ON THE TRANSVERSE STABILITY OF SMOOTH SOLITARY WAVES IN A TWO-DIMENSIONAL CAMASSA–HOLM EQUATION

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Abstract. We consider the propagation of smooth solitary waves in a two-dimensional generalization of the Camassa–Holm equation. We show that transverse perturbations to one-dimensional solitary waves behave similarly to the KP-II theory. This conclusion follows from our two main results: (i) the double eigenvalue of the linearized equations related to the translational symmetry breaks under a transverse perturbation into a pair of the asymptotically stable resonances and (ii) small-amplitude solitary waves are linearly stable with respect to transverse perturbations.

1. Introduction

The Camassa–Holm equation, labelled as *the CH equation*,

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

is a popular model for the dynamics of unidirectional shallow water waves [2, 19] which has been justified mathematically in [7]. It was originally introduced in [12] as a deformation of the integrable KdV equations. The equation models the behavior of shallow water waves both in the setting of solitary and periodic waves. Global solutions exist for initial data with sufficiently gradual slopes and wave breaking occurs in finite time for initial data with steep slopes [5, 6]. There exist smooth and peaked traveling waves both among the spatially solitary and periodic waves [15, 25]. The smooth solitary waves were shown to be spectrally and orbitally stable in the time evolution of the CH equation [10, 22]. Similar stability results were obtained for the traveling periodic waves in [15, 26]. On the other hand, although the peaked traveling waves (both solitary and periodic) are energetically stable in the energy space H^1 [8, 9, 23, 24], the local solutions are only defined in the function space $H^1 \cap W^{1,\infty}$ [11, 27]. It was recently shown that the peaked traveling waves are both spectrally and orbitally unstable in $H^1 \cap W^{1,\infty}$ [21, 28, 33].

As a model for shallow water waves, the CH equation (1.1) is limited to two-dimensional fluid motion confined by a one-dimensional time-dependent surface. Transverse modulations on the water surface can be defined in terms of the two spatial variables $(x, y) \in \mathbb{R}^2$. A generalization of the CH equation with a two-dimensional time-dependent profile $u = u(x, y, t)$ has appeared in the literature only recently. This equation can be written in its simplest dimensionless form as

$$(u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx})_x + u_{yy} = 0. \tag{1.2}$$

Date: July 21, 2023.

It was first derived in [3] as a model in the context of nonlinear elasticity theory. More recently, it was obtained in [16] as a model in the context of incompressible and irrotational shallow water wave theory. We refer to (1.2) as *the CH-KP equation* because it generalizes the CH equation (1.1) in the same way as the Kadomtsev–Petviashvili (KP) equation generalizes the classical Korteweg–de Vries (KdV) equation [20].

In the following we review some mathematical results that have been obtained for the CH-KP equation (1.2) so far. Local existence of solutions was obtained in the space of functions $X^s(\mathbb{R}^2)$ with $s \geq 2$, where

$$X^s(\mathbb{R}^2) := \{u \in H^s(\mathbb{R}^2) : \quad \partial_x^{-1} u \in H^s(\mathbb{R}^2), \;\; \partial_x u \in H^s(\mathbb{R}^2)\},$$

see [16, Theorem 1.1]. The nonlocal operator ∂_x^{-1} can be formally defined as

$$(\partial_x^{-1} f)(x) := \int_{+\infty}^{x} f(x')\, dx'$$

for functions $f(x) : \mathbb{R} \to \mathbb{R}$ that decay to zero as $x \to +\infty$. This nonlocal operator can be used to rewrite (1.2) in the evolution form

$$u_t + (1 - \partial_x^2)^{-1} \left[3uu_x - 2u_x u_{xx} - u u_{xxx} + \partial_x^{-1} u_{yy} \right] = 0. \tag{1.3}$$

The evolution equation (1.3) can be cast in Hamiltonian form

$$u_t = -J F'(u), \tag{1.4}$$

with the skew-adjoint operator $J := \partial_x (1 - \partial_x^2)^{-1}$ and the conserved energy

$$F(u) := \frac{1}{2} \int_{\mathbb{R}^2} \left[u^3 + u u_x^2 + (\partial_x^{-1} u_y)^2 \right] \, dx\, dy. \tag{1.5}$$

It was shown in [16] that $F(u)$ is conserved in time for local solutions in $X^s(\mathbb{R}^2)$ for $s \geq 2$, and so is the momentum

$$E(u) := \frac{1}{2} \int_{\mathbb{R}^2} (u^2 + u_x^2)\, dx\, dy. \tag{1.6}$$

In addition to $F(u)$ and $E(u)$, the mass

$$M(u) := \int_{\mathbb{R}^2} u\, dx\, dy \tag{1.7}$$

is formally conserved in the time evolution of the CH-KP equation (1.2). Various wave breaking criteria were obtained in [16, Theorems 1.2–1.4]. A recent work [38] explored numerical (Galerkin) methods for approximation of solitary waves in the CH–KP equation.

The purpose of this work is to study the transverse stability of perturbed solitary waves in the CH-KP equation (1.2). Line solitary waves are obtained for functions of the form $u(x, y, t) = \phi(x + \gamma y - ct)$ with parameters $\gamma, c \in \mathbb{R}$. In what follows, we will only consider the case $\gamma = 0$ for the traveling wave solutions of the CH equation (1.1).

It was the motivation of the pioneering work [20] to investigate the transverse stability of solitary waves under small slowly varying perturbations. It was discovered that the line solitary waves are transversely unstable in one version of the KP equation and are transversely stable in another version of the KP equation. These versions are now conventionally referred to as the *KP-I* and *KP-II* equations, respectively. The CH-KP equation (1.2) we are considering in the present work corresponds to KP-II.

A rigorous proof of transverse stability of traveling waves in the KP-II equation was completed only recently. Linear and nonlinear stability of the solitary waves have been proven for transversely periodic perturbations in [30] and for decaying perturbations in $\mathbb{R}^2$ [29]. Linear stability of traveling periodic waves was shown in [17] and the nonlinear stability of periodic waves is still an open problem for the KP-II equation.

Asymptotic reductions of other nonlinear systems to the KP-II equation have been explored in the literature. Mizumachi and Shimabukuro used the KP–II equation as an approximation of the Benney–Luke system to prove linear and nonlinear transverse stability of the line solitary waves of small amplitudes [31, 32]. A justification of the asymptotic reduction to the KP-II equation for the two-dimensional Boussinesq equation was done by Gallay and Schneider [13]. In the recent series of papers [14, 18, 34], the KP-II equation was justified as the leading model for a two-dimensional Fermi–Pasta–Ulam system on a square lattice. See also [1] for recent work on transverse stability of line solitary waves in other generalizations of the KP equation.

We can formally obtain the asymptotic reduction of the CH–KP equation to the KP-II equation. Let $k > 0$ be a fixed parameter and consider the slowly varying approximation of small-amplitude perturbations of a constant background in the form

$$u(x, y, t) = k + \varepsilon^2 v(\varepsilon(x - 3kt), \varepsilon^2 y, \varepsilon^3 t). \tag{1.8}$$

By using the chain rule and the evolution form (1.3), we derive the following evolution equation for the variable $v = v(X, Y, T)$ in scaled coordinates as

$$v_T + (1 - \varepsilon^2\partial_X^2)^{-1} \left[2kv_{XXX} + 3vv_X + \partial_X^{-1} v_{YY} - \varepsilon^2(2v_X v_{XX} + vv_{XXX})\right] = 0.$$

The formal truncation at $\varepsilon = 0$ yields the KP-II equation in the form

$$v_T + 2kv_{XXX} + 3vv_X + \partial_X^{-1} v_{YY} = 0. \tag{1.9}$$

For every fixed $k > 0$, the line solitary waves are linearly and nonlinearly stable in the KP-II equation (1.9) [29]. *The main conclusion of this work is that the smooth solitary waves are linearly transversely stable also in the CH-KP equation (1.3).* The nonlinear transverse stability is still an open question, and our results on the linear transverse stability so far are limited to two claims:

- The transverse perturbation breaks the double zero eigenvalue of the linearized equations into a pair of resonances located in the left half-plane. This result is obtained for smooth solitary waves of arbitrary amplitude.
- The line solitary waves are linearly stable with respect to transverse perturbations if the wave amplitude is sufficiently small.

The precise statement of these two results will be given in Section 2, after the traveling waves and their linear stability problems will be described. Sections 3 and 4 contain the proofs of these two main results. Section 5 concludes the paper with a summary and a list of open questions for further studies.

2. Smooth solitary waves

We consider the traveling one-dimensional solitary waves described by solutions to the CH-KP equation (1.2) of the form

$$u(x,y,t) = \phi(x-ct),$$

where $\phi(x) \to k$ as $|x| \to \infty$, for a fixed background parameter $k > 0$. It is well-known [15, 22], see also [10, 25] for earlier results, that such solitary waves exist for $c > 3k$ and have a smooth profile $\phi \in C^\infty(\mathbb{R})$. The following lemma formalizes the result.

Lemma 2.1. *Fix $k > 0$. For every $c > 3k$, there exists a traveling solitary wave solution of the CH equation (1.1) with profile $\phi \in C^\infty(\mathbb{R})$ of the form $\phi(x) = k + \psi(x)$, where ψ is found from the first-order invariant*

$$(\psi')^2 = \psi^2 \, \frac{c-3k-\psi}{c-k-\psi}. \tag{2.1}$$

In particular, $\psi(x) > 0$ for all $x \in \mathbb{R}$, $\psi(x) \to 0$ as $|x| \to \infty$ exponentially fast, and $\psi(x)$ is monotonically decreasing on both sides of its maximum at $\max_{x\in\mathbb{R}} \psi(x) = c - 3k$.

Proof. The traveling wave of the CH equation (1.1) with profile ϕ satisfies the third-order differential equation

$$-c(\phi' - \phi''') + 3\phi\phi' - 2\phi'\phi'' - \phi\phi''' = 0,$$

which can either be integrated directly to give

$$(c-\phi)(\phi - \phi'') + \frac{1}{2}(\phi')^2 - \frac{1}{2}\phi^2 = kc - \frac{3}{2}k^2, \tag{2.2}$$

or first multiplied by $(c - \phi)$ and then integrated to give

$$-(c-\phi)^2(\phi'' - \phi) = k(c-k)^2. \tag{2.3}$$

In both cases, we have fixed the integration constant from the conditions $\phi(x) \to k$ and $\phi'(x), \phi''(x) \to 0$ as $|x| \to \infty$. Multiplying (2.3) by ϕ' and integrating again gives

$$\frac{1}{2}(\phi')^2 - \frac{1}{2}\phi^2 + \frac{k(c-k)^2}{(c-\phi)} = kc - \frac{3}{2}k^2. \tag{2.4}$$

Writing $\phi = k + \psi$, we obtain (2.1) from (2.4).

