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Contents

Chapter 8

Asymptotic methods in soliton stability theory

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Abstract

We describe asymptotic methods for the analysis of soliton stability and related long-term dynamics in nonlinear evolution equations which conserve energy. For these equations there exists a Lyapunov functional which generates stationary soliton solutions through a constrained variational principle. We show that the stability of soliton solutions is determined in many cases by a potential function given by this functional at the stationary soliton solutions. When the potential function has a local minimum in the space of the soliton parameters the soliton solutions are stable. In the opposite case, instability of the soliton solutions takes place and we investigate the structure of the eigenvalues and unstable eigenmodes through a modification of bifurcation analysis. In an extension of this analysis, we propose an asymptotic multi-scale expansion technique and derive several universal finite-dimensional asymptotic equations governing the long-term evolution of unstable solitons of different types. Using these equations, we describe typical scenarios of this instability-induced soliton dynamics and present approximate solutions for the soliton transformation.

8.1 Introduction

In modern nonlinear dynamics, the **existence** and **stability** of equilibrium states is the **primary** starting point for the analysis of dynamical processes described by finite-dimensional systems. Indeed, if the equilibrium states exist and are stable with respect to small perturbations, they can attract certain ranges of initial conditions. However, if an equilibrium state is unstable, small perturbations can lead the dynamical system to other stable stationary or non-stationary attractors, e.g. to periodic orbits or to chaotic motion.

Similar problems are also of crucial importance when dealing with infinite-dimensional systems possesing special steady-state solutions. In particular, we are concerned here with continuous systems describing wave propagation in nonlinear dispersive wave media. For these systems it is known that special localized solutions (solitary waves or solitons) play the role of equilibrium states which realize a balance between the dispersive and nonlinear properties of the wave field. In the last 30 years solitons have been under intense investigation in various branches of contemporary physics and it is unnecessary to mention here the extensive literature devoted to this topic. For our purpose, we recall only the basic and universal fact that *whenever solitons are stable with respect to small perturbations, an arbitrary initial localized pulse evolves into a finite sequence of solitons and an oscillatory tail decaying due to dispersive effects* (see, e.g., [1]). In the opposite case when the solitons are unstable, there is no universal scenario describing soliton transformations. For instance, in this situation *collapse* (formation of singularities in finite time) can occur in the framework of the underlying evolution equations (see, e.g., [2]). Thus, the study of soliton stability is a **fundamental** step in analysing the evolution of localized nonlinear perturbations in such wave systems.

Because of its fundamental importance, the theory of soliton stability began to develop in parallel with the discovery of the remarkable properties of soliton solutions (see, e.g., [3-5]). To the present time, several methods have been shown to be effective in the analysis of soliton stability and they are described in detail in many papers and reviews (see, e.g., [6-13]). We will mention here only the basic ideas and methods used.

First of all, one should separate the problem of solitary wave stability into several different classes according to whether (i) the perturbations have the same spatial dimension as the soliton (*longitudinal* stability), (ii) they have a larger spatial dimension being located along the soliton front (*transverse* stability), or (iii) they are induced by other (e.g. nonconservative) effects resulting in additional external terms to the underlying equations (*structural* stability). In this Chapter we confine ourselves to the first group of problems although the others can be studied by similar methods. Also we consider only *conservative* wave models which have some integrals of motion (e.g. mass, momentum, power and energy) leaving other classes of models, such as diffusive systems, for review elsewhere (see, e.g., [14]).

It is well-known (see, e.g., [15]) that there are generally three different but interrelated definitions of stability, namely (i) *linearized* or spectral, (ii) *energetic* or formal, and (iii) *nonlinear* or Lyapunov stability. Below we give these definitions and review some results which were commonly used in soliton stability problems.

Definition 1.1. Let $u(x,t)$ satisfy a nonlinear evolution equation in the form $u_t = F[u]$ so that an infinitesimal variation δu satisfies the linearized problem $\delta u_t = F'[u]\delta u$, where $F'[u]$ is the Frechet derivative of $F[u]$. Then, the soliton solution $u = u_s$ is called *linearized stable* if the variation δu does not grow for $t > 0$ faster than $O(t)$.

In this definition we have taken into account that in conservative evolution equations the soliton solutions form one- or many-parameter families and, as a result, the linearized problem admits eigenfunctions which grow linearly in time. For example, if $u_s = u_s(x - Vt; V)$ is the travelling-wave soliton solution, the linearized problem has an eigenfunction,

$$\delta u \sim \frac{\partial u_s}{\partial V} - t\frac{\partial u_s}{\partial x}$$

i.e. $\delta u \sim O(t)$ for $t \to \infty$. It is obvious that this linear growth can be easily removed by a renormalization of the soliton parameters, and hence it does not cause real soliton instabilities [4].

Usually, the problem of linearized stability can be reduced with the help of the substitution $\delta u = \psi \exp[\lambda t]$ to study the spectral problem $\mathcal{L}\psi = \lambda\psi$, where $\mathcal{L} = F'[u_s]$ is generally a *non-self-adjoint* operator with variable coefficients. If this spectral problem possesses a (*localized*) eigenmode ψ corresponding to the eigenvalue λ with a *positive* real part, then the infinitesimal perturbation δu grows exponentially in time and instability of the soliton solution occurs. Moreover, even in the case when the zero eigenvalue $\lambda = 0$ is *degenerate*, solitons are linearly unstable but this instability results in power-like growth of the perturbation, i.e. $\delta u \sim O(t^2)$ for $t \to \infty$. We would like to mention that the degeneration of the eigenvalue λ might occur also for nonzero but imaginary values, and this results in linear growth of the perturbation, i.e. $\delta u \sim O(t)$ for $t \to \infty$. This special case represents an exception from Definition 1.1 but, throughout this Chapter, we will discuss only simple cases for soliton stability problems, when the degeneration of the eigenvalue at the imaginary axis is not possible (see also discussion in Section 8.2.3).

