Hamiltonian averaging for solitons with nonlinearity management

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We revisit the averaged equation, derived in Pelinovsky *et al.* [Phys. Rev. Lett. **91**, 240201 (2003)] from the nonlinear Schrödinger (NLS) equation with the nonlinearity management. We show that this averaged equation is valid only at the initial time interval, while a new Hamiltonian averaged NLS equation can be used at longer time intervals. Using the new averaged equation, we construct numerically matter-wave solitons in the context of the Bose-Einstein condensates under the Feshbach resonance management. We show that there is no threshold on the existence of dark solitons of large amplitudes, whereas such a threshold exists for bright solitons.

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We address the nonlinear Schrödinger (NLS) equation with the nonlinearity management, considered in [1,2],

$$iu_t = -u_{xx} + V(x)u + \gamma_0|u|^2u + \frac{1}{\epsilon}\gamma\left(\frac{t}{\epsilon}\right)|u|^2u, \tag{1}$$

where $\epsilon \ll 1$, γ_0 is the averaged nonlinearity coefficient and $\gamma(\tau)$, $\tau = t/\epsilon$ is a mean-zero periodic function with the unit period. The potential $V(x) = \frac{1}{2}\Omega^2x^2$ corresponds to the parabolic magnetic trap for the Bose-Einstein condensates. Although Eq. (1) is written in one spatial dimension (for reasons of simplicity), the generalization of our method and results to multidimensions is straightforward. Additionally, it is worth mentioning that the recent development of trapping and cooling techniques has enabled experimental realizations of quasi-one-dimensional condensates [3], and, thus, the reduction of the fully three-dimensional NLS equation to an effective one-dimensional (1D) model is relevant [4] (see also a rigorous derivation in [5]). Importantly, as the regime of quasi-1D condensates is experimentally tractable, important experimental studies on 1D matter-wave dark [6] and bright [7] solitons have subsequently been performed. Finally, it should be noticed that the model Eq. (1) is also relevant in the context of nonlinear optics (in a layered structure in which Kerr nonlinearity alternates between self-focusing and self-defocusing) as has been discussed in [8].

In [2], we derived an averaged equation for standing waves (solitons) under nonlinearity management, based on the nonlocal transformation that removed the large fast variations of the nonlinearity coefficient $\gamma(\tau)$:

$$u(x,t) = e^{-i\phi(x,t)}v(x,t), \tag{2}$$

where

$$\phi(x,t) = \frac{1}{\epsilon}\int_0^t \gamma\left(\frac{t'}{\epsilon}\right)|v|^2(x,t')dt'. \tag{3}$$

There exists an equivalent local transformation that serves the same purpose,

$$u(x,t) = e^{-i\gamma_{-1}(\tau)|v|^2(x,t)}v(x,t), \quad \tau = \frac{t}{\epsilon}, \tag{4}$$

where $\gamma_{-1}(\tau)$ is the mean-zero antiderivative of $\gamma(\tau)$. By eliminating $|v|_t^2$ from the problem, the local transformation (4) reduces the NLS equation (1) to the form

$$\begin{aligned} iv_t = &-v_{xx} + V(x)v + \gamma_0|v|^2v + 2i\gamma_{-1}(v^2\overline{v_{xx}} + 2|v_x|^2v + v_x^2\overline{v}) \\ &- \gamma_{-1}^2[(|v|_x^2)^2 + 2|v|_{xx}^2|v|^2]v, \end{aligned} \tag{5}$$

where $|v|_x^2$ stands for $(|v|^2)_x$ and γ_{-1}^2 stands for $(\gamma_{-1})^2$. The standard averaging method [9] is applied for decomposition of $v(x,t)$ into a slowly varying part $w(x,t)$ and a small, fast-varying part $v_1(x,\tau)$:

$$v(x,t) = w(x,t) + \epsilon v_1(x,\tau,w(x,t)). \tag{6}$$

From the condition that $v_1(x,\tau,w)$ does not grow in τ, we derive a new averaged NLS equation:

$$iw_t = -w_{xx} + V(x)w + \gamma_0|w|^2w - \mu[(|w|_x^2)^2 + 2|w|^2|w|_{xx}^2]w, \tag{7}$$

where μ is the mean value of $\gamma_{-1}^2(\tau)$. The averaged NLS equation has a standard Hamiltonian form, with the Hamiltonian

$$H = \int_{\mathbb{R}} \left[|w_x|^2 + V(x)|w|^2 + \tfrac{1}{2}\gamma_0|w|^4 + \mu|w|^2(|w|_x^2)^2\right]dx. \tag{8}$$

We would like to comment on the validity of the new Hamiltonian averaged equation (7) in connection to the averaged equation, derived in [2]. Although the averaged equation (6) in [2] gives the condition that the correction $v_1(x,\tau,w)$ does not grow secularly in τ, the nonlocal term (3) in the transformation (2) has a nonzero mean value in the first order of ϵ. The nonzero mean value leads to a linear