A solitary wave with $\psi(x) \to 0$ as $|x| \to \infty$ corresponds to a homoclinic orbit on the phase plane (ψ, ψ') along the level curve (2.1) to the saddle point $(0,0)$. The solitary wave exists if and only if $c - 3k > 0$, because $(0,0)$ is a center point for $c - 3k < 0$ and no homoclinic orbit exists for $c - 3k = 0$. Since $(0,0)$ is a saddle point for $c - 3k > 0$, the convergence rate of $\psi(x) \to 0$ as $|x| \to \infty$ is exponential. The stable and unstable curves at $(0,0)$ do not intersect if $\psi < 0$ and intersect if $\psi > 0$. Hence $\psi(x) > 0$ for all $x \in \mathbb{R}$ and the turning point $x_0 \in \mathbb{R}$ with $\psi'(x_0) = 0$ exists if and only if $\psi(x_0) = c - 3k$. Thus, the profile ψ is monotonically decreasing away from its maximum at $\max_{x\in\mathbb{R}} \psi(x) = c - 3k$. □

Remark 2.2. Due to the translational symmetry of the CH equation we may place the maximum of ψ at $x = 0$ such that $\psi(0) = c - 3k$.

Remark 2.3. Since the scaling (1.8) suggests a reduction of the CH-KP equation (1.3) to the KP-II equation (1.9), the traveling solitary wave of Lemma 2.1 must converge to the traveling solitary wave of the KdV equation

$$v_T + 2kv_{XXX} + 3vv_X = 0. \tag{2.5}$$

Indeed, solving the KdV equation (2.5) for the solitary wave profile with

$$v(X,T) = \text{sech}^2\left(\frac{X-T}{2\sqrt{2k}}\right)$$

gives the formal asymptotic expansion

$$\phi(x) = k + \varepsilon^2 \text{sech}^2\left(\frac{\varepsilon x}{2\sqrt{2k}}\right) + \mathcal{O}(\varepsilon^4), \quad c = 3k + \varepsilon^2, \tag{2.6}$$

where $\varepsilon > 0$ is an arbitrary (small) parameter and x stands for $x - ct$. The asymptotic limit to the solitary wave of small amplitude corresponds to the limit $c \to 3k$ for which $\varepsilon \to 0$. This reduction is made rigorous in Lemma 4.1 below.

In order to set up the linear transverse stability problem for the smooth solitary wave of Lemma 2.1, we consider the decomposition

$$u(x,y,t) = \phi(x-ct) + v(x-ct,y,t)$$

with the perturbation v to the solitary wave profile $\phi \in C^\infty(\mathbb{R})$. After substitution of the decomposition into (1.3) and neglecting the quadratic terms in v, we obtain the linearized equation

$$v_t = J(L - \partial_x^{-2}\partial_y^2)v, \tag{2.7}$$

where $J := \partial_x(1-\partial_x^2)^{-1}$ as in (1.4) and

$$L := c - 3\phi + \phi'' - \partial_x(c-\phi)\partial_x. \tag{2.8}$$

Separation of variables in the linearized equation (2.7) by using normal modes of the form

$$v(x,y,t) = e^{\lambda t}e^{i\eta y}\hat{v}(x),$$

where $\lambda \in \mathbb{C}$ and $\eta \in \mathbb{R}$, yields the spectral stability problem

$$J(L + \eta^2\partial_x^{-2})\hat{v} = \lambda\hat{v}. \tag{2.9}$$

The one-dimensional spectral stability problem is recovered for $\eta = 0$. We can now specify the following definition of transverse spectral stability.

Definition 2.4. *We say that the solitary wave with profile $\phi \in C^\infty(\mathbb{R})$ is transversely spectrally stable if for every $\eta \in \mathbb{R}$ there exists no eigenvalue $\lambda \in \mathbb{C}$ with* $\text{Re}(\lambda) > 0$ *and eigenfunction $\hat{v} \in \text{Dom}(J(L+\eta^2\partial_x^{-2})) \subset L^2(\mathbb{R})$ of the spectral stability problem (2.9).*

A common method to study the linear stability of solitary waves in the KdV equation (2.5) is to use the exponentially weighted space L^2_ν with fixed $\nu > 0$ [4, 36], which is defined as

$$L^2_\nu := \{f(x) : \mathbb{R} \to \mathbb{R} : \quad e^{\nu \cdot} f \in L^2(\mathbb{R})\}. \tag{2.10}$$

If $f \in L^2_\nu$ with $\nu > 0$, then $f(x) \to 0$ as $x \to +\infty$ and so the nonlocal operator ∂_x^{-1} is well-defined. Note however that $f(x)$ does not have to decay and may even be slowly growing as $x \to -\infty$. By using the exponentially weighted space L^2_ν, we rephrase the definition of the transverse spectral stability.

Definition 2.5. *We say that the solitary wave with profile $\phi \in C^\infty(\mathbb{R})$ is transversely asymptotically stable in L^2_ν for some $\nu > 0$ if for every $\eta \in \mathbb{R}$, $\eta \neq 0$ there exists $b > 0$ such that all points λ in the spectrum of the linear operator*

$$J(L + \eta^2 \partial_x^{-2}) : \mathrm{Dom}(J(L + \eta^2 \partial_x^{-2})) \subset L^2_\nu \to L^2_\nu$$

satisfy $\mathrm{Re}(\lambda) \leq -b$.

The fact that $\phi(x) \to k$ as $|x| \to \infty$ exponentially fast greatly simplifies the spectral analysis of our problem. As a result, Weyl's theory implies that the continuous spectrum of $J(L + \eta^2 \partial_x^{-2})$ in L^2_ν is uniquely determined by the purely continuous spectrum of $J(L_0 + \eta^2 \partial_x^{-2})$, where

$$L_0 := c - 3k - (c - k)\partial_x^2. \tag{2.11}$$

In addition, the point spectrum of $J(L + \eta^2 \partial_x^{-2})$ in L^2_ν may contain eigenvalues $\lambda \in \mathbb{C}$ with eigenfunctions $\hat{v} \in \mathrm{Dom}(J(L + \eta^2 \partial_x^{-2}))$.

The first result of this paper is to show that both the continuous spectrum and the two eigenvalues near the origin in the complex plane satisfy the transverse asymptotic stability condition of Definition 2.5 for some $\nu > 0$. The proof is developed in Section 3, where the continuous spectrum is computed with the help of the Fourier transform and the two eigenvalues are computed by using Puiseux expansions [37] in the small parameter η.

Theorem 2.6. *For every $c > 3k$, $\eta \in \mathbb{R}$, and $\nu \in (0, \nu_0)$ with $\nu_0 := \sqrt{\frac{c-3k}{c-k}}$, there exists $b_0 > 0$ such that all points λ in the spectrum of the linear operator $J(L_0 + \eta_2 \partial_x^{-2})$ in L^2_ν satisfy* $\mathrm{Re}(\lambda) \leq -b_0$. *Furthermore, there exists $\eta_0 > 0$ such that the spectrum of the linear operator $J(L + \eta^2 \partial_x^{-2})$ in L^2_ν with $\eta \in (-\eta_0, \eta_0)$ includes a pair of simple eigenvalues $\lambda_\pm(\eta)$ such that for $\eta \neq 0$ we have*

- $\mathrm{Re}(\lambda_+(\eta)) = \mathrm{Re}(\lambda_-(\eta)) < 0,$
- $\mathrm{Im}(\lambda_+(\eta)) = -\mathrm{Im}(\lambda_-(\eta)) > 0,$

and $\lambda_+(0) = \lambda_-(0) = 0$.

Remark 2.7. The result of Theorem 2.6 is consistent with the transverse asymptotic stability with respect to long transverse perturbations in the sense of Definition 2.5 with small $\eta \neq 0$. However, the spectrum of JL in L^2_ν might include more than the continuous spectrum and the double zero eigenvalue. There might exist additional embedded eigenvalues of JL in $L^2(\mathbb{R})$ on the imaginary axis which could become isolated in L^2_ν for $\nu > 0$. The latter possibility has been ruled out for the KdV equation (2.5), see [35, 36]. However, nothing is known about the existence of additional embedded eigenvalues of JL in $L^2(\mathbb{R})$ on $i\mathbb{R}$ for the CH equation (1.1).

The second result of this paper explores the small-amplitude limit of the solitary waves and provides transverse asymptotic stability for solitary waves of small amplitudes in the sense of Definition 2.5. The proof is developed in Section 4 based on estimates for the resolvent equation.

Theorem 2.8. *Let $\lambda_\pm(\eta)$ be the simple eigenvalues of $J(L + \eta^2\partial_x^{-2})$ in L^2_ν for fixed $\nu \in (0, \nu_0)$ found in Theorem 2.6. There exists $\varepsilon_0 > 0$ and $\beta_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon := \sqrt{c - 3k}$, and for every $\eta \in \mathbb{R}$, $\eta \neq 0$, the spectrum of $J(L + \eta^2\partial_x^{-2})$ in L^2_ν is contained in*

$$\mathcal{S} := \{\lambda \in \mathbb{C} : \quad \mathrm{Re}(\lambda) \leq -\beta_0\varepsilon^3\},$$

with the exception of the two simple eigenvalues $\lambda = \lambda_\pm(\eta)$.

Remark 2.9. Since $\mathrm{Re}(\lambda_\pm(\eta)) < 0$ for $\eta \in (-\eta_0, \eta_0)$, $\eta \neq 0$, the solitary waves of small amplitude are transversely asymptotically stable in L^2_ν. By using the Fourier transform in y, the result of Theorem 2.8 also implies the transverse asymptotic stability of these solitary waves with respect to perturbations in $L^2_\nu(\mathbb{R}^2)$, where the weight $\nu \in (0, \nu_0)$ is only applied in the direction of the solitary waves. This yields linear asymptotic stability of solutions to the evolution equation (2.7) in $L^2_\nu(\mathbb{R}^2)$ by semi-group theory.

3. Proof of Theorem 2.6

3.1. **Preliminary results.**

The one-dimensional CH equation (1.1) has the following conserved quantities which play a crucial role in the stability analysis of its traveling solitary and periodic waves [10, 15]:

$$\hat{F}(u) := \frac{1}{2}\int_{\mathbb{R}} (u^3 + uu_x^2 - k^3)\, dx,$$
$$\hat{E}(u) := \frac{1}{2}\int_{\mathbb{R}} (u^2 + u_x^2 - k^2)\, dx,$$
$$\hat{M}(u) := \int_{\mathbb{R}} (u - k)\, dx.$$

The constant values have been subtracted from the integrands to ensure that the integrals converge if $u(x) \to k$ as $|x| \to \infty$ sufficiently fast. These quantities are the one-dimensional analogues of the conserved quantities (1.5), (1.6), and (1.7) of the two-dimensional CH-KP equation (1.2). Using $\hat{F}$, $\hat{E}$, and $\hat{M}$ we define the augmented energy

$$\Lambda_c(u) := -\hat{F}(u) + c\hat{E}(u) - \left(ck - \frac{3}{2}k^2\right)\hat{M}(u).$$

Smooth solutions to the second-order equation (2.2) with the profile $\phi \in C^\infty(\mathbb{R})$ are critical points of Λ_c in the sense that the first variation vanishes:

$$\Lambda_c'(\phi) = -\frac{3}{2}\phi^2 + \frac{1}{2}(\phi')^2 + \phi\phi'' + c\phi - c\phi'' - ck + \frac{3}{2}k^2 = 0.$$

The linear operator L in (2.8) is the Hessian operator of Λ_c at the critical point with the profile $\phi \in C^\infty(\mathbb{R})$. This variational characterization of the traveling wave solutions was explored in the stability analysis in [10, 15], see also [22] for alternative variational characterizations of the traveling wave solutions in the CH equation (1.1).