To estimate the eigenvalue with a positive real part, different techniques have been applied in some particular problems. For instance, in studying the eigenvalue problem for the nonlinear Schrödinger (NLS) equations Vakhitov and Kolokolov [5] applied a Lagrange multiplier method which has been subsequently used by many others. This method allowed them to reduce the original spectral problem to a constrained Sturm–Liouville problem and then find the conditions when the maximum eigenvalue passes through zero and acquires a positive real part. On the other hand, using known functional inequalities (e.g. the Schwartz inequality), Laedke and Spatschek [6] applied a suitable variational principle to the same class of problems in order to estimate domains of linearized stability and instability. Recently, Pego and Weinstein [7] investigated another type of the linearized problem which is related to the generalizations of the Korteweg–de Vries (KdV) and Boussinesq (Bs) equations. They developed new analytical methods suitable for linear differential equations with asymptotically constant coefficients. However, a general method to analyze the linearized stability problem has not yet been proposed.

Definition 1.2. Let a soliton solution $u = u_s$ be a stationary point in the variational problem $\delta\Lambda[u] = 0$, where $\Lambda[u]$ is a Lyapunov functional which is presented by a superposition of a Hamiltonian $H[u]$ and the other constants of motion $N_j[u]$ for $j = 1, 2, ..., n$ in the form $\Lambda = H + \sum_{j=1}^{n} \omega_j N_j$, and $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is a set of parameters of the soliton solutions. Then, the soliton solution is called *energetically stable* if the second variation of the Lyapunov functional at the soliton solution is strictly positive (or strictly negative).

Indeed, if the second variation of the Lyapunov functional is strictly positive a soliton solution realizes a local minimum in a functional space and a small perturbation to the soliton shape will not change drastically the soliton evolution. Energetic stability also implies linearized stability, because the second variation is preserved by the linearized equations. In soliton theory, the introduction of the Lyapunov functional follows directly from the constrained variational problem which produces stationary soliton solutions to the given evolution equation. We mention that the same form of Lyapunov functional is often used to prove nonlinear stability of fluid and plasma equilibria but, in these problems, the search for the (Casimir) functionals N_j can be sometimes rather difficult (see [15]).

The variational principle for soliton solutions was widely used by Kuznetsov and coauthors (see [8] and references therein) to prove the global boundedness from below of the Lyapunov functional at the soliton solution. In some simple cases, when the evolution equations have some scaling symmetries, this can be easily done by means of a method of the functional estimates (see [8]) but this technique does not work in more complicated cases when the soliton solutions are stable only in a local rather than in a global sense.

Definition 1.3. Let operator τ_j denote translations of the family of soliton solutions u_s along parameters of the j-stationary phase. Define the ϵ-vicinity of the soliton orbit as $U_\epsilon = \{u : \inf ||u - \sum_{j=1}^{n} \tau_j u_s|| < \epsilon\}$, where $||u||$ is a quadratic Sobolev norm. The soliton solution $u = u_s$ is called *nonlinearly stable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $u(x,0) \in U_\delta$ then $u(x,t) \in U_\epsilon$ for all $t > 0$.

In other words, this definition implies that the solution being initially near the soliton orbit remains at all later times near this orbit. To extend energetic stability to nonlinear stability it is necessary to take into account nonlinear terms in the Lyapunov functional beyond the second variation terms and then to evaluate them using appropriate norms. For example, using analysis based on the properties of the linearized operators, Weinstein [9] proved for the generalized NLS and KdV equations that the Lyapunov functional has a strict minimum at the soliton solution in the stability domain of a parameter space. General theorems on nonlinear soliton stability were proved by Grillakis *et al.* [10, 11] for a wide but not a general class of evolution equations including the generalized Klein–Gordon equations. These results applied to equations occurring in field theory were recently reviewed by Makhankov *et al.* [12].

These three definitions of soliton stability require different levels of the perturbation amplitudes to preserve the stability of the soliton dynamics, from infinitesimal to small and finite amplitudes. However, for all aforementioned soliton equations it was found that *the linearized, energetic and nonlinear stability of the soliton solutions is determined by concavity of the function $U(\omega) = \Lambda[u_s]$ in the parameter space ω*, where $\Lambda[u_s]$ is the Lyapunov functional expressed at the stationary soliton solution. This *universal* result was also obtained by means of catastrophe theory [13].

The **main objectives** of the present Chapter are to review the known results of the linear stability theory through an asymptotic (bifurcation) approach and to describe a novel *uniform* asymptotic multi-scale method for analysis of *nonlinear* rather than linear stages of solitary wave instabilities. Recently this method was elaborated in a number of papers [16-19] and its applicability seems to be very powerful not only in the problems where the linear stability theory has been considered but also in new problems.

