

It is well-known that peakons in the Camassa–Holm equation are H^1 -orbitally stable thanks to conserved quantities and properties of peakons as constrained energy minimizers. By using the method of characteristics, we prove that piecewise C^1 perturbations to peakons grow in time in spite of their stability in the H^1 -norm. We also show that the linearized stability analysis near peakons contradicts the H^1 -orbital stability result, hence passage from linear to nonlinear theory is false in H^1 .

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1. Introduction

The Camassa–Holm (CH) equation [6]

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad x \in \mathbb{R} \quad (1.1)$$

can be rewritten in the convolution form

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$$u_t + uu_x + \frac{1}{2}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right) = 0, \quad x \in \mathbb{R}, \tag{1.2}$$

where $\varphi(x) = e^{-|x|}$ is the Green function satisfying $(1 - \partial_x^2)\varphi = 2\delta$ with δ being Dirac delta distribution and $(f * g)(x) := \int_{\mathbb{R}} f(x - y)g(y)dy$ being the convolution operator. It is clear that $\varphi \in H^s$ for $s < \frac{3}{2}$, where $H^s := H^s(\mathbb{R})$ is the standard Sobolev space of squared integrable distributions equipped with the norm $\|f\|_{H^s} := \|\langle \cdot \rangle^s \hat{f}\|_{L^2}$ with $\langle x \rangle := \sqrt{1 + x^2}$ and \hat{f} being the Fourier transform of f on \mathbb{R} .

The purpose of function φ is not only to rewrite the evolution problem for the CH equation (1.1) in the convolution form (1.2) that depends on the first derivative of u in x and does not depend on its higher-order derivatives. In addition, φ expresses a particular family of solutions $u(t, x) = c\varphi(x - ct)$ with $c \in \mathbb{R}$ which are referred to as *peakons* (or *peaked solitary waves*). Indeed, the validity of these peakons as solutions to the CH equation in the convolution form (1.2) can be checked directly from the identity

$$-\varphi + \frac{1}{2}\varphi^2 + \frac{3}{4}\varphi * \varphi^2 = 0, \quad x \in \mathbb{R}, \tag{1.3}$$

which is piecewise C^1 on both sides from the peak at $x = 0$.

Cauchy problem for the CH equation (1.1) with the initial data $u_0 \in H^3$ was studied in the series of papers [8–10]. It was shown in Theorem 4.1 in [8] that if $y_0 := (1 - \partial_x^2)u_0$ does not change sign on \mathbb{R} , then the corresponding solution $u \in C(\mathbb{R}^+, H^3)$ exists globally in time. If there exists $x_0 \in \mathbb{R}$ such that $(x - x_0)y_0(x) \geq 0$ on \mathbb{R} , the same conclusion applies (Theorem 4.4 in [8]), whereas if $(x - x_0)y_0(x) \leq 0$ on \mathbb{R} , the local solution breaks in a finite time in the sense that the slope of the solution becomes unbounded from below in a finite time (Theorems 5.1 and 5.2 in [8]).

The condition H^3 on the initial data was used in [8–10] to control the auxiliary quantity y_0 and to extend local solutions to global solutions of the CH equation (1.1). Without these requirements, local well-posedness of the Cauchy problem can be proven for initial data in H^s for every $s > \frac{3}{2}$ [25,31] but cannot be pushed below and at $s = \frac{3}{2}$ because of lack of uniform continuity of the local solution with respect to initial data and the norm inflation [5,19].

A rather sharp sufficient condition on breaking of a local solution in H^s with $s > \frac{3}{2}$ was obtained in [2]. The local solution breaks if there exists $x_0 \in \mathbb{R}$ such that the initial data $u_0 \in H^s$ satisfies $u'_0(x_0) < -|u_0(x_0)|$. Unique global weak solutions $u \in C(\mathbb{R}^+, H^1)$ in energy space H^1 were constructed in [12] (after [11]) under the additional condition that $y_0 := (1 - \partial_x^2)u_0$ has a non-negative finite measure. Non-negativity of $y := (1 - \partial_x^2)u$ is preserved in the time evolution of the CH equation. Note that if $y_0 \geq 0$, then $u_0(x) \geq 0$ and $|u'_0(x)| \leq u_0(x)$ is satisfied at every x for which u'_0 is defined.

Cauchy problem for the CH equation in the convolution form (1.2) with the initial data in H^1 was studied in [3] (similar results appear also in [20]) by means of a coordinate transformation of the quasilinear equation to an equivalent semilinear system. It was proven that the Cauchy problem for the equivalent semilinear system admits a unique global solution (Theorem 1 in [3]), which provides a global conservative solution to the CH equation (Theorem 2 in [3]) such that

$$\begin{cases} u(t, \cdot) \in H^1 & \text{for every } t \in \mathbb{R}^+, \\ \|u(t, \cdot)\|_{H^1} = \|u_0\|_{H^1} & \text{for almost every } t \in \mathbb{R}^+. \end{cases} \tag{1.4}$$

This global conservative solution is consistent with the two conserved quantities of the CH equation:

$$E(u) = \int_{\mathbb{R}} (u^2 + u_x^2) dx, \quad F(u) = \int_{\mathbb{R}} u(u^2 + u_x^2) dx. \tag{1.5}$$

Moreover, it was proven in [4] that the global conservative solution to the CH equation (1.2) is unique for every initial data in H^1 . Continuous dependence from initial data and local well-posedness of the weak solutions

$$u \in C((-T, T), H^1) \cap L^\infty((-T, T), W^{1,\infty}) \cap C^1((-T, T), L^2) \tag{1.6}$$

to the CH equation (1.2) was established very recently in [27] for every $u_0 \in H^1 \cap W^{1,\infty}$, where $W^{1,\infty}$ is the Sobolev space of functions with bounded first derivatives and $T > 0$ is the maximal existence time.

The previous study of stability of peakon solutions in [13,14] relies on the existence of conserved quantities (1.5). It was proven in [13] that the peakon φ is a unique (up to translation) minimizer of $E(u)$ in H^1 subject to the constraint $3F(u) = 2E(u)$ (Proposition 3.6 in [13]), where $E(\varphi) = 2$ and $F(\varphi) = \frac{4}{3}$. Consequently, global smooth solutions $u \in C(\mathbb{R}^+, H^3)$ which are close to φ in H^1 remains close to the translated orbit $\{\varphi(\cdot - a)\}_{a \in \mathbb{R}}$ in H^1 for all $t > 0$ (Theorem 3.1 in [13]).

A different result on stability of peakons was proven in [14], which we reproduce here.

Theorem 1. [14] *Assume existence of a solution $u \in C((0, T), H^1)$ to the CH equation (1.1) with either finite or infinite $T > 0$. For every small $\varepsilon > 0$, if the initial data satisfies*

$$\|u_0 - \varphi\|_{H^1} < \left(\frac{\varepsilon}{3}\right)^4, \tag{1.7}$$

then the solution satisfies

$$\|u(t, \cdot) - \varphi(\cdot - \xi(t))\|_{H^1} < \varepsilon, \quad t \in (0, T), \tag{1.8}$$

where $\xi(t)$ is a point of maximum for $u(t, \cdot)$.

In Theorem 1, the local solution $u \in C((0, T), H^1)$ to the CH equation (1.1) may break in a finite time $T < \infty$ in the sense of Theorem 5.1 in [8]:

$$u_x(t, x) \rightarrow -\infty \quad \text{at some } x \in \mathbb{R} \quad \text{as } t \nearrow T. \tag{1.9}$$

Nevertheless, the H^1 -norm of the solution $u \in C((0, T), H^1)$ remains finite as $t \nearrow T$ thanks to the energy conservation $E(u)$ in time t up to the breaking time T . This implies that the bound (1.8) remains valid in the limit $t \nearrow T$ and allows us to say that *peakons of the CH equation are H^1 orbitally stable*.

Various extensions of the H^1 orbital stability of peakons have been made recently. Orbital stability of peaked periodic waves in the CH equation was proven in [22,23]. Stability of peakons in another integrable equation called the Degasperis–Procesi equation was established in [26] by

extending ideas of [14]. Asymptotic stability of peakons in the class of H^1 functions u with $y := (1 - \partial_x^2)u$ being a non-negative finite measure is proven in [29]. Asymptotic stability of trains of peakons and anti-peakons with y being a sign-indefinite finite measure was constructed recently in [30].