Remark 3.1. Since the linear operator L in (2.8) is the Hessian operator $\Lambda_c''(\phi)$ at the traveling solitary wave with the profile $\phi \in C^\infty(\mathbb{R})$ given by Lemma 2.1, it also arises in the linearization of the CH equation (1.1) given by $v_t = JLv$.

If $\phi = k + \psi$, then

$$\begin{aligned} E_{1\mathrm{D}}(\psi) &:= \hat{E}(\phi) - k\hat{M}(\phi) \\ &= \frac{1}{2}\int_{\mathbb{R}} [(k+\psi)^2 + (\psi')^2 - k^2 - 2k\psi]dx \\ &= \frac{1}{2}\int_{\mathbb{R}} [(\psi')^2 + \psi^2]dx \end{aligned} \tag{3.1}$$

and

$$M_{1\mathrm{D}}(\psi) := \hat{M}(\phi) = \int_{\mathbb{R}} \psi dx. \tag{3.2}$$

The following lemma reports important monotonicity properties of $E_{1\mathrm{D}}(\psi)$ and $M_{1\mathrm{D}}(\psi)$ with respect to the parameter $c \in (3k, \infty)$ for fixed $k > 0$. The proof is based on direct computations.

Lemma 3.2. *For fixed $k > 0$, let ψ be the solitary wave defined by the first-order invariant (2.1). Then, the mappings $c \mapsto M_{1\mathrm{D}}(\psi)$ and $c \mapsto E_{1\mathrm{D}}(\psi)$ are monotonically increasing for every $c \in (3k, \infty)$.*

Proof. Without loss of generality, we place the maximum of ψ at $x = 0$ such that $\psi(0) = c - 3k$, see Remark 2.2. By Lemma 2.1, we have $\psi(x) = \psi(-x) > 0$ for every $x \in \mathbb{R}$ and $\psi'(x) = -\psi'(-x) < 0$ for every $x > 0$. We obtain from (3.2) by explicit computations that

$$\begin{aligned} M_{1\mathrm{D}}(\psi) &= 2\int_0^\infty \psi(x)dx \\ &= 2\int_0^{c-3k} \frac{\sqrt{c-k-\psi}}{\sqrt{c-3k-\psi}} d\psi \\ &= 2\int_0^{c-3k} \frac{\sqrt{2k+z}}{\sqrt{z}} dz \\ &= 8k\int_0^{\xi_0} \sqrt{1+\xi^2} d\xi, \end{aligned}$$

where we have made the substitutions $z = c - 3k - \psi$ and

$$\xi = \frac{\sqrt{z}}{\sqrt{2k}}, \qquad \xi_0 = \frac{\sqrt{c-3k}}{\sqrt{2k}}.$$

The integral is evaluated explicitly to find that

$$M_{1\mathrm{D}}(\psi) = 4k\left[\xi_0\sqrt{1+\xi_0^2} + \mathrm{arcsinh}\xi_0\right],$$

from which it follows that

$$\frac{d}{dc}M_{1\mathrm{D}}(\psi) = 2\sqrt{\frac{c-k}{c-3k}} > 0.$$

Similarly, we find that

$$\begin{aligned} E_{1\mathrm{D}}(\psi) &= 2\int_0^{c-3k} \frac{\psi(c-2k-\psi)}{\sqrt{(c-k-\psi)(c-3k-\psi)}}d\psi \\ &= 2\int_0^{c-3k} \frac{(c-3k-z)(z+k)}{\sqrt{z(z+2k)}}dz, \end{aligned}$$

from which we obtain that

$$\begin{aligned} \frac{d}{dc}E_{1\mathrm{D}}(\psi) &= 2\int_0^{c-3k} \frac{z+k}{\sqrt{z(z+2k)}}dz \\ &= 2\sqrt{z(z+2k)}\Big|_{z=0}^{z=c-3k} \\ &= 2\sqrt{(c-3k)(c-k)} > 0. \end{aligned}$$

Thus, both mappings $c \mapsto M_{1\mathrm{D}}(\psi)$ and $c \mapsto E_{1\mathrm{D}}(\psi)$ are monotonically increasing for every $c \in (3k, \infty)$. □

Remark 3.3. The monotonicity of $c \mapsto E_{1\mathrm{D}}(\psi)$ plays a central role in the proof of the orbital stability of smooth solitary wave in the CH equation (1.1), see [10].

Remark 3.4. For later reference, we also compute $\|\psi\|^2_{L^2}$ by using the same idea as in the proof of Lemma 3.2:

$$\begin{aligned} \|\psi\|^2_{L^2} &= 2\int_0^{c-3k} \frac{\psi\sqrt{c-k-\psi}}{\sqrt{c-3k-\psi}}d\psi \\ &= 2\int_0^{c-3k} \frac{\sqrt{2k+z}(c-3k-z)}{\sqrt{z}}dz \\ &= 8k\int_0^{\xi_0} (c-3k-2k\xi^2)\sqrt{1+\xi^2}d\xi, \end{aligned}$$

from which we obtain

$$\begin{aligned} \|\psi\|^2_{L^2} &= 4k(c-3k)\left[\xi_0\sqrt{1+\xi_0^2} + \mathrm{arcsinh}\xi_0\right] \\ &\quad - 2k^2\left[2\xi_0\sqrt{(1+\xi_0^2)^3} - \xi_0\sqrt{1+\xi_0^2} - \mathrm{arcsinh}\xi_0\right] \\ &= 2k(2c-5k)\left[\xi_0\sqrt{1+\xi_0^2} + \mathrm{arcsinh}\xi_0\right] - 4k^2\xi_0\sqrt{(1+\xi_0^2)^3}. \end{aligned}$$

3.2. **The continuous spectrum of the spectral problem (2.9).** We start by analyzing properties of L. First, L is a self-adjoint Sturm-Liouville operator in $L^2(\mathbb{R})$ with dense domain in $H^2(\mathbb{R})$. The translational symmetry of the CH equation (1.1) implies that

$$L\phi' = 0, \quad \phi' \in \mathrm{Dom}(L) \subset L^2(\mathbb{R}). \tag{3.3}$$

Since ϕ' has only one zero on $\mathbb{R}$, Sturm–Liouville theory implies that the spectrum of L in $L^2(\mathbb{R})$ consists of one simple negative and a simple zero eigenvalue isolated from the strictly positive part of the spectrum. Furthermore, since $\phi \in C^\infty(\mathbb{R})$ is smooth in c, we find by differentiating the traveling wave equation (2.2) with respect to c that

$$L\partial_c\phi = k - \mu, \quad \partial_c\phi \in \mathrm{Dom}(L) \subset L^2(\mathbb{R}), \tag{3.4}$$

where $\mu := \phi - \phi''$. Based on these computations, the following two lemmas specify properties of the linearized operator JL in $L^2(\mathbb{R})$, included here for the sake of completeness, and in the exponentially weighted space L^2_ν for small $\nu > 0$.

Lemma 3.5. *For every $c > 3k$, the spectrum of JL in $L^2(\mathbb{R})$ covers $i\mathbb{R}$ with 0 being an embedded eigenvalue.*

Proof. It follows from (3.3) that $JL\phi' = 0$ with $\phi' \in \mathrm{Ker}(JL) \subset L^2(\mathbb{R})$ so that $0 \in \sigma(JL)$. Because $\phi(x) \to k$ as $|x| \to \infty$ exponentially fast, Weyl's theorem implies that the continuous spectrum of JL is given by the spectrum of JL_0 in $L^2(\mathbb{R})$, where L_0 is given by (2.11). By using the Fourier transform in $L^2(\mathbb{R})$, we obtain that

$$\sigma(JL_0) = \left\{ i\xi(1+\xi^2)^{-1}[c - 3k + (c-k)\xi^2], \quad \xi \in \mathbb{R} \right\} = i\mathbb{R} \quad \text{in} \;\; L^2(\mathbb{R}).$$

Since ϕ is spectrally stable in the time evolution of the CH equation (1.1) [10, 22], no other points of the spectrum of JL in $L^2(\mathbb{R})$ exists outside $i\mathbb{R}$. Thus, the spectrum of JL in $L^2(\mathbb{R})$ is $\sigma(JL) = i\mathbb{R}$ with 0 being an embedded eigenvalue. □

Lemma 3.6. *For every $c > 3k$, there exists $\nu_0 > 0$ such that the continuous spectrum of JL in L^2_ν with $\nu \in (0, \nu_0)$ is strictly negative and the (isolated) zero eigenvalue in L^2_ν is algebraically double.*

Proof. By Weyl's theorem, the continuous spectrum of JL in L^2_ν is given by the spectrum of JL_0 in L^2_ν. Using the Fourier transform we obtain that

$$\sigma(JL_0) = \left\{ (i\xi - \nu)[1 - (i\xi-\nu)^2]^{-1}[c - 3k - (c-k)(i\xi-\nu)^2], \quad \xi \in \mathbb{R} \right\} \quad \text{in} \;\; L^2_\nu.$$

We claim that if $0 < \nu < \nu_0$ with $\nu_0 = \sqrt{\frac{c-3k}{c-k}}$, then

$$\mathrm{Re}(\sigma(JL_0)) < 0 \quad \text{in} \;\; L^2_\nu,$$

where $\mathrm{Re}(\sigma(JL_0))$ coincides with the range of the function $\lambda_r(\xi) : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned} \lambda_r(\xi) &= \mathrm{Re}\left[(i\xi-\nu)[1-(i\xi-\nu)^2]^{-1}[c-3k-(c-k)(i\xi-\nu)^2] \right] \\ &= \mathrm{Re}\left[(c-k)(i\xi-\nu) - 2k(i\xi-\nu)[1-(i\xi-\nu)^2]^{-1} \right] \\ &= -\nu(c-k) - \frac{2k\nu(\nu^2+\xi^2-1)}{(1-\nu^2+\xi^2)^2 + 4\xi^2\nu^2} \end{aligned}$$

Expanding this quantity yields

$$\lambda_r(\xi) = -\frac{\nu}{(1-\nu^2+\xi^2)^2+4\xi^2\nu^2}\left[c-3k+2c\xi^2-2(c-2k)\nu^2+(c-k)(\xi^2+\nu^2)^2\right],$$

which is strictly negative if $\nu > 0$ and

$$(c-k)\nu^4 - 2(c-2k)\nu^2 + c - 3k > 0.$$

The latter constraint is true if $\nu < \nu_0 = \sqrt{\frac{c-3k}{c-k}}$. Note that $\nu_0 \in (0,1)$.