Our approach can be regarded as a '*finite-dimensional analysis*' of soliton instabilities because it uses many results of the stability theory for finite-dimensional Hamiltonian and dissipative systems. As a basis for our analysis, we use a modified soliton perturbation theory to reduce description of the soliton instability development to a finite-dimensional system for an equivalent particle motion.

The fact that solitons are analogous to particles in some dynamical processes is well known. For example, collisions of solitons [20], their dynamics in smooth inhomogeneous fields [21] as well as the internal oscillations of soliton shapes [22] can be approximately described by finite-dimensional Hamiltonian systems. Furthermore, a regular soliton perturbation theory has been elaborated and applied to an approximate description of soliton dynamics under the action of various small perturbations [23]. According to this theory, the action of the perturbations can be reduced in the leading-order (*adiabatic*) approximation to a finite-dimensional system for the variable parameters of the soliton solutions. However, to apply the soliton perturbation theory it is supposed that the solitons are stable in the framework of the nonperturbed problem. As soon as this assumption is not fulfilled and the solitons are unstable, the adiabatic perturbation theory breaks down [24]. In this case, one needs to find higher-order approximations to the equations of the perturbation theory. The latter problem is subject for our studies described in this Chapter.

In the linear approximation, the modified soliton perturbation theory reduces to a bifurcation analysis of the linear eigenvalue problem. The origin of this approach can be found in several previous works. Thus, Zakharov and Rubenchik [25] first used expansions with respect to the small eigenvalue to study the transverse instability of solitons (waveguides) in nonlinear dispersive media. In the two-dimensional case when a cylindrical waveguide was weakly unstable, they modified the asymptotic expansions to take into account a longitudinal instability comparable to a transverse instability. Next, Laedke and Spatschek [6] found the exact but implicit solutions of the linear eigenvalue problem in the so-called *critical* case which arises at the edge of the stability and instability domains. In this case, the unstable perturbation can be represented by a finite Taylor expansion with respect to the evolution time t. Again, quite recently, Pego and Weinstein [26] studied transitions to instability in the generalized KdV and Bs equations applying the Taylor expansions with respect to the small eigenvalue for the analytical functions used in their analysis. As a matter of fact, all these results are closely related to the asymptotic multi-scale

expansion method which we apply here to investigate the soliton stability.

Although our method is rather general, it is best demonstrated through its applications to a number of typical and physically relevant examples. In the following, we present different classes of solitary wave solutions to nonlinear evolution equations including those well known already (e.g. kinks, long-wave and bright solitons) and some new classes, whose study has begun only recently (e.g., dark and coupled solitons).

Example 1.1. Kinks and long-wave solitons

These soliton structures can be described in the framework of the generalized KdV and Boussinesq equations. Typical representatives of these equations are given by

$$u_t + f'(u)u_x + \alpha u_{xxx} - \beta u_{txx} = 0 \qquad (1.1)$$

$$u_{tt} - c_0^2 u_{xx} + [f(u) + u_{xx}]_{xx} = 0 \qquad (1.2)$$

where α and β are non-negative, c_0 is the limiting long-wave speed, the function $f(u)$ has the properties $f(0) = f'(0) = 0$, and $f'(u) = df/du$. For $\alpha = 1$ and $\beta = 0$ equation (1.1) can be regarded as the nonlinear generalization of the KdV equation [27] while for $\alpha = \beta = 1$ it is a generalization of the Benjamin–Bona–Mahoney (BBM) equation [28]. The generalized KdV (**gKdV**) equation (1.1) describes Rossby waves supported by a weak shear flow [29] and long internal waves in a fluid with weakly nonuniform stratification [30], as well as being generally used for the approximate modelling of propagation of strongly nonlinear waves in weakly dispersive media [27]. On the other hand, the generalized Boussinesq (**gBs**) equation (1.2) describes interaction of two counter-propagating nonlinear waves in a nonlinear medium with weak positive dispersion.

The kink and soliton solutions to (1.1) and (1.2) can be found by means of a direct reduction using a travelling-wave coordinate, $u = u_s(x - Vt)$, where V is the velocity. We call the solutions u_s **kinks** if u_s tends to *nonzero* and *different* boundary conditions at infinity (e.g., $u_s \to \pm q$ as $x \to \pm\infty$, where q is constant). If u_s approaches the *same* boundary conditions at infinity we call such solutions **long-wave solitons**. Without loss of generality we consider *zero* boundary conditions for the long-wave soliton solutions so that $u_s \to 0$ as $x \to \pm\infty$. The stability of such soliton solutions was considered in [7,26,31-34].

Example 1.2. Bright and dark solitons

These soliton solutions can be described in the framework of the generalized NLS (**gNLS**) and complex Klein–Gordon (**cKG**) equations which are given by

$$i\Psi_t + \Psi_{xx} + F(|\Psi|^2)\Psi = 0 \tag{1.3}$$

$$\Psi_{tt} - \Psi_{xx} + \left[\omega_0^2 - F(|\Psi|^2)\right]\Psi = 0 \tag{1.4}$$

where we suppose that $F(0) = 0$. In nonlinear optics, equation (1.3) describes propagation of self-guided beams in dielectric waveguides and the function $F(I)$, $I = |\Psi|^2$ is proportional to a nonlinear correction to the refractive index of the optical material [35]. On the other hand, equation (1.4) describes localized structures in field theory (see [12] and references therein).