Multi-peakon solutions were constructed by many analytical and numerical tools [1,7,21,28]. The local characteristic curve $x = \xi(t)$ for the Camassa–Holm equation (1.2) is defined by the equation

$$\frac{d\xi}{dt} = u(t, \xi(t)). \tag{1.10}$$

Since the multipeakon solution is known in the closed form $u(t, x) = \sum_{j=1}^N m_j(t)\varphi(x - x_j(t))$, where $\{x_k(t), m_k(t)\}_{k=1}^N$ satisfies the finite-dimensional Hamiltonian system

$$\frac{dx_k}{dt} = \sum_{j=1}^N m_j \varphi(x_k - x_j), \quad \frac{dm_k}{dt} = - \sum_{j=1}^N m_j m_k \varphi'(x_k - x_j),$$

it is clear that the single peakons move along the local characteristic curves [28] so that

$$\frac{dx_k}{dt} = u(t, x_k(t)), \quad k \in \{1, 2, \dots, N\}.$$

Global conservative solutions to the CH equation with multi-peakons were studied with the inverse scattering transform method in [15], where the long-time behavior of solutions with peakons was investigated (see also [1,24]) and details of collisions between peakons and anti-peakons were given (see also [3,30]).

The purpose of this work is to address the question of stability of a single peakon in the time evolution of the CH equation beyond the H^1 orbital stability result of Theorem 1. We start with the linearized stability analysis and study evolution of the linearized equation around the peakons by using the method of characteristics. The same method is also useful to prove nonlinear instability of piecewise C^1 perturbations to peakons. This instability develops in spite of the H^1 orbital stability result of Theorem 1.

The previous works avoid the question of linearized stability of peakons. It was noticed in [13] that “the nonlinearity plays a dominant role rather than being a higher-order correction”, so that “the passage from the linear to the nonlinear theory is not an easy task, and may even be false”. The authors of [26] added that the peakons are not differentiable in x , which makes it difficult to analyze the spectrum of the linearized operator around the peakons.

By adding a perturbation $v(t, x - t)$ to the single peakon $\varphi(x - t)$ moving with the normalized speed $c = 1$ and dropping the quadratic term in v , we obtain the linearized equation for $v(t, x - t)$ from the CH equation in the convolution form (1.2):

$$v_t + (\varphi - 1)v_x + \varphi'v + \varphi' * \left(\varphi v + \frac{1}{2}\varphi'v_x \right) = 0, \quad x - t \neq 0. \tag{1.11}$$

In what follows, we use x instead of $x - t$ thanks to the translational invariance of the convolution operator. After the change, the location of the peak of φ is placed at $x = 0$. The following theorem represents the first main result of this work.

Theorem 2. For every initial data $v_0 \in H^1$ satisfying $v_0(0) = 0$, there exists a unique global solution $v \in C(\mathbb{R}^+, H^1)$ to the linearized equation (1.11) satisfying $v(t, 0) = 0$,

$$\|v(t, \cdot)\|_{H^1(0, \infty)}^2 = \|v_0\|_{H^1(0, \infty)}^2 + 2(e^t - 1) \int_0^\infty \varphi(s) \left([v_0(s)]^2 + \frac{1}{2} [v_0'(s)]^2 \right) ds \quad (1.12)$$

and

$$\|v(t, \cdot)\|_{H^1(-\infty, 0)}^2 = \|v_0\|_{H^1(-\infty, 0)}^2 + 2(e^{-t} - 1) \int_{-\infty}^0 \varphi(s) \left([v_0(s)]^2 + \frac{1}{2} [v_0'(s)]^2 \right) ds \quad (1.13)$$

for every $t > 0$.

In Theorem 2, we confirm the expectation from [13] that the passage from the linear to the nonlinear theory may be false in H^1 . Indeed, the sharp exponential growth of $\|v(t, \cdot)\|_{H^1(0, \infty)}$ in (1.12) for the solution of the linearized equation (1.11) contradicts the bound (1.8) in Theorem 1 obtained for the solution of the full nonlinear equation (1.2).

On the other hand, by solving the linearized equation (1.11) with a method of characteristics (Lemma 3), we discover the intrinsic instability associated with the peaked profile of the traveling wave φ . This instability is related to the characteristics to the right of the peak for $x > 0$ but not to the left of the peak for $x < 0$ (Lemma 4).

Within the linearized equation (1.11), we also discover that if $v_0(0) \neq 0$, the continuous initial data $v_0 \in H^1$ generates a finite jump discontinuity in the solution $v(t, x)$ at the peak $x = 0$ for every small $t > 0$ (Lemma 2). This finite jump discontinuity is allowed in the domain of the linearized operator associated with the linearized equation (1.11) in L^2 (Remark 1), however it prevents the solution $v(t, \cdot)$ to stay in H^1 for every $t > 0$. Related to this fact, we prove that the single peak of a perturbed peakon in the CH equation in the convolution form (1.2) moves with the speed equal to the local characteristic speed as in (1.10) (Lemma 6). This allows us to show that the constraint on the solution v at the peak required in Theorem 2 is satisfied identically in the time evolution of the full nonlinear equation (1.2) (Remark 3).

Similarly, we show for the linearized equation (1.11) that if $v_0 \in H^1 \cap C^1$ with $v_0(0) = 0$ and $v_0'(0) \neq 0$, then $v(t, \cdot) \notin C^1$ for every small $t > 0$ because of the finite jump discontinuity of $v_x(t, x)$ at $x = 0$ for $t > 0$ (Remark 2). Compared to the constraint on v at the peak, there is no way to maintain a constraint on v_x at the peak and to prevent the finite jump discontinuity of $v_x(t, x)$ at $x = 0$ in the time evolution of the full nonlinear equation (1.2).

The above facts suggest that the L^2 -based spaces like H^1 or H^2 may not be the best spaces to study the intrinsic instability of peakons in the CH equation. Instead, we work in the space of bounded continuous functions on \mathbb{R} which are piecewise continuous differentiable with a finite jump discontinuity of the first derivative at $x = 0$ and bounded first derivatives away from $x = 0$. This space is denoted by C_0^1 :

$$C_0^1 := \{v \in C^0(\mathbb{R}) \cap C^1(\mathbb{R}^-) \cap C^1(\mathbb{R}^+); \quad v, \partial_x v \in L^\infty(\mathbb{R})\}. \quad (1.14)$$

The linear instability result of Theorem 2 is easily extended to prove exponential instability of $\|v_x(t, \cdot)\|_{L^\infty}$ because of the characteristics to the right of the peak (Lemma 4). Unlike the

linear instability in H^1 of Theorem 2, which cannot be true in the nonlinear evolution due to the result of Theorem 1, the linear instability in C_0^1 persists in the full nonlinear equation (1.2) as the nonlinear instability of peakons with respect to piecewise C^1 perturbations (Lemma 9). The following theorem represents the second main result of this work.

Theorem 3. *For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in H^1 \cap C_0^1$ satisfying*

$$\|u_0 - \varphi\|_{H^1} + \|u'_0 - \varphi'\|_{L^\infty} < \delta, \tag{1.15}$$

such that the global conservative solution to the CH equation (1.2) with the initial data u_0 satisfies

$$\|u_x(t_0, \cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty} > 1, \tag{1.16}$$

where $\xi(t)$ is a point of peak of $u(t, \cdot + \xi(t)) \in H^1 \cap C_0^1$ for $t \in [0, t_0]$.

We note that the peakon φ is located on the boundary between global and breaking solutions to the Camassa–Holm equation in the sense that $(1 - \partial_x^2)\varphi = 2\delta$ is zero everywhere except at the peak. It is expected that some perturbations to the peakons lead to global solutions whereas some others break in a finite time in the same sense as (1.9). The nonlinear instability result in Theorem 3 does not distinguish between these two possible scenarios. However, the proof of Lemma 9 shows that the initial data for the growing perturbations near the peak satisfy the same sufficient condition for breaking as in [2] (Remark 5). The bound (1.16) is attained before the breaking time T .