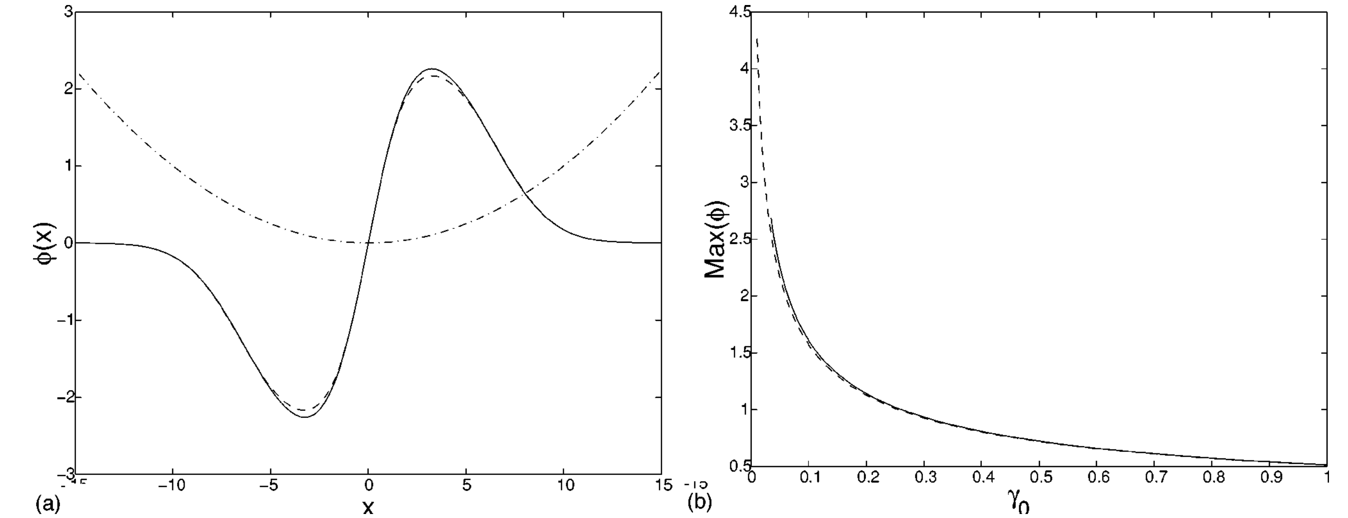

FIG. 1. (a) Example of a dark soliton solution for $\omega=-0.5$, $\Omega^2=0.02$, $\gamma_0=0.1$, and $\gamma_1=2.5\sqrt{\gamma_0}$. The dashed curve shows the result from Eq. (10) and the solid curve shows the result from Eq. (11) of [2]. The potential is shown by the dashed-dotted line. (b) Continuation branch of dark soliton solutions as a function of γ_0 for the same parameters.

growth in τ for $\phi(x,t)$ and to a quadratic growth in τ for the second-order correction term $v_2(x,\tau,w)$ in the extended decomposition:

$$v(x,t) = w(x,t) + \epsilon v_1(x,\tau,w) + \epsilon^2 v_2(x,\tau,w). \tag{9}$$

As a result, the validity of the averaged equation (6) in [2] is destroyed on the time scale of order $\epsilon\tau=t=O(1)$. When dealing with nonlocal transformations such as Eq. (2) and hence with nonlocal integro-differential equations, one cannot use the scalar decompositions (6) and (9), but rather vector decompositions for $v(x,t)$ and $\phi(x,t)$. The modified averaging procedure for the derivation of the Hamiltonian averaged NLS equation (7) from the nonlocal transformation (2) and (3) will be published elsewhere [10].

The comments above explain why the averaged equations (8) and (11) in [2] have no obvious Hamiltonian structure, while the original NLS equation (1) is a Hamiltonian system. Since the numerical approximations of the bound states, shown in Figs. 1–3 of [2], are only valid for the initial time interval, it is important to revisit the numerical approximations of the bound states within the new Hamiltonian averaged NLS equation (7), which is valid on longer time intervals. In the context of Bose-Einstein condensates under the Feshbach resonance management [1], the standing waves correspond to matter-wave bright and dark solitons.