It remains to prove that $0 \in \sigma(JL)$ is a double eigenvalue in L^2_ν. Since $\phi'(x) \to 0$ as $|x| \to \infty$ exponentially fast, we have $\phi' \in L^2_\nu$ for sufficiently small $\nu > 0$. The Wronskian between two solutions $\{f_1, f_2\}$ of $Lf = 0$ is asymptotically constant at infinity and nonzero since

$$W(f_1,f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \frac{W_0}{c-\phi}, \quad x \in \mathbb{R},$$

where W_0 is a nonzero constant. If one solution $f_1 := \phi'$ decays exponentially at infinity, the other (linearly independent) solution f_2 grows exponentially at infinity. Hence

$$\ker L = \text{span}(\phi') \quad \text{in} \;\; L^2_\nu.$$

Furthermore, since ϕ is even, L is parity preserving. There exists an even solution f_0 to the inhomogeneous equation $Lf_0 = 1$ and since L converges to L_0 at infinity, f_0 is non-decaying at infinity. Since $JLf = 0$ implies $Lf = C$ for some constant $C \in \mathbb{R}$ and $f = Cf_0 \notin L^2_\nu$ is non-decaying if $C \neq 0$, it follows that

$$\ker(JL) = \ker(L) = \text{span}(\phi') \quad \text{in} \;\; L^2_\nu.$$

In order to study the algebraic multiplicity of the zero eigenvalue, we consider solutions of $JLf = \phi'$. Since it follows from (3.4) that $JL\partial_c\phi = -\phi'$ and $\partial_c\phi \in L^2_\nu$, we have

$$\ker((JL)^2) = \text{span}(\phi', \partial_c\phi) \quad \text{in} \;\; L^2_\nu.$$

The zero eigenvalue of JL is algebraically double if and only if there exists no $f \in L^2_\nu$ such that $JLf = \partial_c\phi$, or equivalently,

$$Lf = \partial_x^{-1}\partial_c\mu, \tag{3.5}$$

where $\partial_x^{-1}\partial_c\mu \in L^2_\nu$. If the eigenfunctions of L are defined in L^2_ν, then the adjoint eigenfunctions are defined in $L^2_{-\nu}$ due to the transformation $L \mapsto L_\nu := e^{\nu x} L e^{-\nu x}$ for eigenfunctions in the weighted space L^2_ν, see [4, 36]. As a result, the inner product in L^2_μ is equivalent to the inner product in L^2, i.e.

$$\forall f \in L^2_\nu,\;\; \forall g \in L^2_{-\nu}: \quad \langle f, g\rangle_{L^2_\nu} := \langle e^{\nu x} f, e^{-\nu x} g\rangle_{L^2} = \langle f, g\rangle_{L^2}. \tag{3.6}$$

In what follows, we drop the subscript L^2 for the inner product in L^2. To provide the existence of solutions $f \in L^2_\nu$ of the linear inhomogeneous equation (3.5), we check the Fredholm condition given by

$$\langle \phi', \partial_x^{-1}\partial_c\mu\rangle = -\langle (\phi - k), \partial_c\mu\rangle = -\frac{d}{dc}E_{1\text{D}}(\psi), \tag{3.7}$$

where $E_{1D}(\psi)$ is given by (3.1) and integration by parts gives no contribution at infinity since $\phi(x) \to k$ as $|x| \to \infty$ exponentially fast. By Lemma 3.2, the right-hand side is strictly negative so that no $f \in L^2_\nu$ exists such that $JLf = \partial_c \phi$. Hence, $0 \in \sigma(JL)$ is a double eigenvalue in L^2_ν. □

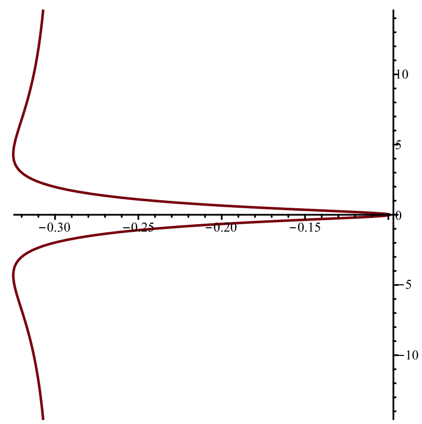

FIGURE 3.1. A plot of $\lambda(\xi)$, $\xi \in \mathbb{R}$ in the complex plane for $k = 1$, $c = 4$, $\eta = 0.01$, and $\nu = 0.1$

Based on Lemma 3.6, we can study properties of the spectral stability problem (2.9) with transverse wave number $\eta \in \mathbb{R}$, $\eta \neq 0$. The continuous spectrum of $J(L + \eta^2 \partial_x^{-2})$ in L^2_ν coincides with the purely continuous spectrum of $J(L_0 + \eta^2 \partial_x^{-2})$ in L^2_ν, which can be obtained by using the Fourier transform in x. The spectrum $\sigma(L_0 + \eta^2 \partial_x^{-2})$ in L^2_ν is defined by the range of the function $\lambda(\xi) : \mathbb{R} \to \mathbb{C}$ given by

$$\lambda(\xi) = (i\xi - \nu)[1 - (i\xi - \nu)^2]^{-1} \left[c - 3k - (c-k)(i\xi - \nu)^2 + \eta^2 (i\xi - \nu)^{-2} \right]. \qquad (3.8)$$

Figure 3.1 gives a plot of $\lambda(\xi)$ for specific values of k, c, η, and ν. The plot suggests that $\sigma(L_0 + \eta^2 \partial_x^{-2})$ in L^2_ν is located in the left half-plane bounded away from zero. The following lemma proves this property.

Lemma 3.7. *For every* $c > 3k$, $\eta \in \mathbb{R}$ *and* $\nu \in (0, \nu_0)$, *where* $\nu_0 := \sqrt{\frac{c-3k}{c-k}}$, *we have* $\mathrm{Re}(\lambda(\xi)) < 0$ *for all* $\xi \in \mathbb{R}$.

Proof. The expression (3.8) can be simplified in the form:

$$\lambda(\xi) = (c-k)(i\xi - \nu) - 2k(i\xi - \nu)[1 - (i\xi - \nu)^2]^{-1} + \eta^2 (i\xi - \nu)^{-1} [1 - (i\xi - \nu)^2]^{-1}.$$

Computing the real part and using $\lambda_r(\xi)$ from the proof of Lemma 3.6, we obtain

$$\mathrm{Re}(\lambda(\xi)) = \lambda_r(\xi) - \frac{\eta^2 \nu (1 - \nu^2 + 3\xi^2)}{(\xi^2 + \nu^2)[(1 - \nu^2 + \xi^2)^2 + 4\xi^2 \nu^2]}.$$

Since $\lambda_r(\xi) < 0$ for $\nu \in (0, \nu_0)$ with $\nu_0 := \sqrt{\frac{c-3k}{c-k}}$ and $\nu_0 \in (0,1)$, we have $\mathrm{Re}(\lambda(\xi)) < 0$ for all $\xi \in \mathbb{R}$. □

3.3. **Splitting of the double zero eigenvalue in L^2_ν for $\eta \neq 0$.** By Lemma 3.6, 0 is a double (isolated) eigenvalue of JL in L^2_ν for small $\nu > 0$. When $\eta \neq 0$ in (2.9), the translational symmetry is broken and the double zero eigenvalue may split into two complex eigenvalues of $J(L + \eta^2 \partial_x^{-2})$. Since it is isolated away from the continuous spectrum of $J(L + \eta^2 \partial_x^{-2})$ for every $\eta \in \mathbb{R}$ and small $\nu > 0$ by Lemma 3.7, the splitting can be studied by using perturbative methods in powers of η.

The following lemma states that when $\eta \neq 0$ the double zero eigenvalue of JL in L^2_ν for small $\nu > 0$ splits into a pair of eigenvalues of $J(L + \eta^2 \partial_x^{-2})$ located in the left half of the complex plane. The result holds for solitary waves of arbitrary amplitude and is derived by means of Puiseux expansions in η. Together with Lemma 3.7, this proves the result of Theorem 2.6.

Lemma 3.8. *There exists $\nu_0 > 0$ such that for every fixed $\nu \in (0, \nu_0)$ there exists $\eta_0 > 0$ such that the spectrum of $J(L + \eta^2 \partial_x^{-2})$ in L^2_ν for $\eta \in (-\eta_0, \eta_0)$ contains a pair of simple eigenvalues $\lambda_\pm(\eta)$ such that for $\eta \neq 0$ we have*

- $\mathrm{Re}(\lambda_+(\eta)) = \mathrm{Re}(\lambda_-(\eta)) < 0$,
- $\mathrm{Im}(\lambda_+(\eta)) = -\mathrm{Im}(\lambda_-(\eta)) > 0$,

and $\lambda_+(0) = \lambda_-(0) = 0$.

Proof. By Lemma 3.7, there exists $\nu_0 > 0$ such that for every fixed $\nu \in (0, \nu_0)$, the double zero eigenvalue of JL in L^2_ν is isolated from its continuous spectrum of $J(L + \eta^2 \partial_x^{-2})$ in L^2_ν. Since $\eta^2 \partial_x^{-2}$ is a bounded analytic perturbation to the unbounded operator L in L^2_ν for $\nu > 0$, the eigenvalues of $J(L + \eta^2 \partial_x^{-2})$ in L^2_ν are continuous functions of η.

By Lemma 3.6, the zero eigenvalue of JL in L^2_ν is geometrically simple and algebraically double. Hence we use Puiseux expansions [37] in order to trace the eigenvalues $\lambda_\pm(\eta)$ satisfying $\lambda_\pm(\eta) \to 0$ as $\eta \to 0$ with respect to small but nonzero η. Solutions of the spectral stability problem (2.9) with $\lambda = \lambda(\eta)$ are expanded as

$$\begin{aligned}\hat{v} &= v_0 + v_1 \eta + v_2 \eta^2 + v_3 \eta^3 + \mathcal{O}(\eta^4),\\ \lambda(\eta) &= \lambda_1 \eta + \lambda_2 \eta^2 + \lambda_3 \eta^3 + \mathcal{O}(\eta^4).\end{aligned}$$

where $v_0, v_1, v_2, v_3 \in L^2_\nu$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ are to be determined. We obtain at different orders in powers of η that

$$\begin{aligned}\mathcal{O}(1):&\quad JLv_0 = 0,\\ \mathcal{O}(\eta):&\quad JLv_1 = \lambda_1 v_0,\\ \mathcal{O}(\eta^2):&\quad JLv_2 = \lambda_2 v_0 + \lambda_1 v_1 - (1-\partial_x^2)^{-1}\partial_x^{-1} v_0\\ \mathcal{O}(\eta^3):&\quad JLv_3 = \lambda_3 v_0 + \lambda_2 v_1 + \lambda_1 v_2 - (1-\partial_x^2)^{-1}\partial_x^{-1} v_1.\end{aligned}$$

With arbitrary normalization, we can set $v_0 = \phi'$ and $v_1 = -\lambda_1 \partial_c \phi$ due to computations in the proof of Lemma 3.6. Then, at the order of $\mathcal{O}(\eta^2)$, we write $v_2 = -\lambda_2 \partial_c \phi + \hat{v}_2$, where $\hat{v}_2$ satisfies

$$JL\hat{v}_2 = -\lambda_1^2 \partial_c \phi - (1-\partial_x^2)^{-1}(\phi - k).$$

After inverting J in L^2_ν with $\nu > 0$ we rewrite this linear inhomogeneous equation in the equivalent form

$$L\hat{v}_2 = -\lambda_1^2 \partial_x^{-1} \partial_c \mu - \partial_x^{-1}(\phi - k).$$