The soliton solutions are given by a function involving two parameters, $\Psi = \Psi_s(x - 2Vt)\exp[i\Omega t]$, where $2V$ is the soliton velocity and Ω is the soliton propagation constant. We call these solutions **bright solitons** if $|\Psi_s|^2$ approaches zero as $|x| \to \infty$ and **dark solitons** if $|\Psi|^2 \to q$, where q is the intensity of the continuous-wave background. The stability of bright solitons has been considered in details in many papers [5,9-10,36-38] while the study of stability of dark solitons started only recently [18,39-41].

Example 1.3. Coupled solitons

Coupled solitons are described by models consisting of two and more components which support individually one of the soliton solutions described above. Here we consider only one example of these models, the **coupled NLS** equations given by

$$i\Psi_{1t} + \Psi_{1xx} + 2\sigma_1\left(|\Psi_1|^2 + \rho|\Psi_2|^2\right)\Psi_1 = 0 \tag{1.5a}$$

$$i\Psi_{2t} + \Psi_{2xx} + 2\sigma_2\left(\rho|\Psi_1|^2 + |\Psi_2|^2\right)\Psi_2 = 0 \tag{1.5b}$$

where ρ is a coupling parameter and $\sigma_1, \sigma_2 = \pm 1$. These equations describe interaction between two optical pulses with different polarizations or different carrier frequencies in a birefrigent fibre [42]. The coefficient ρ usually takes values from 2/3 to 2, while the signs of σ_1 and σ_2 are positive (negative) if the corresponding pulses propagate in the anomalous (normal) regime of the birefrigent fibre.

The coupled solitons to (1.5) are expressed by the three-parameter substitution, $\Psi_j = \Psi_{js}(x - 2Vt)\exp[i\Omega_j t]$ for $j = 1, 2$. According to the signs σ_1 and σ_2 there are possible coupled states of bright-bright, dark-bright and dark-dark solitons [43-45]. Only the stability of coupled bright-bright solitons has been previously considered [46-48]. We mention also that similar coupled (multiply-charged) solitons were investigated for coupled complex Klein–Gordon (many-component field) equations [11, 12, 49].

8.2 Linear stability theory

8.2.1 Stability of equilibrium states in finite dimensions

In this Section we recall the well-known results of linear stability theory for finite-dimensional conservative or dissipative systems [50, 51]. First, we consider a system of n particles with the conserved energy,

$$E = K(q, \dot{q}) + U(q) = \frac{1}{2}\sum_{i,j} M_{ij}(q)\dot{q}_i\dot{q}_j + U(q) \tag{2.1}$$

where $q = (q_1, q_2, ..., q_n)$ are generalized coordinates, $\dot{q} = (\dot{q}_1, \dot{q}_2, ..., \dot{q}_n)$ are generalized velocities, $K(q, \dot{q})$ stands for the kinetic energy, and $U(q)$ is the potential energy. In nonrelativistic mechanics, the kinetic energy is usually supposed to be a *positive definite* quadratic form in the velocities [50, 51].

The system of particles can form an *equilibrium state* if the potential function $U(q)$ has a local extremum for some $q = q_0$, so that

$$\left.\frac{\partial U}{\partial q_j}\right|_{q=q_0} = 0, \quad \text{for all } j$$

The stability of this equilibrium state is determined by the following theorem.

Theorem 2.1. Let a system of particles be defined by the energy (2.1) with positive definite kinetic energy and have an equilibrium state at the point $q = q_0$. (i) The equilibrium state is nonlinearly stable if the function $U(q)$ has a strict minimum at the point $q = q_0$. (ii) The equilibrium state is linearized and energetically stable if the quadratic form generated by the matrix with the coefficients

$$u_{ij} = \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{q=q_0}$$

is positive definite.

Indeed, the second variation of the energy at the equilibrium state is given by a superposition of the kinetic energy and potential energy quadratic forms,

$$\delta^2 E = \frac{1}{2}\sum_{i,j} m_{ij}\delta\dot{q}_i\delta\dot{q}_j + \frac{1}{2}\sum_{i,j} u_{ij}\delta q_i\delta q_j \tag{2.2}$$

where $\delta q_j = q_j - q_{j0}$ and $m_{ij} = M_{ij}(q_0)$. Then, if the potential energy quadratic form is positive definite (as the kinetic energy quadratic form is supposed to be) the positive definite property of the second variation $\delta^2 E$ follows. In this case, the function $U(q)$ has a local minimum in the equilibrium state and the equations for the particle motion describe small-amplitude oscillations of n particles with n real frequencies [50]. Furthermore, if the second quadratic form in (2.2) is only non-negative definite (i.e. it can take zero values for certain nonzero δq_j), the equilibrium state is linearized and energetically unstable because of solutions which grow secularly in time. However, even in this case, nonlinear stability can still be proved if the potential energy function has a strict minimum at the extremal point.