Note that the criterion (1.9) for breaking in [8] and the sufficient condition for breaking in [2] are derived for smooth solutions to the Camassa–Holm equation. Although perturbations to the peakons are not smooth in $H^1 \cap C_0^1$, we have justified the same breaking criterion in the method of characteristics for the solutions in $H^1 \cap C_0^1$ (Lemma 7). Moreover, since $C_0^1 \subset W^{1,\infty}$, our solution coincides for $t \in (0, T)$ with the local solution (1.6) constructed in [27]. The maximal existence time $T > 0$ of the local solution (1.6) may be finite because of the blow-up of the $W^{1,\infty}$ norm. This blow-up of the $W^{1,\infty}$ norm agrees well with the instability criterion (1.16) used in Theorem 3.

A general global conservative solution to the CH equation (1.2) satisfies (1.4) and may have finite jumps of the energy $E(u)$ in (1.5) at some time instances, which correspond to collision between peakons and anti-peakons [3]. The initial data $u_0 \in H^1 \cap C_0^1$ in Theorem 3 exclude anti-peakons and hence no jumps of the energy $E(u)$ occur in the time evolution of the local solution (1.6) at least for $t \in [0, T)$.

Study of the instability of peakons in the CH equation is inspired by the recent work [16–18] on smooth and peaked periodic waves in the reduced Ostrovsky equation,

$$u_t + uu_x = \partial_x^{-1}u, \tag{1.17}$$

which is another generalization of the inviscid Burgers equation $u_t + uu_x = 0$. While smooth periodic waves are linearly and nonlinearly stable [16,17], peaked periodic waves were found to be linearly unstable because the L^2 -norm of the perturbation grows exponentially in time [18]. The linear instability was found from the solution of the truncated linearized equation obtained by the method of characteristics and from the estimates on the solution of the full linearized

equation. Nonlinear instability was not studied in [18] due to the lack of global well-posedness results on solutions of the reduced Ostrovsky equation (1.17) in H^1 .

Compared to [18], we show here that the full linearized equation (1.11) can be solved by method of characteristics without truncation and that the nonlinear instability of peakons in the CH equation (1.2) can be concluded from the linear instability of perturbations in $H^1 \cap C_0^1$.

The remainder of the article is organized as follows. Linearized evolution near a single peakon is studied in Section 2, where the proof of Theorem 2 is given. Nonlinear evolution of piecewise C^1 perturbations to a single peakon is studied in Section 3, where the proof of Theorem 3 is given. Section 4 concludes the article with ideas for further work.

2. Linearized evolution near a single peakon

Let us first simplify the linearized equation (1.11) by using the following elementary result.

Lemma 1. *Assume that $v \in H^1$. Then, for every $x \in \mathbb{R}$,*

$$\varphi' * \left(\varphi v + \frac{1}{2} \varphi' v_x \right) (x) = -\varphi'(x)v(x) + \varphi'(x)v(0) - \varphi(x) \int_0^x v(y) dy. \quad (2.1)$$

Proof. Since integrals of absolutely integrable functions are continuous and since $\varphi, \varphi', v, v_x \in L^2$, the map

$$x \mapsto \varphi' * \left(\varphi v + \frac{1}{2} \varphi' v_x \right)$$

is continuous for every $x \in \mathbb{R}$. Now, $H^1(\mathbb{R})$ is continuously embedded into the space of bounded continuous functions on \mathbb{R} decaying to zero at infinity and thus, $v \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Integrating by parts yields the following explicit expression for every $x \in \mathbb{R}$:

$$\begin{aligned} (\varphi' * \varphi' v_x) (x) &= (\varphi'' * \varphi' v) (x) - (\varphi' * \varphi'' v) (x) \\ &= (\varphi * \varphi' v) (x) - 2\varphi'(x)v(x) - (\varphi' * \varphi v) (x) + 2\varphi'(x)v(0), \end{aligned}$$

which simplifies the left-hand side of (2.1) to the form:

$$\varphi' * \left(\varphi v + \frac{1}{2} \varphi' v_x \right) (x) = -\varphi'(x)v(x) + \varphi'(x)v(0) + \frac{1}{2} (\varphi * \varphi' v) (x) + \frac{1}{2} (\varphi' * \varphi v) (x).$$

Furthermore, we obtain

$$\begin{aligned} \frac{1}{2} (\varphi * \varphi' v) (x) + \frac{1}{2} (\varphi' * \varphi v) (x) &= -\frac{1}{2} \int_{\mathbb{R}} \varphi(x-y)\varphi(y)v(y) [\text{sign}(y) + \text{sign}(x-y)] dy \\ &= -\int_0^x \varphi(x-y)\varphi(y)v(y) dy \end{aligned}$$

$$= -\varphi(x) \int_0^x v(y)dy,$$

which completes the proof of (2.1). □

The Cauchy problem for the linear equation (1.11) can be written in the evolution form:

$$\begin{cases} \frac{dv}{dt} = Av, & t > 0, \\ v|_{t=0} = v_0, \end{cases} \tag{2.2}$$

where the linearization near the single peakon φ for a perturbation v in H^1 is given by

$$(Av)(x) = [1 - \varphi(x)]v'(x) + \varphi(x) \int_0^x v(y)dy - v(0)\varphi'(x), \quad x \neq 0, \tag{2.3}$$

thanks to Lemma 1 and the translational invariance.

In order to define a strong solution to the Cauchy problem (2.2), we consider the operator $A : \text{dom}(A) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ with the maximal domain given by

$$\text{dom}(A) = \{v \in L^2(\mathbb{R}) : Av \in L^2(\mathbb{R})\}. \tag{2.4}$$

Since $\varphi(x) \rightarrow 0$ as $|x| \rightarrow \infty$ exponentially fast and $\varphi' \in L^2(\mathbb{R})$, $\text{dom}(A)$ is equivalent to

$$\text{dom}(A) \equiv \{v \in L^2(\mathbb{R}) : (1 - \varphi)v' \in L^2(\mathbb{R}), |v(0)| < \infty\}.$$

Remark 1. Since $\varphi(0) = 1$, H^1 is continuously embedded into $\text{dom}(A)$ but it is not equivalent to $\text{dom}(A)$. In particular, if $v \in H^1(\mathbb{R}^+) \cap H^1(\mathbb{R}^-)$ with a finite jump discontinuity at $x = 0$ then $v \in \text{dom}(A)$ but $v \notin H^1$. However, the representation (2.1) in Lemma 1 does not hold for solutions in $\text{dom}(A)$ with a finite jump discontinuity at $x = 0$.

Let us consider the linearized Cauchy problem (2.2) in the space C_0^1 defined by (1.14). The following lemma shows that unless $v_0 \in C_0^1$ satisfies the constraint $v_0(0) = 0$, solutions to the Cauchy problem (2.2) do not remain in C_0^1 for $t \neq 0$.

Lemma 2. Assume that $v_0 \in \text{dom}(A) \cap C_0^1$ with $v_0(0) \neq 0$ and that there exists a strong solution $v \in C(\mathbb{R}^+, \text{dom}(A))$ to the Cauchy problem (2.2). Then, $v(t, \cdot) \notin C_0^1$ for every $t \in (0, t_0)$ with $t_0 > 0$ sufficiently small.

Proof. Assume that $v(t, \cdot) \in C_0^1$ for $t \in [0, t_0]$ with some $t_0 > 0$ and obtain a contradiction. If $v(t, \cdot) \in C_0^1$, then it follows from (2.2) and (2.3) for every $t \in (0, t_0)$ that

$$\lim_{x \rightarrow \pm 0} \frac{dv}{dt}(t, x) = \pm v(t, 0),$$

and since $v(0, 0) = v_0(0) \neq 0$, we have $v(t, \cdot) \notin C(\mathbb{R})$ for every $t \in (0, t_0)$ with $t_0 > 0$ sufficiently small because of the finite jump discontinuity at $x = 0$. Consequently, $v(t, \cdot) \notin C_0^1$ for every $t \in (0, t_0)$. \square

Due to the reason in Lemma 2, we set $v_0(0) = 0$ for the initial condition $v_0 \in H^1 \cap C_0^1$ in the Cauchy problem (2.2). Note that $\text{dom}(A) \cap C_0^1 \equiv H^1 \cap C_0^1$. If $v \in H^1 \cap C_0^1$ and $v(0) = 0$, then $(Av)(x)$ in (2.3) is continuous at $x = 0$ and its definition can be extended for every $x \in \mathbb{R}$ by

$$(Av)(x) = [1 - \varphi(x)]v'(x) + \varphi(x) \int_0^x v(y)dy, \quad x \in \mathbb{R}. \tag{2.5}$$

The next result shows that the condition $v_0(0) = 0$ on $v_0 \in H^1 \cap C_0^1$ is not only necessary but also sufficient for existence of the unique global solution in $H^1 \cap C_0^1$ to the Cauchy problem (2.2) with (2.5).