By using (3.6) we check the Fredholm condition for the existence of solutions $\hat{v}_2 \in L^2_\nu$:

$$\lambda_1^2 \langle \phi', \partial_x^{-1} \partial_c \mu \rangle + \langle \phi', \partial_x^{-1}(\phi - k) \rangle = 0.$$

Note that $\partial_x^{-1}(\phi - k) = \int_{+\infty}^x (\phi - k)$, so the second term gives after integration by parts

$$\begin{aligned}\langle \phi', \partial_x^{-1}(\phi - k) \rangle &= (\phi - k) \int_{+\infty}^x (\phi - k) dx' \Big|_{x \to -\infty}^{x \to +\infty} - \int_{-\infty}^{\infty} (\phi - k)^2 dx \\ &= -\|\phi - k\|^2_{L^2} = -\|\psi\|^2_{L^2}.\end{aligned}$$

On the other hand, the first term is evaluated with the help of (3.7). Since $\frac{d}{dc} E_{1\mathrm{D}}(\psi) > 0$ by Lemma 3.2, we obtain that

$$\lambda_1^2 = -\frac{\langle \phi', \partial_x^{-1}(\phi - k) \rangle}{\langle \phi', \partial_x^{-1} \partial_c \mu \rangle} = -\frac{\|\psi\|^2_{L^2}}{\frac{d}{dc} E_{1\mathrm{D}}(\psi)} < 0. \tag{3.9}$$

Thus, we have two roots for $\lambda_1 \in i\mathbb{R}$, which determine two simple eigenvalues $\lambda = \lambda_\pm(\eta)$. At the leading order, we have $\mathrm{Im}(\lambda_+(\eta)) = -\mathrm{Im}(\lambda_-(\eta)) > 0$ and the complex-conjugate symmetry of eigenvalues is preserved since J and L are real-valued.

At the next order $\mathcal{O}(\eta^3)$ we write $v_3 = -\lambda_3 \partial_c \phi + \hat{\phi}_3$, where $\hat{v}_3$ satisfies

$$JL\hat{v}_3 = \lambda_1 \left[\hat{v}_2 + (1 - \partial_x^2)^{-1} \partial_x^{-1} \partial_c \phi - 2\lambda_2 \partial_c \phi \right],$$

which, after inverting J in L^2_ν with $\nu > 0$, gives

$$L\hat{v}_3 = \lambda_1 \left[(1 - \partial_x^2) \partial_x^{-1} \hat{v}_2 + \partial_x^{-2} \partial_c \phi - 2\lambda_2 \partial_x^{-1} \partial_c \mu \right].$$

By using (3.6) we check the Fredholm condition for the existence of solutions $\hat{v}_3 \in L^2_\nu$:

$$\begin{aligned} 2\lambda_2 &= \frac{\langle \phi', \partial_x^{-1} \left[(1 - \partial_x^2) \hat{v}_2 + \partial_x^{-1} \partial_c \phi \right] \rangle}{\langle \phi', \partial_x^{-1} \partial_c \mu \rangle} \\ &= \frac{\langle \phi - k, (1 - \partial_x^2) \hat{v}_2 \rangle + \langle \phi - k, \partial_x^{-1} \partial_c \phi \rangle}{\frac{d}{dc} E_{1\mathrm{D}}(\psi)}. \end{aligned}$$

For the first term in the numerator, we use (3.4) and obtain

$$\begin{aligned} \langle \phi - k, (1 - \partial_x^2) \hat{v}_2 \rangle &= \langle \mu - k, \hat{v}_2 \rangle = -\langle L \partial_c \phi, \hat{v}_2 \rangle = -\langle \partial_c \phi, L \hat{v}_2 \rangle \\ &= \lambda_1^2 \langle \partial_c \phi, \partial_x^{-1} \partial_c \mu \rangle + \langle \partial_c \phi, \partial_x^{-1}(\phi - k) \rangle. \end{aligned}$$

We use the even parity of ϕ for which $\int_{+\infty}^{x}(\phi-k)dx' = -\frac{1}{2}\int_{-\infty}^{\infty}(\phi-k)dx' + \int_0^x(\phi-k)dx'$, where the second term is odd, and obtain

$$\begin{aligned}
\langle \phi - k, \partial_x^{-1}\partial_c\phi\rangle &= -\frac{1}{2}\int_{-\infty}^{\infty}(\phi-k)dx\left(\int_{-\infty}^{\infty}\partial_c\phi dx\right) = -\frac{1}{2}M_{1\mathrm{D}}(\psi)\frac{d}{dc}M_{1\mathrm{D}}(\psi),\\
\langle \partial_c\phi, \partial_x^{-1}(\phi-k)\rangle &= -\frac{1}{2}\left(\int_{-\infty}^{\infty}\partial_c\phi dx\right)\int_{-\infty}^{\infty}(\phi-k)dx = -\frac{1}{2}M_{1\mathrm{D}}(\psi)\frac{d}{dc}M_{1\mathrm{D}}(\psi),\\
\langle \partial_c\phi, \partial_x^{-1}\partial_c\mu\rangle &= -\frac{1}{2}\left(\int_{-\infty}^{\infty}\partial_c\phi dx\right)\int_{-\infty}^{\infty}\partial_c\mu dx = -\frac{1}{2}\left(\frac{d}{dc}M_{1\mathrm{D}}(\psi)\right)^2,
\end{aligned}$$

which then yields

$$\begin{aligned}
2\lambda_2 &= \frac{\frac{d}{dc}M_{1\mathrm{D}}(\psi)}{\frac{d}{dc}E_{1\mathrm{D}}(\psi)}\left[\frac{\|\psi\|_{L^2}^2}{2\frac{d}{dc}E_{1\mathrm{D}}(\psi)}\frac{d}{dc}M_{1\mathrm{D}}(\psi) - M_{1\mathrm{D}}(\psi)\right]\\
&= \frac{\frac{d}{dc}M_{1\mathrm{D}}(\psi)}{2\left(\frac{d}{dc}E_{1\mathrm{D}}(\psi)\right)^2}\left[\|\psi\|_{L^2}^2\frac{d}{dc}M_{1\mathrm{D}}(\psi) - 2M_{1\mathrm{D}}(\psi)\frac{d}{dc}E_{1\mathrm{D}}(\psi)\right], \qquad (3.10)
\end{aligned}$$

where we have used (3.9) for λ_1^2.

In order to identify the sign of λ_2, we recall from Lemma 3.2 that the mappings $c \mapsto M_{1\mathrm{D}}(\psi)$ and $c \mapsto E_{1\mathrm{D}}(\psi)$ are monotonically increasing. Hence, the sign of λ_2 is equivalent to the sign of

$$\begin{aligned}
&\|\psi\|_{L^2}^2\frac{d}{dc}M_{1\mathrm{D}}(\psi) - 2M_{1\mathrm{D}}(\psi)\frac{d}{dc}E_{1\mathrm{D}}(\psi)\\
&= \frac{4k\sqrt{c-k}}{\sqrt{c-3k}}\left[(7k-2c)(\xi_0\sqrt{1+\xi_0^2} + \mathrm{arcsinh}\xi_0) - 2k\xi_0\sqrt{(1+\xi_0^2)^3}\right], \quad \xi_0 := \frac{\sqrt{c-3k}}{\sqrt{2k}},
\end{aligned}$$

where we have substituted explicit expressions from Lemma 3.2 and Remark 3.4. Since $c > 3k$, we obtain

$$\begin{aligned}
&\|\psi\|_{L^2}^2\frac{d}{dc}M_{1\mathrm{D}}(\psi) - 2M_{1\mathrm{D}}(\psi)\frac{d}{dc}E_{1\mathrm{D}}(\psi)\\
&\quad \leq \frac{4k^2\sqrt{c-k}}{\sqrt{c-3k}}\left[\xi_0\sqrt{1+\xi_0^2} + \mathrm{arcsinh}\xi_0 - 2\xi_0\sqrt{(1+\xi_0^2)^3}\right],\\
&\quad = -\frac{4k^2\sqrt{c-k}}{\sqrt{c-3k}}\xi_0\sqrt{1+\xi_0^2}\left[1 + 2\xi_0^2 - \frac{\log(\xi_0+\sqrt{1+\xi_0^2})}{\xi_0\sqrt{1+\xi_0}^2}\right],
\end{aligned}$$

where we have used $\mathrm{arcsinh}\xi_0 = \log(\xi_0 + \sqrt{1+\xi_0^2})$. Since $\log(\xi_0+\sqrt{1+\xi_0^2}) < \xi_0\sqrt{1+\xi_0^2}$ for every $\xi_0 > 0$, the expression in the bracket is positive so that $\lambda_2 < 0$. This yields $\mathrm{Re}(\lambda_+(\eta)) = \mathrm{Re}(\lambda_-(\eta)) < 0$ at the leading order and hence for sufficiently small $\eta \neq 0$. $\square$

Remark 3.9. In the KdV limit (2.6) as $c \to 3k$, we can simplify the expressions (3.9) and (3.10) for λ_1 and λ_2 to obtain

$$\lambda_1^2 = -\frac{\sqrt{2k}}{2\sqrt{c-3k}}\left[4(c-3k)\xi_0 - \frac{8}{3}k\xi_0^3 + \mathcal{O}(\xi_0^5)\right] \sim -\frac{4}{3}(c-3k)$$

and

$$2\lambda_2 = \frac{k}{(c-3k)^2}\left[4(3k-c)\xi_0 - \frac{8}{3}k\xi_0^3 + \mathcal{O}(\xi_0^5)\right] \sim -\frac{8\sqrt{2k}}{3\sqrt{c-3k}},$$

where we have used the explicit expressions in the proof of Lemma 3.2 and the asymptotic limit $\xi_0 \to 0$. Extracting the postive square root for λ_1 yields the expansion for $\lambda_\pm(\eta)$ in the form

$$\lambda_\pm(\eta) = \pm\frac{2i}{\sqrt{3}}\sqrt{c-3k}\eta - \frac{4}{3}\frac{\sqrt{2k}}{\sqrt{c-3k}}\eta^2 + \mathcal{O}(\eta^3).$$

Using the KP-II scaling (1.8) and (2.6) with $\eta = \varepsilon^2\Upsilon$ and $c - 3k = \varepsilon^2$, we obtain

$$\varepsilon^{-3}\lambda_\pm(\varepsilon^2\Upsilon) = \pm\frac{2i}{\sqrt{3}}\Upsilon - \frac{4}{3}\sqrt{2k}\Upsilon^2 + \mathcal{O}(\Upsilon^3),$$

which is the asymptotic expansion of the exact expression of the pair of eigenvalues $\Lambda_\pm(\Upsilon)$ of the corresponding linearized operator for the KP-II equation (1.9),

$$\Lambda_\pm(\Upsilon) = \pm\frac{2i}{\sqrt{3}}\Upsilon\sqrt{1 \pm \frac{4i}{\sqrt{3}}\sqrt{2k}\Upsilon}, \tag{3.11}$$

see [29].