Next, we also include certain dissipative effects described by Rayleigh's dissipative function $F(q,\dot{q})$. In the quasi-linear approximation, this function can also be presented by a quadratic form with respect to the generalized velocities [50],

$$F = \frac{1}{2}\sum_{i,j} K_{ij}(q)\dot{q}_i\dot{q}_j \tag{2.3}$$

The equations of motion are now

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = -\frac{\partial F}{\partial \dot{q}_j} \tag{2.4}$$

where L is the Lagrangian given by $L = K(q,\dot{q}) - U(q)$. The dissipative function F is proportional to the rate of energy damping according to the balance equation following from (2.1),(2.3) and (2.4),

$$\frac{dE}{dt} = -2F$$

Therefore, we suppose for global dissipation that the quadratic form in (2.3) is positive definite. Then, one can prove (see [51]) that the existence and stability of equilibrium states are given by the same conditions as above. Thus, we conclude that both in conservative and certain dissipative finite-dimensional systems with positive definite $K(q,\dot{q})$ and $F(q,\dot{q})$ the stability of equilibrium states is uniquely determined by concavity of the potential function $U(q)$. This main result has a direct analogy in the stability theory of solitons.

8.2.2 Stability of soliton solutions of evolution equations

In this Section we review some results for soliton stability theory. Let us consider a nonlinear evolution equation with a conserved energy written in the abstract

Hamiltonian form [10, 11],

$$u_t = JH'[u] \tag{2.5}$$

where $H[u]$ is an energy functional (Hamiltonian), $H'[u]$ is the variational (Frechet) derivative of $H[u]$, and J is a skew-symmetric linear operator. As we mentioned in the Introduction (see Definition 1.2), soliton solutions $u = u_s$ of equations of this type depend on one or more parameters ω_j ($j = 1, 2, ..., n$) which are associated with additional integrals of motion N_j. These solutions are stationary points of the Hamiltonian H for fixed invariants N_j which can be written as the following *constrained variational problem*,

$$\delta\Lambda[u] = \delta\left(H[u] + \sum_{j=1}^{n}\omega_j N_j[u]\right) = 0 \quad \text{for} \quad u = u_s \tag{2.6}$$

Here $\Lambda[u]$ is a Lyapunov functional while the parameters ω_j play the role of Lagrangian multipliers. The variational problem (2.6) leads to differential relations between the integral invariants H and N_j evaluated at the stationary soliton solutions. We adopt the subscript 's' for these invariants. Indeed, varying (2.6) with respect to the soliton parameters we obtain the differential relations,

$$\frac{\partial H_s}{\partial \omega_i} + \sum_{j=1}^{n}\omega_j\frac{\partial N_{js}}{\partial \omega_i} = 0 \tag{2.7a}$$

Then, symmetric relations between N_{is} and N_{js} follow from (2.7a) after elimination of H_s,

$$\frac{\partial N_{is}}{\partial \omega_j} = \frac{\partial N_{js}}{\partial \omega_i} \tag{2.7b}$$

The theory of soliton stability is based on a study of the properties of the second variation $\delta^2\Lambda = \Lambda[u_s + \delta u] - \Lambda[u_s] = \frac{1}{2}\langle\delta u, \Lambda''[u_s]\delta u\rangle + o(||\delta u||^2)$, where $\langle u, w\rangle$ is a proper inner product (see, e.g., formula (2.12) below). This second variation $\delta^2\Lambda$ generates the linearized equations,

$$\delta u_t = J\Lambda''[u_s]\delta u \tag{2.8}$$

Usually the operator $\Lambda''[u_s]$ is a self-adjoint operator of Sturm–Liouville type. Then, the first and important step is to study the eigenfunctions to the spectral problem $\Lambda''[u_s]\phi = \mu\phi$ which generally contains a set of *neutral* modes (i.e. eigenfunctions for $\mu = 0$) generated by translational symmetries of the soliton solutions. For different evolution equations there are known the following three general types of spectrum of the operator $\Lambda''[u_s]$. They are shown schematically in Fig.1.

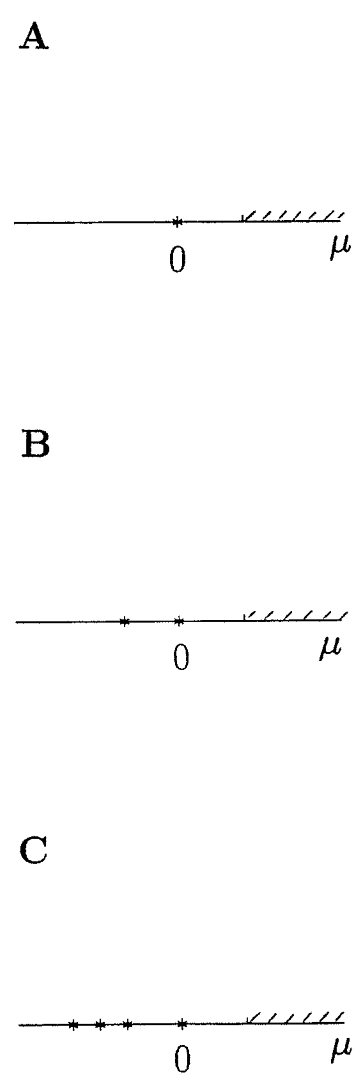

Fig.1. Three characteristic types of spectrum of the operator $\Lambda''[u_s]$. Here μ is a real eigenvalue for the corresponding spectral problem, stars ($*$) and dashed ($///$) regions depict localized modes and branches of the continuous spectrum, respectively.

A. The neutral modes are *ground states* (i.e. even and nodeless solutions of the spectral problem) and the rest of the spectrum has only positive values for μ.

B. There exist only one mode for negative μ, neutral modes for $\mu = 0$ and the rest of the spectrum has positive values for μ.