Lemma 3. *Assume that $v_0 \in H^1 \cap C_0^1$ with $v_0(0) = 0$. There exists the unique global solution $v \in C(\mathbb{R}, H^1 \cap C_0^1)$ to the Cauchy problem (2.2) with (2.5) satisfying $v(t, 0) = 0$ for every $t \in \mathbb{R}$.*

Proof. We solve the evolution problem (2.2) with (2.5) by using the method of characteristics piecewise for $x > 0$ and $x < 0$. The family of characteristic curves $X(t, s)$ satisfying the initial-value problem

$$\begin{cases} \frac{dX}{dt} = \varphi(X) - 1, \\ X|_{t=0} = s, \end{cases} \tag{2.6}$$

are uniquely defined for every $s \in \mathbb{R}$ thanks to the Lipschitz continuity of φ on \mathbb{R} . The peak location at $X = 0$ is the critical point which remains invariant under the time flow of the initial-value problem (2.6). For $s \in \mathbb{R}$ and $t \in \mathbb{R}$, we obtain the family of characteristic curves in the exact form

$$X(t, s) = \begin{cases} \log [1 + (e^s - 1)e^{-t}], & s > 0, \\ 0, & s = 0, \\ -\log [1 + (e^{-s} - 1)e^t], & s < 0, \end{cases} \tag{2.7}$$

with $\lim_{s \rightarrow 0^\pm} X(t, s) = 0$ for every $t \in \mathbb{R}$. Let us define

$$w_0(x) := \int_0^x v_0(y)dy, \quad w(t, x) := \int_0^x v(t, y)dy, \tag{2.8}$$

then $w_0 \in C^1(\mathbb{R})$ with $w_0(0) = w_0'(0) = 0$. We are looking for the solution $w(t, \cdot) \in C^1(\mathbb{R})$ with $w(t, 0) = w_x(t, 0) = 0$ for every $t \in \mathbb{R}$. Substituting $v = w_x$ into $v_t = (1 - \varphi)v_x + \varphi w$ yields

$$w_{tx} + (\varphi - 1)w_{xx} - \varphi w = 0, \quad x \in \mathbb{R},$$

which can be integrated as follows:

$$w_t + (\varphi - 1)w_x - \varphi'w = \text{const} = 0, \quad x \in \mathbb{R},$$

where the integration constant is zero thanks to the boundary condition $w(t, 0) = 0$ for every $t \in \mathbb{R}$. Along each characteristic curve satisfying (2.6), $W(t, s) := w(t, X(t, s))$ satisfies the initial-value problem:

$$\begin{cases} \frac{dW}{dt} = \varphi'(X(t, s))W, \\ W|_{t=0} = w_0(s), \end{cases} \tag{2.9}$$

which can be solved uniquely in the exact form:

$$W(t, s) = w_0(s)X_s(t, s), \quad X_s(t, s) = \begin{cases} \frac{1}{1+(e^t-1)e^{-s}}, & s > 0, \\ \frac{1}{1+(e^{-t}-1)e^s}, & s < 0, \end{cases} \tag{2.10}$$

with $\lim_{s \rightarrow 0^\pm} W(t, s) = 0$ for every $t \in \mathbb{R}$ thanks to $w_0(0) = 0$. Note the different limits $\lim_{s \rightarrow 0^\pm} X_s(t, s) = e^{\mp t}$ if $t \neq 0$, hence $X(t, \cdot) \in C_0^1$ if $t \neq 0$. Also note that

$$X_s(t, s) > 0, \quad \text{for every } s \neq 0 \tag{2.11}$$

and

$$X_s(t, s) \rightarrow 1 \quad \text{as } |s| \rightarrow \infty. \tag{2.12}$$

Finally, we solve (2.2) with (2.5) by using (2.7) and (2.10). Along each characteristic curve satisfying (2.6), $V(t, s) := v(t, X(t, s))$ satisfies the initial-value problem:

$$\begin{cases} \frac{dV}{dt} = \varphi(X(t, s))W(t, s), \\ V|_{t=0} = v_0(s), \end{cases} \tag{2.13}$$

which can be solved uniquely in the exact form:

$$V(t, s) = v_0(s) + w_0(s)Y(t, s), \quad Y(t, s) = \begin{cases} \frac{(e^t-1)e^{-s}}{1+(e^t-1)e^{-s}}, & s > 0, \\ \frac{(1-e^{-t})e^s}{1+(e^{-t}-1)e^s}, & s < 0, \end{cases} \tag{2.14}$$

with $\lim_{s \rightarrow 0^\pm} V(t, s) = 0$ for every $t \in \mathbb{R}$ thanks to $w_0(0) = v_0(0) = 0$. Furthermore, $V(t, s)$ is continuously differentiable in s piecewise for $s > 0$ and $s < 0$ for every $t \in \mathbb{R}$ since $v_0 \in C_0^1$ and $w_0 \in C^1$. Hence, $V(t, \cdot) \in C_0^1$ for every $t \in \mathbb{R}$. Also, thanks to the properties (2.11) and (2.12), we have $V(t, \cdot) \in H^1$ for every $t \in \mathbb{R}$ since $v_0 \in H^1$ and $w_0\varphi \in H^1$.

Finally, thanks again to the properties (2.11), (2.12), and $X(t, 0) = 0$ for every $t \in \mathbb{R}$, the change of coordinates $(t, s) \rightarrow (t, X)$ is a C_0^1 invertible transformation so that the solution $v(t, \cdot) = V(t, s(t, \cdot))$ belongs to $H^1 \cap C_0^1$ and satisfies $v(t, 0) = V(t, 0) = 0$ for every $t \in \mathbb{R}$. \square

By analyzing the exact solution $v(t, \cdot) \in H^1 \cap C_0^1$ of Lemma 3 in t , we show that $v_x(t, \cdot)$ grows in t due to characteristics with $x > 0$. At the same time, if we add an additional constraint $v_0 \in L^1$ on the initial data, we can also show that $v(t, \cdot)$ remains bounded in the supremum norm as $t \rightarrow \infty$.

Lemma 4. Assume that $v_0 \in H^1 \cap C_0^1 \cap L^1$ with $v_0(0) = 0$ and $\alpha := \lim_{x \rightarrow 0^+} v_0'(x) \neq 0$. Then, we have for every $t \geq 0$:

$$\|v(t, \cdot)\|_{L^\infty(0, \infty)} \leq \|v_0\|_{L^\infty(0, \infty)} + \|v_0\|_{L^1(0, \infty)}, \quad \|v(t, \cdot)\|_{L^\infty(-\infty, 0)} \leq 2\|v_0\|_{L^\infty(-\infty, 0)} \tag{2.15}$$

and

$$\|v_x(t, \cdot)\|_{L^\infty(0, \infty)} \geq \alpha e^t, \quad \|v_x(t, \cdot)\|_{L^\infty(-\infty, 0)} \leq \|v_0'\|_{L^\infty(-\infty, 0)} + 2\|v_0\|_{L^\infty(-\infty, 0)}. \tag{2.16}$$

Proof. If $v_0 \in H^1 \cap C_0^1$ with $v_0(0) = 0$, we have $V(t, \cdot) \in H^1 \cap C_0^1$ for every $t \in \mathbb{R}$ by Lemma 3. If in addition $v_0 \in L^1$, then $w_0 \in L^\infty$ thanks to the bound $\|w_0\|_{L^\infty} \leq \|v_0\|_{L^1}$.