Remark 3.10. The continuous spectrum of $J(L + \eta^2\partial_x^{-2})$ in L^2_ν deforms to $i\mathbb{R}$ as $\nu \to 0$, which can be seen by taking the limit $\nu \to 0$ in equation (3.8). On the other hand, the location of the simple eigenvalues $\lambda_\pm(\eta)$ is independent of ν for $\eta \in (-\eta_0, \eta_0)$ and $\nu \in (0, \nu_0)$ as follows from (3.9) and (3.10). As a result, the continuous spectrum crosses the location of the simple eigenvalues for some $\nu_1 \in (0, \nu_0)$ that depends on $\eta \neq 0$. Consequently, as is shown in [36], the simple eigenvalues of $J(L + \eta^2\partial_x^{-2})$ in L^2_ν for $\nu \in (\nu_1, \nu_0)$ are no longer eigenvalues of $J(L + \eta^2\partial_x^{-2})$ in L^2_ν for $\nu \in (0, \nu_1)$ and in $L^2(\mathbb{R})$, because they are associated with the eigenfunctions growing exponentially as $x \to -\infty$. Such points are referred to as *resonances* of the linear operator $J(L + \eta^2\partial_x^{-2})$, see [36].

4. Proof of Theorem 2.8

4.1. **Preliminary results.** We consider the spectral stability problem in the form (2.9). Writing $\phi = k + \psi$ and $c = 3k + \gamma$, we can rewrite the spectral problem (2.9) in the equivalent form

$$\partial_x(1-\partial_x^2)^{-1}\left(\gamma - 3\psi + \psi'' - \partial_x(\gamma - \psi)\partial_x - 2k\partial_x^2 + \eta^2\partial_x^{-2}\right)\hat{v} = \lambda\hat{v}. \tag{4.1}$$

In order to analyze the spectral problem (4.1) in the limit of small-amplitude solitary waves, we give a rigorous proof of the approximation result in Remark 2.3 and justify the asymptotic approximation (2.6). The following lemma presents this asymptotic result.

Lemma 4.1. *There exists $\varepsilon_0 > 0$ and $C_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ the solitary wave solution of Lemma 2.1 satisfying $\psi(0) = c - 3k$ and $\psi'(0) = 0$ can be written in the form*

$$\psi(x) = \varepsilon^2 \Psi_{\rm KdV}(X) + \varepsilon^4 \tilde{\Psi}(X), \quad X = \varepsilon x, \quad c = 3k + \varepsilon^2, \tag{4.2}$$

where

$$\Psi_{\rm KdV}(X) := \mathrm{sech}^2\left(\frac{X}{2\sqrt{2k}}\right) \quad \text{and} \quad \|\tilde{\Psi}\|_{L^\infty} \leq C_0.$$

Proof. Substituting $\psi(x) = \varepsilon^2 \Psi(X)$, $X = \varepsilon x$, and $c = 3k + \varepsilon^2$ into (2.1) yields the first-order invariant

$$(\Psi')^2 = \Psi^2 \frac{1 - \Psi}{2k + \varepsilon^2(1 - \Psi)},$$

for some $\Psi \in H^2(\mathbb{R})$. The function $\Psi_{\rm KdV}$ is a solution of the above equation in the limit $\varepsilon \to 0$. To prove (4.2) we differentiate the first-order invariant and obtain the second-order equation in the form $F(\Psi, \varepsilon^2) = 0$, where $F(\Psi, \varepsilon^2) : H^2(\mathbb{R}) \times \mathbb{R} \to L^2(\mathbb{R})$ is the operator function given by

$$F(\Psi, \varepsilon^2) := -\Psi'' + \Psi \frac{k(2 - 3\Psi) + \varepsilon^2(1 - \Psi)^2}{(2k + \varepsilon^2(1 - \Psi))^2}.$$

It is clear that F is a C^1 function near $(\Psi_{\rm KdV}, 0)$ satisfying

$$F(\Psi_{\rm KdV}, 0) = -\Psi_{\rm KdV}'' + \frac{1}{4k}\Psi_{\rm KdV}(2 - 3\Psi_{\rm KdV}) = 0$$

and

$$D_\Psi F(\Psi_{\rm KdV}, 0) = -\partial_x^2 + \frac{1}{2k}(1 - 3\Psi_{\rm KdV}).$$

Since 0 is a simple eigenvalue of $D_\Psi F(\Psi_{\rm KdV}, 0)$ with odd eigenfunction $\Psi_{\rm KdV}'$, and the rest of its spectrum is bounded away from 0, the operator $D_\Psi F(\Psi_{\rm KdV}, 0)$ is invertible in the subspace of even functions in $H^2(\mathbb{R})$. By the implicit function theorem, there exists a unique C^1 mapping $\varepsilon^2 \mapsto \Psi(\cdot, \varepsilon^2) \in H^2(\mathbb{R})$ which yields the unique even solution of $F(\Psi(\cdot, \varepsilon^2), \varepsilon^2) = 0$ for small ε^2 such that $\Psi(\cdot, \varepsilon^2) \to \Psi_{\rm KdV}$ as $\varepsilon^2 \to 0$. The decomposition (4.2) follows from the C^1 property of this mapping and the continuous embedding of $H^2(\mathbb{R})$ into $L^\infty(\mathbb{R})$. $\square$

The KP-II scaling (1.8) and (2.6) corresponds to

$$\lambda = \varepsilon^3 \Lambda, \quad \eta = \varepsilon^2 \Upsilon, \quad \gamma = \varepsilon^2, \quad x = \varepsilon^{-1} X, \quad \hat{v}(x) = \hat{V}(X). \tag{4.3}$$

By Lemma 4.1, we can also write

$$\psi(x) = \varepsilon^2 \Psi(X), \quad \Psi := \Psi_{\rm KdV} + \varepsilon^2 \tilde{\Psi}, \quad c = 3k + \varepsilon^2. \tag{4.4}$$

The spectral problem (4.1) can then be rewritten as

$$\partial_X (1 - \varepsilon^2 \partial_X^2)^{-1} \left(L_{\rm KdV} + \varepsilon^2 L_{\rm pert} + \Upsilon^2 \partial_X^{-2}\right) \hat{V} = \Lambda \hat{V}, \tag{4.5}$$

where

$$L_{\rm KdV} := 1 - 3\Psi_{\rm KdV} - 2k\partial_X^2, \quad L_{\rm pert} := \Psi'' - \partial_X(1 - \Psi)\partial_X - 3\tilde{\Psi}.$$

Since

$$\nu_0 = \frac{\sqrt{c-3k}}{\sqrt{c-k}} = \frac{\varepsilon}{\sqrt{2k+\varepsilon^2}}$$

in Lemma 3.7, we need to rescale the exponential weight ν as $\nu = \varepsilon\rho$ and replace the weighted space (2.10) by

$$L^2_\rho := \{F(X) : \mathbb{R} \to \mathbb{R} : \quad e^{\rho \cdot} F \in L^2(\mathbb{R})\}.$$

The parameter ρ is fixed in $(0, \rho_0)$, where $\rho_0 := 1/\sqrt{2k}$. In order to prove Theorem 2.8, we consider the resolvent equations obtained from the spectral stability problem (4.1) in the original variables and (4.5) in the scaled variables. The two resolvent equations are used in two different regions:

- *the high-frequency region* with $|\eta| \geq K_0^2 \varepsilon^2$ for sufficiently large $K_0 > 0$;
- *the low-frequency region* with $|\eta| \leq K^2 \varepsilon^2$ for every fixed $K > 0$.

Combining the two regions covers the entire range of η values since K can be taken to be greater than K_0. Estimates in Lemma 4.6 and Lemma 4.8 below prove the result of Theorem 2.8.

4.2. **The high-frequency region.** We start with the following result, which is a generalization of [31, Lemma 3.1] obtained for the linearized KP-II equation and extended here for the spectral problem (4.5).

Proposition 4.2. *For every $\rho \in (0, \rho_0)$ there exist $\varepsilon_0 > 0$ and $\beta_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, $\Upsilon \in \mathbb{R}$, and every $\Lambda \in \mathbb{C}$ satisfying $\mathrm{Re}(\Lambda) > -\beta_0$, we have*

$$\| \left(\Lambda - \partial_X (1 - \varepsilon^2 \partial_X^2)^{-1} (1 - (2k + \varepsilon^2)\partial_X^2 + \Upsilon^2 \partial_X^{-2})\right)^{-1} \|_{L^2_\rho \to L^2_\rho} \leq (\mathrm{Re}(\Lambda) + \beta_0)^{-1}. \quad (4.6)$$

Moreover, there exists $C > 0$ such that

$$\begin{aligned} &\|\partial_X (1 - \varepsilon^2 \partial_X^2)^{-1} \left(\Lambda - \partial_X (1 - \varepsilon^2 \partial_X^2)^{-1} (1 - (2k + \varepsilon^2)\partial_X^2 + \Upsilon^2 \partial_X^{-2})\right)^{-1} \|_{L^2_\rho \to L^2_\rho} \\ &\quad \leq C \left(\mathrm{Re}(\Lambda) + \beta_0\right)^{-1/2}. \end{aligned} \quad (4.7)$$

if $\mathrm{Re}(\Lambda) > -\frac{1}{2}\beta_0$.

Proof. Since the operators in the estimates (4.6) and (4.7) have constant coefficients, we can use the Fourier transform in X and introduce the spectral function

$$\Lambda(\Xi) := (i\Xi - \rho)[1 - \varepsilon^2 (i\Xi - \rho)^2]^{-1} [1 - (2k + \varepsilon^2)(i\Xi - \rho)^2 + \Upsilon^2 (i\Xi - \rho)^{-2}],$$

for $\Upsilon \in \mathbb{R}$. The function $\Lambda(\Upsilon)$ is a scaled version of the function $\lambda(\xi)$ in (3.8). We deduce the explicit expression as in the proof of Lemmas 3.6 and 3.7:

$$\begin{aligned} \mathrm{Re}\,(\Lambda(\Xi)) = -\rho \Big[1 &+ \frac{2k(3\Xi^2 - \rho^2 + \varepsilon^2 (\Xi^2 - \rho^2)^2)}{1 + 2\varepsilon^2(\Xi^2 - \rho^2) + \varepsilon^4 (\Xi^2 + \rho^2)^2} \\ &+ \frac{\Upsilon^2 (1 + 3\varepsilon^2 \Xi^2 - \varepsilon^2 \rho^2)}{(\Xi^2 + \rho^2)[1 + 2\varepsilon^2 (\Xi^2 - \rho^2) + \varepsilon^4 (\Xi^2 + \rho^2)^2]} \Big]. \end{aligned} \quad (4.8)$$

Since

$$1 - 2\varepsilon^2 \rho^2 \leq 1 + 2\varepsilon^2 (\Xi^2 - \rho^2) + \varepsilon^4 (\Xi^2 + \rho^2)^2 \leq [1 + \varepsilon^2 (\Xi^2 + \rho^2)]^2,$$

we have

$$\begin{aligned} -\mathrm{Re}(\Lambda(\Xi)) &\geq \rho\left[1-2k\rho^2+\frac{2k(-\varepsilon^2\rho^4+3\Xi^2+\varepsilon^2\Xi^4+\varepsilon^4\rho^2(\Xi^2+\rho^2)^2)}{1+2\varepsilon^2(\Xi^2-\rho^2)+\varepsilon^4(\Xi^2+\rho^2)^2}\right] \\ &\geq \rho\left[1-2k\rho^2-\frac{2k\varepsilon^2\rho^4}{1-2\varepsilon^2\rho^2}+\frac{2k[\Xi^2(3+\varepsilon^2\Xi^2)+\varepsilon^4\rho^2(\Xi^2+\rho^2)^2]}{[1+\varepsilon^2(\Xi^2+\rho^2)]^2}\right] \end{aligned} \tag{4.9}$$

uniformly for all $\Upsilon \in \mathbb{R}$. Therefore, there exists $\rho_0 = 1/\sqrt{2k}$ such that for every $\rho \in (0, \rho_0)$ there exists $\varepsilon_0 > 0$ and $\beta_0 > 0$ such that $-\mathrm{Re}\Lambda(\Xi) \geq \beta_0$ for every $\varepsilon \in (0, \varepsilon_0)$ uniformly for all $\Xi \in \mathbb{R}$. For instance, we can choose

$$\beta_0 := \rho\left[1-2k\rho^2-\frac{2k\varepsilon_0^2\rho^4}{1-2\varepsilon_0^2\rho^2}\right] > 0$$

for a suitable choice of $\varepsilon_0 > 0$. Hence, for every $\Lambda \in \mathbb{C}$ satisfying $\mathrm{Re}(\Lambda) > -\beta_0$, we have

$$|\Lambda - \Lambda(\Xi)| \geq (\mathrm{Re}(\Lambda) + \beta_0)$$

and the bound (4.6) holds from standard Fourier estimates.