C. For negative μ there exist two or more modes or even an infinite-dimensional subspace of eigenfunctions.

The case **A** occurs, for example, for kink solutions to nonlinear evolution equations and, in this case, it is easy to prove that the quadratic form $\langle \delta u, \Lambda''[u_s]\delta u\rangle$ is always positive, i.e. the soliton solutions are absolutely stable [8, 10]. The case **B** is well-known for long-wave and bright solitons [9, 10] which have effectively only one parameter, say $\omega \equiv \omega_1$ and $N \equiv N_1$. In this case, the following theorem was proved by Grillakis *et al.* [10,Theorem 2].

Theorem 2.2. Define functions of ω of the form $U(\omega) = \Lambda[u_s]$ and $N_s(\omega) = N[u_s]$ and let the spectral problem $\Lambda''[u_s]\phi = \mu\phi$ satisfy condition **B**. Then, (i) the soliton solution $u = u_s$ with the parameter $\omega = \omega_0$ is nonlinearly stable if the function $U(\omega)$ has a strict minimum at the point $\omega = \omega_0$ and (ii) the soliton solution is linearized and energetically stable if the concavity of U is positive, i.e.

$$\left.\frac{d^2U}{d\omega^2}\right|_{\omega=\omega_0} = \left.\frac{dN_s}{d\omega}\right|_{\omega=\omega_0} > 0$$

In the last formula we have used the differential relations (2.7a). Thus, it follows from Theorem 2.2 that the Lyapunov functional evaluated at the soliton solution serves as an effective potential function which determines the stability of soliton solutions by its concavity at the extremal point. We would like to mention that there are some examples when the spectrum of $\Lambda''[u_s]$ has only one mode for negative μ but the rest of the spectrum is located in $\mu > 0$ starting from a neutral nonlocalized mode with $\mu = 0$. For instance, this occurs for dark soliton solutions [52]. Although the rigorous stability theory for this case has not yet been completely studied, our preliminary results (see Example 2.3.2) indicate that Theorem 2.2 seems to be valid even for this modification of condition **B**.

Finally, the case **C** is known, for example, for coupled solitons [11], 'higher-order' solitons with a number of nodes [54] as well as for long-wave solitons in some systems of generalized Bs equations [55]. Under certain restrictions (see [11] for details) Theorem 2.2 can be generalized for this case as well so that a strict minimum of the function $U(\omega) = \Lambda[u_s]$ defined now in a space of n parameters ω_j ($j = 1, 2, ..., n$) determines the nonlinear stability of soliton solutions. However, when the restrictions described in [11] are not satisfied the simple criteria based on the function $U(\omega)$ may not work and instability might arise even if this function has a strict minimum in the parameter space [54, 55].

In the rest of this Section we describe briefly some examples of soliton stability problems.

Example 2.2.1. Kinks and long-wave solitons

Soliton solutions to the gKdV equation (1.1) can be found through the substitution $u = u_s(x - Vt)$, where u_s satisfies the stationary-wave equation,

$$(\alpha + \beta V)u_{sxx} - Vu_s + f(u_s) = 0 \qquad (2.9)$$

The existence of kink and/or long-wave soliton solutions is determined by the particular form of the function $f(u)$ and the values of the parameter V. For instance, the following assumptions provide some sufficient conditions for their existence.

Assumption 2.1. (*existence of kinks*). Equation (2.9) admits solutions with the boundary conditions $u_s \to \pm q$ as $x \to \pm\infty$ if:
i) $V = V(q) = q^{-1}f(q)$ and V is confined to the interval $(-\alpha\beta^{-1}, 0)$;
ii) the function $f(u)$ is odd and satisfies the condition $qf'(q) < f(q) < 0$.

Assumption 2.2. (*existence of long-wave solitons*). Equation (2.9) admits solutions with the boundary conditions $u_s \to 0$ as $x \to \pm\infty$ if:
i) V is either positive or negative but bounded from above as $V < -\alpha\beta^{-1}$;
ii) there exists at least one value $u = u^*$ such that $\int_0^{u^*} f(u)du - \frac{1}{2}Vu^{*2} = 0$.

The geometrical meaning of these conditions is obvious from the construction of an equivalent potential $W(u_s) = -\frac{1}{2}Vu_s^2 + \int_0^{u_s} f(u)du$. We present this potential in Figs.2(a,b) for the nonlinear function $f(u) = \sigma u^3$, where $\sigma = \pm 1$. The kink solutions shown in Fig.2(c) correspond to the trajectory 's' [see Fig.2(a)] connecting two saddle points in the potential $W(u_s)$. These solutions exist for $\sigma = -1$ and are described by the explicit function,

$$u_s = q\tanh[\kappa x]$$

where $\kappa = q/\sqrt{2(\alpha - \beta q^2)}$ and $V = V(q) = -q^2$. The long-wave solitons shown in Fig.2(d) for $\sigma = +1$ correspond to a homoclinic orbit [see Fig.2(b), trajectory 's']. These long-wave solitons exist for $\sigma = +1$ and are expressed by the function,