Using the standard standing wave ansatz $w(x,t) = \phi(x)e^{i\omega t}$, we find the ODE problem for $\phi(x)$:

$$-\phi'' + \omega\phi + V(x)\phi + \gamma_0\phi^3 - 4\mu[2\phi^3(\phi')^2 + \phi^4\phi''] = 0. \tag{10}$$

For the time-dependent nonlinearity coefficient, we use the sinusoidal function $\gamma(\tau)=\gamma_1\sin(2\pi\tau)$, such that $\mu = \gamma_1^2/(8\pi^2)$. Figure 1(a) shows the profile of the dark soliton

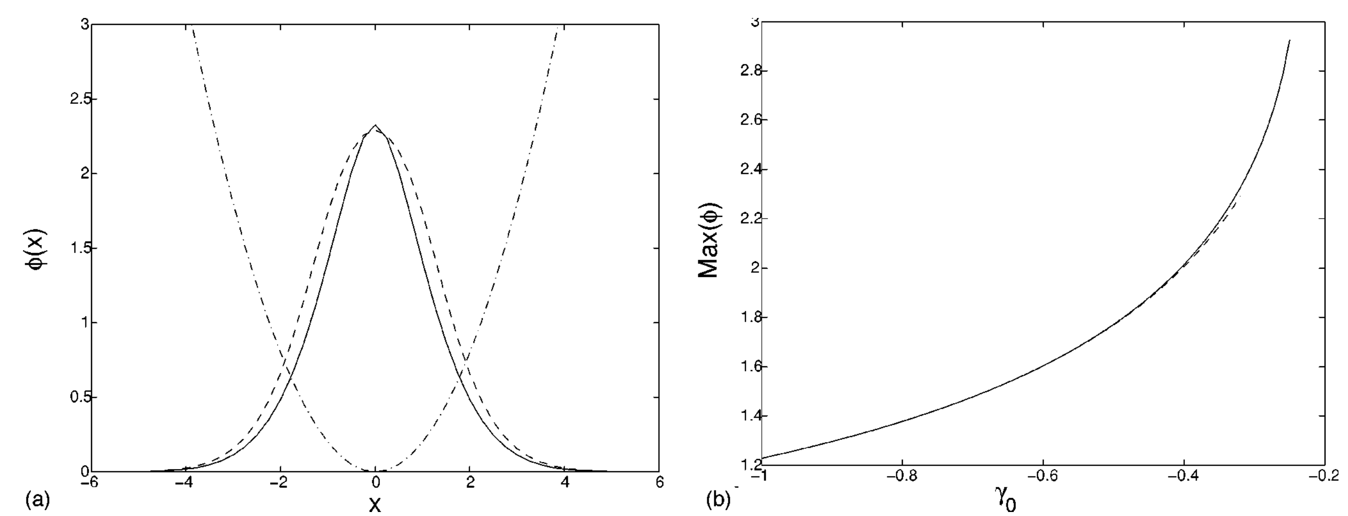

FIG. 2. Same as in Fig. 1, but for a bright soliton solution branch. The case of $\omega=0.5$, $\Omega^2=0.4$, $\gamma_0=-0.32$, and $\gamma_1=1$ is shown in the left panel.

$\phi(x)$ from Eq. (10) (dashed curve) for $\omega=-0.5$, $\Omega^2=0.02$, $\gamma_0=0.1$, and $\gamma_1=2.5\sqrt{\gamma_0}$. The solid line shows the bound state of Eq. (11) in [2] for the same parameters. Solution families of these two equations are continued on Fig. 1(b) as a function of γ_0 for the same parameter set.

It is clearly seen from Figs. 1(a) and 1(b) that the dark soliton solutions remain structurally very close to the ones obtained in [2]. However, there is a nontrivial difference occuring for large amplitudes of the dark solitons as $\gamma_0\to 0$. In that case, contrary to what was numerically predicted on the basis of Eq. (11) of [2], there is *no threshold* for the existence of dark solitons of large amplitudes, i.e., such solutions may exist for arbitrarily small γ_0. It should be noted that in all the numerical results of [2], γ_1 was scaled by $\sqrt{\gamma_0}$ (e.g., $\gamma_1=0.5$ in [2] should be read $\gamma_1=0.5\sqrt{\gamma_0}$).

Figures 2(a) and 2(b) show similar results for the bright soliton profile $\phi(x)$ for $\omega=0.5$, $\Omega^2=0.4$, $\gamma_0=-0.32$, and $\gamma_1=1$. The difference between solutions of Eq. (10) (dashed curve) and of Eq. (11) in [2] (solid curve) is, again, only observable for bright solitons of large amplitudes as $\gamma_0\to 0$. Contrary to the dark soliton case, there exists a *threshold* on the amplitudes of the bright soliton solutions in the new averaged equation (10), i.e., the solution family terminates at a nonzero value of γ_0.

We conclude that the new Hamiltonian averaged NLS equations (7) and (10) improve the averaging results of [2] for longer time intervals. Additionally, there is, typically, no threshold for the existence of dark soliton solutions of large amplitudes, whereas such a threshold typically exists for bright soliton solutions. Nevertheless, the differences between previous averaged equations in [2] and new averaged equations (7) and (10) are not observable in the case of moderate values of the average nonlinearity coefficient. This fact justifies the very good agreement between the direct numerical simulations of the NLS equation (1) and the results of the averaging approximation (see Figs. 4 and 5 in [2]).

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