For the bound (4.7), we obtain from (4.9) that there exists $\gamma_0 > 0$ such that

$$-\mathrm{Re}(\Lambda(\Xi)) \geq \beta_0 + \frac{\gamma_0 \Xi^2}{1+\varepsilon^2(\Xi^2+\rho^2)}.$$

For instance, we can choose $\gamma_0 := 2k\rho$ since $\varepsilon_0\rho_0 < 1$. Hence for every $\Lambda \in \mathbb{C}$ satisfying $\mathrm{Re}(\Lambda) > -\frac{1}{2}\beta_0$, we have

$$|\Lambda - \Lambda(\Xi)| \geq \frac{1}{2}\beta_0 + \frac{\gamma_0 \Xi^2}{1+\varepsilon^2(\Xi^2+\rho^2)}. \tag{4.10}$$

Since there exists $C_0 \in (0,1)$ such that

$$1+2\varepsilon^2(\Xi^2-\rho^2)+\varepsilon^4(\Xi^2+\rho^2)^2 \geq C_0[1+\varepsilon^2(\Xi^2+\rho^2)]^2,$$

we obtain

$$\begin{aligned} \frac{|i\Xi-\rho|}{|1-\varepsilon^2(i\Xi-\rho)^2||\Lambda-\Lambda(\Xi)|} &\leq \frac{C\sqrt{\Xi^2+\rho^2}}{|1+\varepsilon^2(\Xi^2+\rho^2)||\Lambda-\Lambda(\Xi)|} \\ &\leq \frac{C}{\sqrt{1+\varepsilon^2(\Xi^2+\rho^2)}\sqrt{|\Lambda-\Lambda(\Xi)|}} \\ &\leq C(\mathrm{Re}(\Lambda)+\beta_0)^{-1/2}, \end{aligned} \tag{4.11}$$

for some generic constants $C > 0$ uniformly in $\Xi \in \mathbb{R}$. The bound (4.7) follows again from Fourier theory. □

In order to complete the estimates in the high-frequency region, we obtain a modified version of Proposition 4.2.

Proposition 4.3. *Let $\varepsilon_0 > 0$ and $\beta_0 > 0$ be the same as in Proposition 4.2. There are $K_0 > 0$ and $C_0 > 0$ such that for every $\Lambda \in \mathbb{C}$ satisfying $\mathrm{Re}(\Lambda) > -\frac{1}{2}\beta_0$ and every $\Upsilon \in \mathbb{R}$ satisfying $|\Upsilon| \geq K_0^2$, we have*

$$\begin{aligned}
&\|\partial_X(1-\varepsilon^2\partial_X^2)^{-1}\left(\Lambda - \partial_X(1-\varepsilon^2\partial_X^2)^{-1}(1-(2k+\varepsilon^2)\partial_X^2 + \Upsilon^2\partial_X^{-2})\right)^{-1}\|_{L^2_\rho \to L^2_\rho} \\
&\qquad \leq CK_0^{-1}\left(\mathrm{Re}(\Lambda)+\beta_0\right)^{-1/2}. \qquad (4.12)
\end{aligned}$$

Proof. This follows from the bounds on $\Lambda(\Xi)$ obtained in the proof of Proposition 4.2. If $|\Xi| \geq K_0$ and $K_0 > 0$ is sufficiently large, then it follows from (4.10) that for every $\Upsilon \in \mathbb{R}$, we have

$$|\Lambda - \Lambda(\Xi)| \geq \frac{\gamma_0 K_0^2}{1+\varepsilon^2(\Xi^2+\rho^2)}.$$

On the other hand, if $|\Xi + i\rho| \leq K_0$ and $|\Upsilon| \geq K_0^2 \geq K_0|\Xi + i\rho|$, then it follows from (4.8) that

$$|\Lambda - \Lambda(\Xi)| \geq \frac{\rho\Upsilon^2(1+3\varepsilon^2\Xi^2 - \varepsilon^2\rho^2)}{(\Xi^2+\rho^2)[1+\varepsilon^2(\Xi^2-\rho^2)]^2} \geq \frac{\rho K_0^2}{1+\varepsilon^2(\Xi^2+\rho^2)}.$$

Then, similarly to (4.11), we obtain

$$\frac{|i\Xi - \rho|}{|1-\varepsilon^2(i\Xi-\rho)^2||\Lambda - \Lambda(\Xi)|} \leq \frac{C}{\sqrt{1+\varepsilon^2(\Xi^2+\rho^2)}\sqrt{|\Lambda-\Lambda(\Xi)|}} \leq CK_0^{-1}(\mathrm{Re}(\Lambda)+\beta_0)^{-1/2},$$

for some generic constant $C > 0$ uniformly in $\Xi \in \mathbb{R}$. This justifies the bound (4.16). □

The resolvent equation in the original variables is obtained from the spectral problem (4.1) with $\gamma = \varepsilon^2$ in the form:

$$(\lambda - A_0 - A_1 - A_2)\, u = f, \qquad f \in L^2_\nu, \tag{4.13}$$

where

$$\begin{aligned}
A_0 &:= \partial_x(1-\partial_x^2)^{-1}(\varepsilon^2 - (2k+\varepsilon^2)\partial_x^2 + \eta^2\partial_x^{-2}),\\
A_1 &:= \partial_x(1-\partial_x^2)^{-1}\partial_x\psi\partial_x,\\
A_2 &:= \partial_x(1-\partial_x^2)^{-1}(-3\psi + \psi'').
\end{aligned}$$

Using this notation we obtain the following corollary of Proposition 4.3 which gives the bounds in original variables.

Corollary 4.4. *For every $\lambda \in \mathbb{C}$ satisfying $\mathrm{Re}(\lambda) > -\frac{1}{2}\beta_0\varepsilon^3$ with some $\beta_0 > 0$ and every $\eta \in \mathbb{R}$ satisfying $|\eta| \geq K_0^2\varepsilon^2$ with sufficiently large $K_0 > 0$ we find that*

$$\|(\lambda - A_0)^{-1}\|_{L^2_{\varepsilon\rho}\to L^2_{\varepsilon\rho}} \leq C\varepsilon^{-3}, \tag{4.14}$$

$$\|\partial_x(1-\partial_x^2)^{-1}(\lambda - A_0)^{-1}\|_{L^2_{\varepsilon\rho}\to L^2_{\varepsilon\rho}} \leq C\varepsilon^{-2}, \tag{4.15}$$

and

$$\|\partial_x(1-\partial_x^2)^{-1}(\lambda - A_0)^{-1}\|_{L^2_{\varepsilon\rho}\to L^2_{\varepsilon\rho}} \leq CK_0^{-1}\varepsilon^{-2}. \tag{4.16}$$

Remark 4.5. Since the continuous spectrum of $\varepsilon^{-3}A_0$ in L^2_ρ is bounded away from $i\mathbb{R}$ by the ε-independent constant β_0, and $\varepsilon^{-3}A_1$ is a relatively bounded perturbation to $\varepsilon^{-3}A_0$ of order $\mathcal{O}(\varepsilon^2)$ due to the scaling (4.4), the estimates (4.14), (4.15), and (4.16) apply also for $(\lambda - A_0 - A_1)^{-1}$ instead of $(\lambda - A_0)^{-1}$. Hence, we will use

$$\|(\lambda - A_0 - A_1)^{-1}\|_{L^2_{\varepsilon\rho}\to L^2_{\varepsilon\rho}} \le C\varepsilon^{-3}, \tag{4.17}$$

$$\|\partial_x(1-\partial_x^2)^{-1}(\lambda - A_0 - A_1)^{-1}\|_{L^2_{\varepsilon\rho}\to L^2_{\varepsilon\rho}} \le C\varepsilon^{-2}, \tag{4.18}$$

and

$$\|\partial_x(1-\partial_x^2)^{-1}(\lambda - A_0 - A_1)^{-1}\|_{L^2_{\varepsilon\rho}\to L^2_{\varepsilon\rho}} \le CK_0^{-1}\varepsilon^{-2}. \tag{4.19}$$

instead of (4.14), (4.15), and (4.16).

The following lemma uses the fact that the operator A_2 in (4.13) is small compared to the operator $A_0 + A_1$ in $L^2_{\varepsilon\rho}$ due to the KP-II scaling (4.3) and (4.4), see the estimate (4.22) below. As a result, we obtain the following resolvent estimate in the high-frequency region.

Lemma 4.6. *For every $\rho \in (0,\rho_0)$ there exists $\varepsilon_0 > 0$, $\beta_0 > 0$, and $K_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$, $\eta \in \mathbb{R}$ satisfying $|\eta| \ge K_0^2\varepsilon^2$, and $\lambda \in \mathbb{C}$ satisfying $\mathrm{Re}(\lambda) > -\beta_0\varepsilon^3$, there exists a unique solution $u \in \mathrm{Dom}(A_0) \subset L^2_{\varepsilon\rho}$ to the resolvent equation (4.13) with $f \in L^2_{\varepsilon\rho}$ such that*

$$\|u\|_{L^2_{\varepsilon\rho}} \le C\varepsilon^{-3}\|f\|_{L^2_{\varepsilon\rho}}, \tag{4.20}$$

for some $C > 0$ independently of $f \in L^2_{\varepsilon\rho}$ and ε.