$$u_s = \sqrt{2V}\,\text{sech}[\kappa x]$$

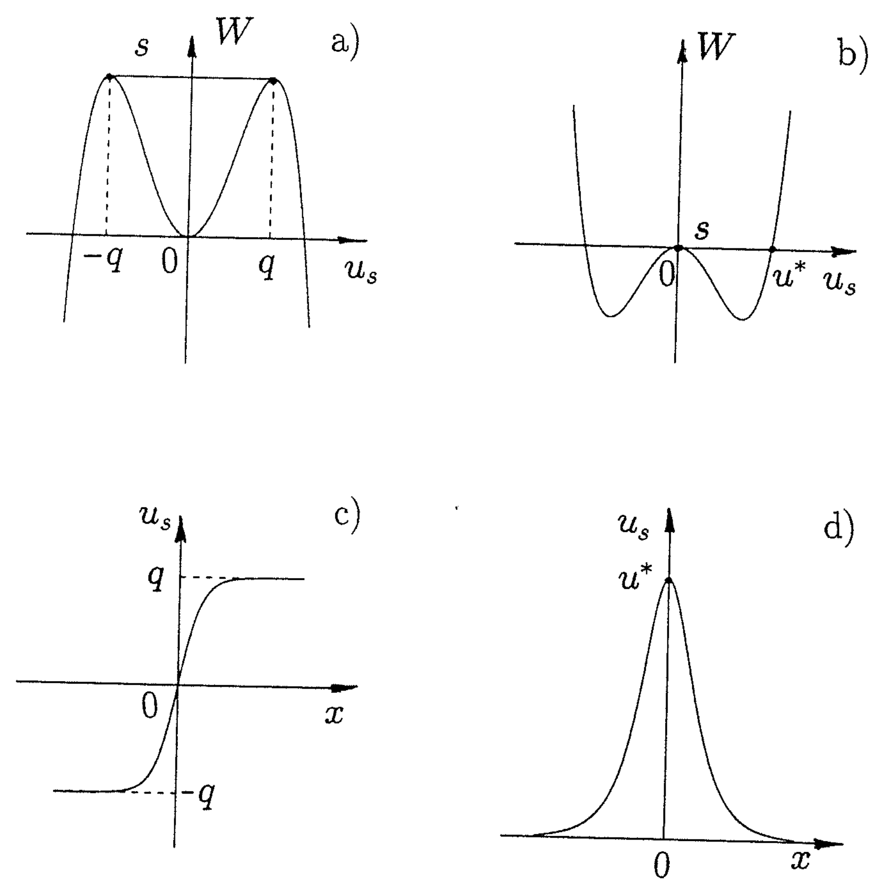

Fig.2. An equivalent potential $W(u_s)$ for the stationary-wave equation (2.9) with $f(u) = \sigma u^3$, where $\sigma = -1$ (a) and $\sigma = +1$ (b), as well as the kink (c) and long-wave soliton (d) solutions corresponding to the separatrix trajectories 's'. Parameters u^* and q stand for soliton amplitude and boundary conditions for the kink solutions, respectively.

where $\kappa = \sqrt{V}/\sqrt{\alpha+\beta V}$ and V is strictly positive. We note that in the general case the function $u_s(x)$ is *odd* for kink solutions and *even* for long-wave solitons.

The stationary-wave equation (2.9) can be obtained from the variational problem (2.6) for $n=1$ and the Lyapunov functional has the form $\Lambda = H[u] + VP[u]$, where the energy H and momentum P are given by

$$H = \frac{1}{2}\int_{-\infty}^{+\infty}\left(\alpha u_x^2 - 2\int_q^u f(u)du\right)dx \tag{2.10a}$$

$$P = \frac{1}{2}\int_{-\infty}^{+\infty}\left(u^2 - q^2 + \beta u_x^2\right)dx \tag{2.10b}$$

The linearized problem (2.8) can be written in the operator form,

$$(\delta u)_t - \beta(\delta u)_{txx} = (\mathcal{L}\delta u)_x \tag{2.11}$$

where $\mathcal{L} = -(\alpha+\beta V)\partial_x^2 - f'(u_s) + V$. The second variation $\delta^2\Lambda$ can be expressed as a quadratic form $\delta^2\Lambda = \frac{1}{2}\langle\delta u, \mathcal{L}\delta u\rangle$, where the inner product is defined by

$$\langle u, w\rangle = \int_{-\infty}^{+\infty} uw dx \tag{2.12}$$

A neutral mode of the operator $\mathcal{L}$ is given by the spatial translation u_{sx}. For the kink solutions this translation is a nodeless function and, therefore, the operator $\mathcal{L}$ satisfies condition **A** (see Fig.1). As a result, the quadratic form $\delta^2\Lambda$ is positive definite, which follows also from the explicit representation,

$$\delta^2\Lambda = \frac{1}{2}(\alpha+\beta V)\int_{-\infty}^{+\infty}\left[u_{sx}\left(\frac{\delta u}{u_{sx}}\right)_x\right]^2 dx > 0 \tag{2.13}$$

Thus, the kink solutions to (1.1) are absolutely stable with respect to small perturbations. The same conclusion also follows for kinks in other evolution equations such as the gBs equation (1.2) or the (scalar) Klein–Gordon equation [8, 10]. As a matter of fact, the kink solutions to (1.1) or (1.2) have no free parameters because the parameter q is fixed by the boundary conditions and $V = V(q)$. Such (stable) solutions can be regarded as *0-parameter* solitons.