We obtain by elementary computations:

$$\max_{t \geq 0, s \geq 0} \frac{(e^t - 1)e^{-s}}{1 + (e^t - 1)e^{-s}} = 1, \quad \max_{t \geq 0, s \leq 0} \frac{(e^{-t} - 1)se^s}{1 + (e^{-t} - 1)e^s} = 1. \tag{2.17}$$

While the proof of the first equality is obvious, the proof of the second equality consists of several steps. For fixed $s \leq 0$, the function

$$Z(t, s) := \frac{(e^{-t} - 1)se^s}{1 + (e^{-t} - 1)e^s}$$

is monotonically increasing in t so that

$$Z_{\max}(s) := \max_{t \geq 0} Z(t, s) = \lim_{t \rightarrow \infty} Z(t, s) = \frac{|s|e^{-|s|}}{1 - e^{-|s|}}$$

Then, $Z_{\max}(s)$ is monotonically decreasing in $|s|$ attaining the maximal value 1 as $|s| \rightarrow 0$. It follows from (2.14) and (2.17) that

$$|V(t, s)| \leq |v_0(s)| + |w_0(s)|, \quad t \geq 0, \quad s \geq 0 \tag{2.18}$$

and

$$|V(t, s)| \leq |v_0(s)| + \left| \frac{w_0(s)}{s} \right|, \quad t \geq 0, \quad s \leq 0, \tag{2.19}$$

where it follows from (2.8) that

$$\left| \frac{w_0(s)}{s} \right| \leq \|v_0\|_{L^\infty}.$$

Bounds (2.15) follow from the estimates (2.18) and (2.19).

By using the chain rule $U(t, s) = V_s(t, s)/X_s(t, s)$ for $U(t, s) := v_x(t, X(t, s))$, we obtain from (2.14) the exact solution for $s > 0$ and $t > 0$:

$$U(t, s) = v_0'(s) [1 + (e^t - 1)e^{-s}] + v_0(s)(e^t - 1)e^{-s} - w_0(s) \frac{(e^t - 1)e^{-s}}{1 + (e^t - 1)e^{-s}}. \tag{2.20}$$

If $\alpha := \lim_{x \rightarrow 0^+} v'_0(x) \neq 0$, then the first bound in (2.16) follows from the estimate

$$\|U(t, \cdot)\|_{L^\infty(0, \infty)} \geq \lim_{s \rightarrow 0^+} U(t, s) = \alpha e^t, \quad t \geq 0, \tag{2.21}$$

thanks to $w_0(0) = v_0(0) = 0$.

Similarly, we obtain from (2.14) the exact solution for $s < 0$ and $t > 0$:

$$U(t, s) = v'_0(s) [1 + (e^{-t} - 1)e^s] + v_0(s)(1 - e^{-t})e^s + w_0(s) \frac{(1 - e^{-t})e^s}{1 + (e^{-t} - 1)e^s}. \tag{2.22}$$

By using the second equality in (2.17), we obtain from (2.22) that

$$|U(t, s)| \leq |v'_0(s)| + |v_0(s)| + \left| \frac{w_0(s)}{s} \right|, \quad t \geq 0, \quad s \leq 0, \tag{2.23}$$

which yields the second bound in (2.16). \square

Remark 2. If $v_0 \in C^1$ with $v_0(0) = 0$ and $v'_0(0) \neq 0$, then $\lim_{s \rightarrow 0^\pm} U(t, s) = v'_0(0)e^{\pm t}$ so that $U(t, \cdot) \notin C^1$ for $t \neq 0$. Consequently, $v(t, \cdot) \notin C^1$ for $t \neq 0$.

The exponential growth of $v_x(t, \cdot)$ as $t \rightarrow \infty$ in Lemma 4 discovers the linear instability of the peakon in the supremum norm on v_x . The same instability is observed in the H^1 norm, as formulated in Theorem 2. Below we prove equalities (1.12) and (1.13) in Theorem 2 for $v_0 \in H^1$ with $v_0(0) = 0$ without additional requirements $v_0 \in C^1_0$ and $v_0 \in L^1$.

Proof of Theorem 2. By working on $[0, \infty)$, we use the exact solutions in (2.14) and (2.20), integrate by parts, and obtain

$$\|v(t, \cdot)\|_{L^2(0, \infty)}^2 = \int_0^\infty \frac{[v_0(s)]^2}{1 + (e^t - 1)e^{-s}} ds + \int_0^\infty \frac{[w_0(s)]^2 (e^t - 1)e^{-s}}{[1 + (e^t - 1)e^{-s}]^3} ds \tag{2.24}$$

and

$$\begin{aligned} \|v_x(t, \cdot)\|_{L^2(0, \infty)}^2 &= \int_0^\infty [v'_0(s)]^2 [1 + (e^t - 1)e^{-s}] ds + 2 \int_0^\infty [v_0(s)]^2 (e^t - 1)e^{-s} ds \\ &\quad + \int_0^\infty \frac{[v_0(s)]^2 (e^t - 1)e^{-s}}{1 + (e^t - 1)e^{-s}} ds - \int_0^\infty \frac{w_0^2(s) (e^t - 1)e^{-s}}{[1 + (e^t - 1)e^{-s}]^3} ds. \end{aligned} \tag{2.25}$$

In order to obtain (2.24), we use (2.14) and the chain rule to write

$$\|v(t, \cdot)\|_{L^2(0, \infty)}^2 = \int_0^\infty \left[v_0(s) + w_0(s) \frac{(e^t - 1)e^{-s}}{1 + (e^t - 1)e^{-s}} \right]^2 \frac{ds}{1 + (e^t - 1)e^{-s}}. \tag{2.26}$$

We expand the square and integrate the middle term by parts with the boundary conditions $w_0(0) = 0$ and $\lim_{s \rightarrow \infty} [w_0(s)]^2 e^{-s} = 0$ thanks to the Hölder inequality:

$$[w_0(x)]^2 = \left(\int_0^x v_0(s) ds \right)^2 \leq x \|v_0\|_{L^2(0, \infty)}^2, \quad x > 0.$$

This transforms the middle term to the form

$$\int_0^\infty \frac{2v_0 w_0 (e^t - 1) e^{-s}}{[1 + (e^t - 1) e^{-s}]^2} ds = \int_0^\infty \frac{w_0^2 (e^t - 1) e^{-s}}{[1 + (e^t - 1) e^{-s}]^2} ds - 2 \int_0^\infty \frac{w_0^2 (e^t - 1)^2 e^{-2s}}{[1 + (e^t - 1) e^{-s}]^3} ds \quad (2.27)$$

Substituting (2.27) into (2.26) yields (2.24).

Computations for (2.25) are similar but longer with three terms integrated by parts with the boundary conditions $v_0(0) = 0$ and $\lim_{s \rightarrow \infty} v_0(s) = 0$. Indeed, we write

$$\begin{aligned} \|v_x(t, \cdot)\|_{L^2(0, \infty)}^2 &= \int_0^\infty [v_0'(s) (1 + (e^t - 1) e^{-s}) + v_0(s) (e^t - 1) e^{-s} \\ &\quad - w_0(s) \frac{(e^t - 1) e^{-s}}{1 + (e^t - 1) e^{-s}}]^2 \frac{ds}{1 + (e^t - 1) e^{-s}}. \end{aligned} \quad (2.28)$$

The first cross term gives

$$2 \int_0^\infty v_0 v_0' (e^t - 1) e^{-s} ds = \int_0^\infty v_0^2 (e^t - 1) e^{-s} ds. \quad (2.29)$$

The second cross term gives

$$\begin{aligned} -2 \int_0^\infty w_0 v_0' \frac{(e^t - 1) e^{-s}}{1 + (e^t - 1) e^{-s}} ds &= 2 \int_0^\infty v_0^2 \frac{(e^t - 1) e^{-s}}{1 + (e^t - 1) e^{-s}} ds \\ &\quad - 2 \int_0^\infty w_0 v_0 \frac{(e^t - 1) e^{-s}}{1 + (e^t - 1) e^{-s}} ds \\ &\quad + 2 \int_0^\infty w_0 v_0 \frac{(e^t - 1)^2 e^{-2s}}{[1 + (e^t - 1) e^{-s}]^2} ds. \end{aligned} \quad (2.30)$$

The last term in (2.30) cancel out the third cross term in (2.28). The second term in (2.30) is also integrated by parts to the form:

$$\begin{aligned}
 -2 \int_0^\infty w_0 v_0 \frac{(e^t - 1)e^{-s}}{1 + (e^t - 1)e^{-s}} ds &= - \int_0^\infty w_0^2 \frac{(e^t - 1)e^{-s}}{1 + (e^t - 1)e^{-s}} ds \\
 &+ \int_0^\infty w_0^2 \frac{(e^t - 1)^2 e^{-2s}}{[1 + (e^t - 1)e^{-s}]^2} ds. \tag{2.31}
 \end{aligned}$$

Substituting (2.29), (2.30), and (2.31) into (2.28) yields (2.25). Finally, summing up (2.24) and (2.25) yields (1.12).