Proof. We use the resolvent identity

$$(\lambda - A_0 - A_1 - A_2)^{-1} = [I - (\lambda - A_0 - A_1)^{-1}A_2]^{-1}(\lambda - A_0 - A_1)^{-1}.$$

It follows from the bound (4.17) that we only need to show that the operator

$$I - (\lambda - A_0 - A_1)^{-1}A_2$$

is invertible with a bounded inverse in $L^2_{\varepsilon\rho}$, which is true if $\|(\lambda - A_0 - A_1)^{-1}A_2\|_{L^2_{\varepsilon\rho}\to L^2_{\varepsilon\rho}}$ is small. Since the decomposition (4.4) implies that

$$\|(-3\psi + \psi'')f\|_{L^2_{\varepsilon\rho}} \le C\varepsilon^2\|f\|_{L^2_{\varepsilon\rho}}, \tag{4.21}$$

it follows from the bound (4.18) that the smallness of $\|(\lambda - A_0 - A_1)^{-1}A_2\|_{L^2_{\varepsilon\rho}\to L^2_{\varepsilon\rho}}$ cannot be deduced from smallness of ε. Nevertheless, if we use the estimates (4.19) and (4.21), then we obtain

$$\|(\lambda - A_0 - A_1)^{-1}A_2\|_{L^2_{\varepsilon\rho}\to L^2_{\varepsilon\rho}} \le C_0K_0^{-1} \tag{4.22}$$

for some $C_0 > 0$. If $K_0 > 0$ is sufficiently large, the norm is small and the operator $I - (\lambda - A_0 - A_1)^{-1}A_2$ is invertible with a bounded inverse in $L^2_{\varepsilon\rho}$. The bound (4.20) follows from (4.17). $\square$

4.3. **The low-frequency region.** We first consider the two eigenvalues $\lambda_\pm(\eta)$ of the spectral problem (4.1) in L^2_ν for small $\eta \neq 0$, see Lemma 3.8. By Remark 3.9, the expansion of $\varepsilon^{-3}\lambda_\pm(\varepsilon^2\Upsilon)$ in Υ agrees with the exact expression (3.11) known for the KP-II equation (1.9). The following lemma states that the same correspondence holds for every Υ if ε is sufficiently small.

Lemma 4.7. *Let $\Lambda_\pm(\Upsilon)$ be given by (3.11) for every $\Upsilon \in \mathbb{R}$. For every $\rho \in (0,\rho_0)$, there exists $\varepsilon_0 > 0$ and $C_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$ the spectral problem (4.5) admits eigenvalues $\varepsilon^{-3}\lambda_\pm(\varepsilon^2\Upsilon)$ in L^2_ρ such that*

$$|\varepsilon^{-3}\lambda_\pm(\varepsilon^2\Upsilon) - \Lambda_\pm(\Upsilon)| \leq C_0\varepsilon^2.$$

Proof. By bootstrapping arguments, an eigenfunction $\hat{V}$ of the spectral problem (4.5) in L^2_ρ satisfies that

$$\hat{V} \in \mathrm{Dom}(\partial_X(1-\varepsilon^2\partial_X^2)^{-1}(L_{\mathrm{KdV}} + \Upsilon^2\partial_X^{-2})) \subset L^2_\rho$$

if and only if

$$\hat{V} \in \mathrm{Dom}(\partial_X(L_{\mathrm{KdV}} + \Upsilon^2\partial_X^{-2})) \subset L^2_\rho.$$

Hence we can rewrite the spectral problem (4.5) for the eigenfunction $\hat{V}$ in L^2_ρ in the equivalent form

$$\partial_X\left(L_{\mathrm{KdV}} + \varepsilon^2 L_{\mathrm{pert}} + \Upsilon^2\partial_X^{-2}\right)\hat{V} = \Lambda(1-\varepsilon^2\partial_X^2)\hat{V}. \tag{4.23}$$

Since $(\Lambda_\pm(\Upsilon), U_\pm) \in \mathbb{C}\times L^2_\rho$ are solutions of the truncated problem

$$\partial_X\left(L_{\mathrm{KdV}} + \Upsilon^2\partial_X^{-2}\right)U_\pm = \Lambda_\pm(\Upsilon)U_\pm, \tag{4.24}$$

we can write the decomposition $\Lambda = \Lambda_\pm(\Upsilon) + \varepsilon^2\tilde{\Lambda}$, $\hat{V} = U_\pm + \varepsilon\tilde{U}$ and obtain the perturbed problem for $(\tilde{\Lambda}, \tilde{U})$ given by

$$\begin{aligned}\partial_X&\left(L_{\mathrm{KdV}} + \varepsilon^2 L_{\mathrm{pert}} + \Upsilon^2\partial_X^{-2}\right)\tilde{U} - (\Lambda_\pm(\Upsilon) + \varepsilon^2\tilde{\Lambda})(1-\varepsilon^2\partial_X^2)\tilde{U}\\ &= -L_{\mathrm{pert}}U_\pm - \Lambda_\pm\partial_X^2 U_\pm + \tilde{\Lambda}(1-\varepsilon^2\partial_X^2)U_\pm.\end{aligned}$$

This equation is routinely solved by using the method of Lyapunov–Schmidt reduction with $\tilde{\Lambda}$ being uniquely defined from the condition that $\tilde{U} \in \mathrm{Dom}(\partial_X(L_{\mathrm{KdV}} + \Upsilon^2\partial_X^{-2})) \subset L^2_\rho$ satisfy the orthogonality condition to the adjoint eigenfunction for the eigenvalue $\Lambda_\pm(\Upsilon)$. See Lemma 3.4 and Corollary 3.5 in [31] for details. □

The resolvent equation in the scaled variables is obtained from the spectral stability problem (4.5) in the form

$$\left(\Lambda - \partial_X(1-\varepsilon^2\partial_X^2)^{-1}\left(L_{\mathrm{KdV}} + \varepsilon^2 L_{\mathrm{pert}} + \Upsilon^2\partial_X^{-2}\right)\right)U = F, \qquad F \in L^2_\rho. \tag{4.25}$$

The following lemma uses the smallness of $\varepsilon^2 L_{\mathrm{pert}}$ and the formalism from [31] in order to obtain the resolvent estimate in the low-frequency region.

Lemma 4.8. *For every $\rho \in (0, \rho_0)$ there exists $\varepsilon_0 > 0$, $\beta_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, $\Upsilon \in \mathbb{R}$ and $\Lambda \in \mathbb{C}$ satisfying $\mathrm{Re}(\Lambda) > -\beta_0$ and $\Lambda \neq \varepsilon^{-3}\lambda_\pm(\varepsilon^2\Upsilon)$, there exists a unique solution*

$$U \in \mathrm{Dom}(\partial_X(1-\varepsilon^2\partial_X^2)^{-1}(L_{\mathrm{KdV}} + \Upsilon^2\partial_X^{-2})) \subset L^2_\rho$$

of the resolvent equation (4.25) for every $F \in L^2_\rho$ satisfying

$$\|U\|_{L^2_\rho} \leq C\|F\|_{L^2_\rho} \tag{4.26}$$

for $C > 0$.

Proof. Let Q_{KP} be the projection operator for the spectral problem (4.24) which reduces L^2_ρ to the subspace orthogonal to the two adjoint eigenfunctions for the eigenvalues $\Lambda_\pm(\Upsilon)$. It follows from Proposition 3.2 in [31] (proven in [29]) that there exists $\beta_0 > 0$ and $C_0 > 0$ such that for every $\Lambda \in \mathbb{C}$ satisfying $\mathrm{Re}(\Lambda) > -\beta_0$ and every $F \in L^2_\rho$, we have

$$\|(\Lambda - \partial_X(L_{\mathrm{KdV}} + \Upsilon^2\partial_X^{-2}))^{-1}Q_{\mathrm{KP}}F\|_{L^2_\rho} \leq C_0\|F\|_{L^2_\rho}. \tag{4.27}$$

By the proximity result of Lemma 4.7, we can introduce $\mathcal{Q}$, the projection operator for the spectral problem (4.23) which reduces L^2_ρ to the subspace orthogonal to the two adjoint eigenfunctions for the eigenvalues $\varepsilon^{-3}\lambda_\pm(\varepsilon^2\Upsilon)$. The bound (4.27) and the proximity result suggest that there exists $\beta_0 > 0$ and $C_0 > 0$ such that for every $\Lambda \in \mathbb{C}$ satisfying $\mathrm{Re}(\Lambda) > -\beta_0$ and every $F \in L^2_\rho$, we have

$$\|(\Lambda - \mathcal{M})^{-1}\mathcal{Q}F\|_{L^2_\rho} \leq C_0\|F\|_{L^2_\rho}, \tag{4.28}$$

where

$$\mathcal{M} := \partial_X(1-\varepsilon^2\partial_X^2)^{-1}(L_{\mathrm{KdV}} - \varepsilon^2\partial_X(1-\Psi)\partial_X + \Upsilon^2\partial_X^{-2}).$$

Writing again the resolvent identity as

$$\begin{aligned}&(\Lambda - \partial_X(1-\varepsilon^2\partial_X^2)^{-1}(L_{\mathrm{KdV}} + \varepsilon^2 L_{\mathrm{pert}} + \Upsilon^2\partial_X^{-2}))^{-1})^{-1}\\ &\quad = [I - \varepsilon^2(\Lambda - \mathcal{M})(\Psi'' - 3\tilde{\Psi})]^{-1}(\Lambda - \mathcal{M})^{-1}\end{aligned}$$

and using smallness of ε^2, we obtain the invertibility of the near-identity operator

$$[I - \varepsilon^2(\Lambda - \mathcal{M})(\Psi'' - 3\tilde{\Psi})] : L^2_\rho \to L^2_\rho$$

for every $\Lambda \in \mathbb{C}$ satisfying $\mathrm{Re}(\Lambda) > -\beta_0$. The bound (4.26) on the unique solution U to the resolvent equation (4.25) follows from the bound (4.28). □

5. Conclusion

We have derived two results, which suggest that the transverse perturbations to the one-dimensional solitary waves of the CH equation (1.1) are stable in the time evolution of the CH-KP equation (1.2), similar to the KP-II theory. First, we proved that the double zero eigenvalue of the linearized equation related to the translational symmetry breaks under a transverse perturbation into a pair of the asymptotically stable resonances, which are isolated eigenvalues in the exponentially weighted L^2 space. Second, we considered the small-amplitude solitary waves governed by the perturbed KP-II equation and proved their linear stability under transverse perturbations.

We conclude the paper with a list of further questions. First, nonlinear stability of small-amplitude solitary waves of CH-KP is an open question, see [32] for such analysis in the Benney–Luke equation. Second, peaked traveling waves of the CH equation (1.1) exist but they are linearly and nonlinearly unstable in the time evolution in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, see [21, 33]. It would be interesting to see how the peaked profile of the solitary waves breaks under transverse perturbations and whether cusps (waves with infinite slopes at their maximum) would form in finite time. Third, transverse stability of smooth periodic waves and transverse instability of peaked periodic waves can be studied based on the stability analysis of the periodic waves in the one-dimensional model, see [15] and [28]. Finally, hydrodynamical applications of the obtained results are interesting in their own right within modeling of shallow water waves in seas and oceans [16].

Acknowledgement. This project was developed in June 2022 during a Research in Teams stay at the Erwin Schrödinger Institute, Vienna. The authors thank members of stuff of the ESI for support during this work. D. E. Pelinovsky acknowledges the funding of this study provided by Grants No. FSWE-2020-0007 and No. NSH-70.2022.1.5.

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