Computations on $(-\infty, 0]$ are similar from the exact solutions in (2.14) and (2.22). Integration by parts yield

$$\|v(t, \cdot)\|_{L^2(-\infty, 0)}^2 = \int_{-\infty}^0 \frac{[v_0(s)]^2}{1 + (e^{-t} - 1)e^s} ds + \int_{-\infty}^0 \frac{[w_0(s)]^2 (e^{-t} - 1)e^s}{[1 + (e^{-t} - 1)e^s]^3} ds \tag{2.32}$$

and

$$\begin{aligned}
 \|v_x(t, \cdot)\|_{L^2(-\infty, 0)}^2 &= \int_{-\infty}^0 [v'_0(s)]^2 [1 + (e^{-t} - 1)e^s] ds + 2 \int_{-\infty}^0 [v_0(s)]^2 (e^{-t} - 1)e^s ds \\
 &+ \int_{-\infty}^0 \frac{[v_0(s)]^2 (e^{-t} - 1)e^s}{1 + (e^{-t} - 1)e^s} ds - \int_{-\infty}^0 \frac{w_0^2(s) (e^{-t} - 1)e^s}{[1 + (e^{-t} - 1)e^s]^3} ds. \tag{2.33}
 \end{aligned}$$

Summing up (2.32) and (2.33) yields (1.13). □

3. Nonlinear evolution near a single peakon

Global existence and uniqueness of the conservative solution (1.4) to the Cauchy problem for the CH equation in the convolution form (1.2) with initial data $u_0 \in H^1$ was proven in [3,4]. Continuous dependence of the conservative solution (1.6) for initial data $u_0 \in H^1 \cap W^{1,\infty}$ was proven in [27]. We rewrite the Cauchy problem in the form:

$$\begin{cases} u_t + uu_x + Q[u] = 0, & t > 0, \\ u|_{t=0} = u_0, \end{cases} \tag{3.1}$$

where

$$Q[u](x) := \frac{1}{2} \int_{\mathbb{R}} \varphi'(x - y) \left([u(y)]^2 + \frac{1}{2} [u'(y)]^2 \right) dy, \quad x \in \mathbb{R}. \tag{3.2}$$

The following lemma shows that $Q[u](x)$ is continuous for every $x \in \mathbb{R}$ if $u \in H^1$.

Lemma 5. For every $u \in H^1$, we have $Q[u] \in C(\mathbb{R})$.

Proof. We can rewrite (3.2) in the explicit form:

$$Q[u](x) = \frac{1}{2}e^x \int_x^\infty e^{-y} \left([u(y)]^2 + \frac{1}{2}[u'(y)]^2 \right) dy - \frac{1}{2}e^{-x} \int_{-\infty}^x e^y \left([u(y)]^2 + \frac{1}{2}[u'(y)]^2 \right) dy.$$

Each integral is a continuous function for every $x \in \mathbb{R}$ since it is given by an integral of the absolutely integrable function. Hence $Q[u]$ is continuous on \mathbb{R} . \square

The unique global conservative solution (1.4) to the Cauchy problem (3.1) satisfies the weak formulation

$$\int_0^\infty \int_{\mathbb{R}} \left(u\psi_t + \frac{1}{2}u^2\psi_x - Q[u]\psi \right) dxdt + \int_{\mathbb{R}} u_0(x)\psi(0, x)dx = 0, \tag{3.3}$$

where the equality is true for every test function $\psi \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$. We consider the class of solutions with a single peak placed at the point $\xi(t)$ so that $u(t, \cdot + \xi(t)) \in C_0^1$ for every $t \in [0, T)$, where C_0^1 is defined by (1.14) and $T > 0$ is the maximal existence time of such solutions. The following lemma shows that the single peak moves with its local characteristic speed as in (1.10).

Lemma 6. Assume that there exists $T > 0$ (finite or infinite) such that the weak global conservative solution (1.4) to the equation (3.3) satisfies $u \in C([0, T), H^1)$ and $u(t, \cdot + \xi(t)) \in C_0^1$ for every $t \in [0, T)$ with a single peak located at $x = \xi(t)$. Then, $\xi \in C^1(0, T)$ satisfies

$$\frac{d\xi}{dt} = u(t, \xi(t)), \quad t \in (0, T). \tag{3.4}$$

Proof. Integrating (3.3) by parts on $(-\infty, \xi(t))$ and $(\xi(t), \infty)$ and using the fact that $u(t, \cdot) \in C(\mathbb{R})$ and $u(t, \cdot + \xi(t)) \in C_0^1$ for every $t \in [0, T)$, we obtain the following equations piecewise outside the peak’s location:

$$u_t(t, x) + u(t, x)u_x(t, x) + Q[u](t, x) = 0, \quad \pm[x - \xi(t)] > 0, \quad t \in (0, T). \tag{3.5}$$

Since $u(t, \cdot) \in H^1$ for every $t \in \mathbb{R}^+$, $Q[u]$ is a continuous function of x on \mathbb{R} for every $t \in \mathbb{R}^+$ by Lemma 5. Therefore, it follows from (3.5) that

$$[u_t]^\pm + u(t, \xi(t))[u_x]^\pm = 0, \quad t \in (0, T), \tag{3.6}$$

where $[u_x]^\pm = \lim_{x \rightarrow \xi(t)^+} u_x(t, x) - \lim_{x \rightarrow \xi(t)^-} u_x(t, x)$ is the jump of u_x across the peak location at $x = \xi(t)$. On the other hand, since $u(t, \cdot + \xi(t)) \in C_0^1$ for $t \in [0, T)$, we differentiate $u(t, \xi(t))$ continuously on both sides from $x = \xi(t)$ and define

$$\dot{u}^\pm(t) := \lim_{x \rightarrow \xi(t)^\pm} \left[u_t + \frac{d\xi}{dt}u_x \right], \quad t \in (0, T). \tag{3.7}$$

Since $u(t, \xi(t))$ is continuous for $t \in [0, T)$, we have $\dot{u}^+(t) = \dot{u}^-(t)$ for every $t \in (0, T)$. Therefore, it follows from (3.7) that

$$[u_t]_-^+ + \frac{d\xi}{dt}[u_x]_-^+ = 0, \quad t \in (0, T). \tag{3.8}$$

Since $[u_x]_-^+ \neq 0$ if $u \notin C^1(\mathbb{R})$, then it follows from (3.6) and (3.8) that $\xi(t)$ satisfies (3.4). Since $u \in C((0, T) \times \mathbb{R})$ due to Sobolev embedding of $H^1(\mathbb{R})$ into $C(\mathbb{R})$ and the assumption that $u \in C([0, T), H^1)$, then $\xi \in C^1(0, T)$. \square

In order to study the nonlinear evolution near a single peakon, we decompose the weak global conservative solution (1.4) as the following sum of the peakon and its small perturbation:

$$u(t, x) = \varphi(x - t - a(t)) + v(t, x - t - a(t)), \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}, \tag{3.9}$$

where $v(t, \cdot) \in H^1$ for every $t \in \mathbb{R}^+$ is the perturbation and $a(t)$ is the deviation of the peak position of the perturbed peakon from its unperturbed position moving with the unit speed. If there exists $T > 0$ such that $v(t, \cdot) \in C_0^1$ for every $t \in [0, T)$, the global conservative solution (1.4) consisting of a single peakon satisfies $u \in C([0, T), H^1)$ and $u(t, \cdot + \xi(t)) \in C_0^1$ for every $t \in [0, T)$, that is, it has a single peak at $\xi(t) = t + a(t)$. By Lemma 6, $a \in C^1(0, T)$ satisfies the equation

$$\frac{da}{dt} = v(t, 0), \quad t \in (0, T). \tag{3.10}$$

Substituting (3.9) and (3.10) into the Cauchy problem (3.1) yields the following Cauchy problem for the peaked perturbation v to the peakon φ :

$$\begin{cases} v_t = (1 - \varphi)v_x + \varphi w + (v|_{x=0} - v)v_x - Q[v], & t \in (0, T), \\ v|_{t=0} = v_0, \end{cases} \tag{3.11}$$

where $w(t, x) = \int_0^x v(t, y)dy$ and the linear evolution has been simplified by using (2.1) in Lemma 1. Note that $x - t - a(t)$ in (3.9) becomes x in (3.11) thanks to the translational invariance of the system (3.1) with (3.2). Also note that we have canceled the v -independent term in (3.11) by using the peakon equation

$$(1 - \varphi)\varphi' - Q(\varphi) = 0, \quad x \neq 0,$$

which is a derivative of the stationary equation (1.3).

Remark 3. The dynamical equation (3.10) cancels the last term in the linearization at the peakon (2.3) without additional requirement of $v|_{x=0} = 0$ imposed in Lemmas 3 and 4.

Related to the Cauchy problem (3.11), we define the family of characteristic coordinates $X(t, s)$ satisfying the initial-value problem:

$$\begin{cases} \frac{dX}{dt} = \varphi(X) - 1 + v(t, X) - v(t, 0), & t \in (0, T), \\ X|_{t=0} = s. \end{cases} \tag{3.12}$$

Along each characteristic curve parameterized by s , let us define $V(t, s) := v(t, X(t, s))$. It follows from (3.11) and (3.12) that $V(t, s)$ on each characteristic curve $x = X(t, s)$ satisfies the initial-value problem:

$$\begin{cases} \frac{dV}{dt} = \varphi(X)w(t, X) - Q[v](X), & t \in (0, T), \\ V|_{t=0} = v_0(s). \end{cases} \quad (3.13)$$

The following lemma transfers well-posedness theory for differential equations to the existence, uniqueness, and smoothness of the family of characteristic coordinates and the solution surface.

Lemma 7. *Assume $v_0 \in H^1 \cap C_0^1$. There exists the maximal existence time $T > 0$ (finite or infinite) such that the family of characteristic coordinates $[0, T) \times \mathbb{R} \ni (t, s) \mapsto X \in \mathbb{R}$ to (3.12) and the solution surface $[0, T) \times \mathbb{R} \ni (t, s) \mapsto V \in \mathbb{R}$ to (3.13) exist and are unique as long as $v_x(t, \cdot) \in L^\infty(\mathbb{R})$ for $t \in [0, T)$. Moreover, X and V are C^1 in t and C_0^1 in s for every $(t, s) \in [0, T) \times \mathbb{R}$.*

Proof. A simple extension of the proof of Lemma 5 implies that if $f \in H^1 \cap C_0^1$, then $Q[f] \in C_0^1$. Every C_0^1 function is Lipschitz continuous at $x = 0$. In addition, since $v(t, \cdot) \in H^1$ for every $t > 0$, then $v(t, \cdot) \in L^\infty(\mathbb{R})$, hence the function $v(t, \cdot)$ is globally Lipschitz continuous when it is locally Lipschitz continuous. Since φ have the same properties on \mathbb{R} and $w(t, \cdot) \in C^1(\mathbb{R})$ if $v(t, \cdot) \in C(\mathbb{R})$, the right-hand-sides of systems (3.12) and (3.13) are global Lipschitz continuous functions of X as long as the solution $v(t, \cdot) \in H^1$ remains $v(t, \cdot) \in C_0^1$ for $t \in [0, T)$ with some (finite or infinite) $T > 0$. Existence and uniqueness of the classical solutions $X(\cdot, s) \in C^1(0, T)$ and $V(\cdot, s) \in C^1(0, T)$ for every $s \in \mathbb{R}$ follows from the ODE theory. By the continuous dependence theorem, $X(t, \cdot) \in C_0^1$ and $V(t, \cdot) \in C_0^1$ for every $t \in [0, T)$.

Let us now show that $v(t, \cdot) \in C_0^1$ for $t \in [0, T)$ if $v_x(t, \cdot) \in L^\infty(\mathbb{R})$ for $t \in [0, T)$. By differentiating (3.12) piecewise for $s > 0$ and $s < 0$, we obtain

$$\begin{cases} \frac{dX_s}{dt} = [\varphi'(X) + v_x(t, X)] X_s, & t \in (0, T), \\ X_s|_{t=0} = 1, \end{cases} \quad (3.14)$$

with the solution

$$X_s(t, s) = \exp \left(\int_0^t [\varphi'(X(t', s)) + v_x(t', X(t', s))] dt' \right).$$

If $v_x(t, \cdot) \in L^\infty(\mathbb{R})$ for $t \in [0, T)$, then $X_s(t, s) > 0$ for $t \in [0, T)$ piecewise for $s > 0$ and $s < 0$, hence the change of coordinates $(t, s) \rightarrow (t, X)$ is a C_0^1 invertible transformation. As a result, $V(t, \cdot) \in C_0^1$ implies that $v(t, \cdot) \in C_0^1$ for $t \in [0, T)$. \square

Since $\varphi(0) = 1$ and $v(t, \cdot) \in C(\mathbb{R})$ for every $t \in \mathbb{R}^+$, $X = 0$ is a critical point of the initial-value problem (3.12). Therefore, the unique solution of Lemma 7 for $s = 0$ satisfies $X(t, 0) = 0$. This limiting characteristic curve separates the family of characteristic curves with $s > 0$ and $s < 0$. Since $V(t, s)$ is C^1 in t for every $s \in \mathbb{R}$ and Lipschitz continuous at $s = 0$ by Lemma 7, the limiting value $V_0(t) := V(t, 0)$ is C^1 in t and satisfies

$$\frac{dV_0}{dt} = -Q[v](0), \tag{3.15}$$

where we have used $w(t, 0) = 0$.

In order to control $v_x(t, \cdot) \in L^\infty(\mathbb{R})$ for $t \in [0, T)$ needed in the condition of Lemma 7, we differentiate (3.11) in x and obtain

$$\begin{cases} v_{xt} = (1 - \varphi)v_{xx} - \varphi'v_x + \varphi v + \varphi'w + (v|_{x=0} - v)v_{xx} - \frac{1}{2}v_x^2 + v^2 - P[v], \\ v_x|_{t=0} = v'_0, \end{cases} \tag{3.16}$$

where we have used $(1 - \partial_x^2)\varphi = 2\delta$ and have defined

$$P[v](x) := \frac{1}{2} \int_{\mathbb{R}} \varphi(x - y) \left([v(y)]^2 + \frac{1}{2}[v'(y)]^2 \right) dy, \quad x \in \mathbb{R}. \tag{3.17}$$

It follows from (3.12) and (3.16) that $U(t, s) := v_x(t, X(t, s))$ on each characteristic curve satisfies the initial-value problem:

$$\begin{cases} \frac{dU}{dt} = -\varphi'(X)U + \varphi(X)V + \varphi'(X)w(t, X) - \frac{1}{2}U^2 + V^2 - P[v](X), & t \in (0, T), \\ U|_{t=0} = v'_0(s). \end{cases} \tag{3.18}$$

Although $\varphi'(X)$ has a jump discontinuity at $X = 0$, the regions \mathbb{R}^+ and \mathbb{R}^- for s are separated from each other thanks to the fact that the limiting characteristic curve at $X = 0$ corresponds to the critical point of the initial-value problem (3.12). As a result, we consider the initial-value problem (3.18) separately for $s > 0$ and $s < 0$.

Lemma 8. Assume $v_0 \in H^1 \cap C^1_0$ and $T > 0$ (finite or infinite) be given by Lemma 7. There exist unique solutions $[0, T) \times \mathbb{R}^+ \ni (t, s) \mapsto U^+ \in \mathbb{R}$ and $[0, T) \times \mathbb{R}^- \ni (t, s) \mapsto U^- \in \mathbb{R}$ to (3.18) as long as $v_x(t, \cdot) \in L^\infty(\mathbb{R})$ for $t \in [0, T)$. Moreover, U^\pm are C^1 in t and s for every $(t, s) \in [0, T) \times \mathbb{R}^\pm$.

Proof. Similarly to the proof of Lemma 5, it follows that if $f \in H^1$, then $P[f] \in C^1(\mathbb{R})$. In addition, $\varphi'(X(t, s))$ are C^1 functions of t and s for $t \in [0, T)$ and separately for $s \in \mathbb{R}^+$ and $s \in \mathbb{R}^-$. Existence and uniqueness of solutions $[0, T) \times \mathbb{R}^+ \ni (t, s) \mapsto U^+ \in \mathbb{R}$ and $[0, T) \times \mathbb{R}^- \ni (t, s) \mapsto U^- \in \mathbb{R}$ to (3.18) follow from the ODE theory. By the continuous dependence theorem, U^\pm are C^1 in t and s for every $(t, s) \in [0, T) \times \mathbb{R}^\pm$. \square

By Lemma 8, we are allowed to define the one-sided limits $U_0^\pm(t) := \lim_{s \rightarrow 0^\pm} U^\pm(t, s)$ for $t \in [0, T)$. The functions U_0^\pm are C^1 in t and satisfy for $t \in (0, T)$:

$$\frac{dU_0^\pm}{dt} = \pm U_0^\pm + V_0 - \frac{1}{2}(U_0^\pm)^2 + V_0^2 - P[v](0). \tag{3.19}$$

By analyzing the time evolution (3.15) and (3.19), we finally prove the nonlinear instability of peaked perturbations to the H^1 -orbitally stable peakon.

Lemma 9. For every $\delta > 0$, there exist $t_0 > 0$ and $v_0 \in H^1 \cap C_0^1$ satisfying

$$\|v_0\|_{H^1} + \|v_0'\|_{L^\infty} < \delta, \tag{3.20}$$

such that the unique solution $v(t, \cdot) \in H^1 \cap C_0^1$ to the Cauchy problem (3.11) in Lemmas 7 and 8 satisfies

$$\|v_x(t_0, \cdot)\|_{L^\infty} > 1. \tag{3.21}$$

Proof. We deduce the nonlinear instability of the peakon by looking at the limiting characteristics to the peak location at $s = 0$. By Lemma 4, the linear exponential instability is related to the characteristics to the right of the peak for $s > 0$ but not to the left of the peak for $s < 0$. Therefore, we shall use equation (3.19) for U_0^+ only.

Combining (3.15) and (3.19) for U_0^+ together yields the following equation for $t \in (0, T)$:

$$\frac{d}{dt}(V_0 + U_0^+) = (V_0 + U_0^+) + V_0^2 - \frac{1}{2}(U_0^+)^2 + F_0, \tag{3.22}$$

where

$$\begin{aligned} F_0 &:= -Q[v](0) - P[v](0) \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} [\varphi'(y) - \varphi(y)] \left([v(t, y)]^2 + \frac{1}{2}[v_y(t, y)]^2 \right) dy, \\ &= - \int_0^{+\infty} e^{-y} \left([v(t, y)]^2 + \frac{1}{2}[v_y(t, y)]^2 \right) dy, \end{aligned}$$

and we have used $\varphi(x) = e^{-|x|}$ in the last equality. By using an integrating factor, we rewrite (3.22) in the equivalent form

$$\frac{d}{dt} [e^{-t}(V_0 + U_0^+)] = e^{-t} \left[V_0^2 - \frac{1}{2}(U_0^+)^2 + F_0 \right] \leq e^{-t} V_0^2,$$

where the last inequality is due to $F_0 \leq 0$. Integrating the differential inequality yields the bound:

$$V_0(t) + U_0^+(t) \leq e^t \left[V_0(0) + U_0^+(0) + \int_0^t e^{-\tau} V_0^2(\tau) d\tau \right], \quad t \in [0, T). \tag{3.23}$$

Since $\varphi'(x) \rightarrow \mp 1$ as $x \rightarrow \pm 0$ and $\varphi(x) \rightarrow 0$ as $|x| \rightarrow \infty$, whereas $v(t, \cdot)$ is small in the C_0^1 norm at least for $t \in [0, T)$, the coordinate $\xi(t) = t + a(t)$ for the peak’s location coincides with the location of the maximum of $u(t, \cdot)$ in the decomposition (3.9) for $t \in [0, T)$. By Theorem 1, for every small $\varepsilon > 0$, if $\|v_0\|_{H^1} < \varepsilon^4/81$, then

$$\|v(t, \cdot)\|_{H^1} < \varepsilon, \quad t \in (0, T).$$

By Sobolev’s embedding, we have

$$|V_0(t)| \leq \|v(t, \cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}} \|v(t, \cdot)\|_{H^1} < \varepsilon. \tag{3.24}$$

Let us assume that the initial data $v_0 \in H^1 \cap C_0^1$ satisfies $v_0(0) = 0$ and

$$\alpha := \lim_{x \rightarrow 0^+} v'_0(x) = -\|v'_0\|_{L^\infty} = -2\varepsilon^2. \tag{3.25}$$

The initial bound (3.20) is consistent with (3.25) if for every small $\delta > 0$, the small value of ε satisfies

$$\left(\frac{\varepsilon}{3}\right)^4 + 2\varepsilon^2 < \delta,$$

which just specifies ε in terms of δ . With these constraints on v_0 , the bound (3.23) yields

$$V_0(t) + U_0^+(t) \leq e^t [\alpha + \varepsilon^2] = -\varepsilon^2 e^t,$$

or equivalently, $|V_0(t) + U_0^+(t)| \geq \varepsilon^2 e^t$. Hence, for every small $\varepsilon > 0$ there exists sufficiently large

$$\tau := \log(2) - 2 \log(\varepsilon) > 0,$$

such that $|V_0(\tau) + U_0^+(\tau)| \geq 2$. This implies $|U_0^+(\tau)| > 1$ thanks to the bound (3.24).

If $T < \tau$, then $v_x(t, x) \rightarrow -\infty$ for some $x \in \mathbb{R}$ as $t \nearrow T$ by the condition of Lemma 7. Therefore, there exists $t_0 \in (0, T)$ such that the bound (3.21) is true. If $T > \tau$, then the differential equation (3.22) is valid for $t \in [0, T)$ by Lemmas 7 and 8 so that the bound (3.21) is true at $t_0 = \tau$ thanks to the bound

$$\|v_x(t, \cdot)\|_{L^\infty} = \|U(t, \cdot)\|_{L^\infty} \geq |U_0^+(t)|, \quad t \in [0, T).$$

In both cases, the bound (3.21) is proven. \square

Remark 4. The result of Lemma 9 gives the proof of Theorem 3 thanks to the representation (3.9) with the peak’s location defined at $\xi(t) = 1 + a(t)$.

Remark 5. If $a(0) = 0$ without loss of generality, the initial data $u_0 \in H^1 \cap C_0^1$ is represented by $u_0(x) = \varphi(x) + v_0(x)$ so that $\lim_{x \rightarrow 0} u_0(x) = 1 + v_0(0)$ whereas $\lim_{x \rightarrow 0^+} u'_0(x) = -1 + \lim_{x \rightarrow 0^+} v'_0(x)$. The initial data in the proof of Lemma 9 satisfies $v_0(0) = \mathcal{O}(\varepsilon^4)$ whereas $\lim_{x \rightarrow 0^+} v'_0(x) = -2\varepsilon^2$. Since ε is small, the initial data satisfies $\lim_{x \rightarrow 0^+} u'_0(x) < -u_0(0)$, which is the sufficient condition for breaking in a finite time obtained in [2]. Hence, the local solution used in Lemma 9 to prove nonlinear instability of peakons breaks in a finite (rather than infinite) time $T > 0$ in the $W^{1,\infty}$ norm.

4. Conclusion

We have shown that the passage from linear to nonlinear stability of peakons in H^1 is false for the Camassa–Holm equation because the linear result of Theorem 2 gives linearized instability in H^1 whereas the result of Theorem 1 gives nonlinear stability of peakons in H^1 . On the other hand, we show that the linearized instability in $H^1 \cap C_0^1$ persists as the nonlinear instability result of Theorem 3. The latter result is natural for the Camassa–Holm equation where smooth solutions may break in a finite time with the slopes becoming unbounded from below.

We conclude the paper with possible extensions of our main results. It is quite natural to prove instability of the peaked periodic waves with respect to the peaked periodic perturbations in the framework of the CH equation (1.1). Furthermore, the same instability is likely to hold for peakons in the Degasperis–Procesi equation, although the linearized evolution may not be as simple as the one for the CH equation. Finally, the method of characteristics is likely to work to prove nonlinear instability of peaked periodic wave in the reduced Ostrovsky equation (1.17), which has been an open problem up to now.

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