In the case of long-wave solitons, the neutral mode u_{sx} has one node and the representation (2.13) is no longer valid because the integral is divergent at $x=0$. According to the oscillation theorem, there exists only one eigenfunction with a negative eigenvalue for the spectrum of $\mathcal{L}$ which is nodeless. Therefore, the operator $\mathcal{L}$ for long-wave solitons satisfies the condition **B** (see Fig.1) of Theorem 2.2. According to this theorem, stability of the long-wave solitons is

determined by a concavity of the function $U = U(V) = \Lambda[u_s]$ and, therefore, the long-wave soliton is nonlinearly stable if the momentum $P_s = P_s(V) = P[u_s]$ is increasing [31, 33]. A similar result can be obtained for the gBs equation (1.2) and also for some other models supporting *1-parameter* long-wave soliton solutions [7, 26]. For the example of the nonlinear function $f(u) = u^3$, the momentum $P_s(V)$ [see (2.10b)] is given by

$$P_s = 2\sqrt{\frac{V}{\alpha+\beta V}}\left(\alpha + \frac{4}{3}\beta V\right)$$

which is an increasing function of V for $V > 0$. Therefore, the long-wave solitons for the gKdV equation (1.1) with the nonlinear function $f(u) = u^3$ are stable.

Example 2.2.2. Bright and dark solitons

The stationary solutions to the gNLS equation (1.3) are given by the two-parameter substitution $\Psi = \Psi_s(x - 2Vt)\exp[i\Omega t]$ which reduces (1.3) to the equation,

$$\Psi_{sxx} - 2iV\Psi_{sx} + \left(F(|\Psi_s|^2) - \Omega\right)\Psi_s = 0 \tag{2.14}$$

Using separation of variables $\Psi_s = \Phi\exp[i\Theta]$, where Φ and Θ are real, we find from (2.14) a simple equation for $\Theta(x)$,

$$\Theta_x = V - \frac{q}{\Phi^2}\nu \tag{2.15}$$

where ν is the integration constant. The function Φ satisfies the second-order differential equation,

$$\Phi_{xx} + \left[F(\Phi^2) + V^2 - \nu^2 - \Omega\right]\Phi + \nu^2\left(1 - \frac{q^2}{\Phi^4}\right)\Phi = 0 \tag{2.16}$$

with boundary conditions $\Phi^2 \to 0$ for bright solitons and $\Phi^2 \to q$ for dark solitons as $|x| \to \infty$. Sufficient conditions for the existence of these soliton solutions are given by the following assumptions.

Assumption 2.3. (*existence of bright solitons*). Equation (2.16) admits solutions with the boundary conditions $\Phi^2 \to 0$ as $|x| \to \infty$ if:
i) $\nu = 0$ and $\Omega = \omega + V^2$, where ω is positive;
ii) there exists at least one value $I = I^* > 0$ such that $\int_0^{I^*} F(I)dI - \omega I^* = 0$.

Assumption 2.4. (*existence of dark solitons*). Equation (2.16) admits solutions with the boundary conditions $\Phi^2 \to q$ as $|x| \to \infty$ if:
i) $\Omega = F(q) + V^2 - \nu^2$;
ii) $\nu^2 < c^2 = -\frac{1}{2}qF'(q)$ and, therefore, $F'(q) < 0$;
iii) there exists at least one value $I = I^* > 0$ such that

$$\int_q^{I^*} [F(I) - F(q)]dI + \nu^2(I^* - q)^2/I^* = 0$$

The separatrix trajectories representing the bright and dark soliton solutions are denoted as 's' in Figs.3(a,b) for the equivalent potential $W(I) = \int_q^I [F(I) - F(q)]dI + \nu^2(I-q)^2/I$, where $I = \Phi^2$. We have evaluated this potential for the nonlinear function, $F(I) = \sigma I$, where $\sigma = \pm 1$. The bright (for $\sigma = +1$) and dark (for $\sigma = -1$) soliton solutions are shown in Figs.3(c,d), respectively. For the given nonlinear function, these solutions can be found from (2.16) in the analytical form,

$$\Phi = \sqrt{2\omega}\ \mathrm{sech}[\sqrt{\omega}x]$$

for bright solitons and

$$\Phi = \left(q - \frac{1}{2}\left(c^2 - \nu^2\right)\mathrm{sech}^2[\kappa x]\right)^{1/2}$$

for dark solitons, where $\kappa = \sqrt{c^2 - \nu^2}$ and $c^2 = \frac{1}{2}q$.

The bright and dark soliton solutions can be generally found from the variational problem (2.6) for $n = 3$ with the Lyapunov functional in the form

$$\Lambda = H[\Psi] + VP[\Psi] + \Omega N[\Psi] + q\nu S[\Psi] \quad (2.17)$$

Here the energy H and the constants of motion P (momentum), N (power), and S (phase shift) are defined as

$$H = \frac{1}{2}\int_{-\infty}^{+\infty}\left(|\Psi_x|^2 - \int_q^{|\Psi|^2} F(I)dI\right)dx \quad (2.18a)$$

$$P = \frac{i}{2}\int_{-\infty}^{+\infty}(\Psi^*\Psi_x - \Psi\Psi_x^*)\,dx \quad (2.18b)$$

$$N = \frac{1}{2}\int_{-\infty}^{+\infty}\left(|\Psi|^2 - q\right)dx \quad (2.18c)$$

and

$$S = -\frac{i}{2}\int_{-\infty}^{+\infty}\left(\frac{1}{\Psi}\Psi_x - \frac{1}{\Psi^*}\Psi_x^*\right)dx \quad (2.18d)$$

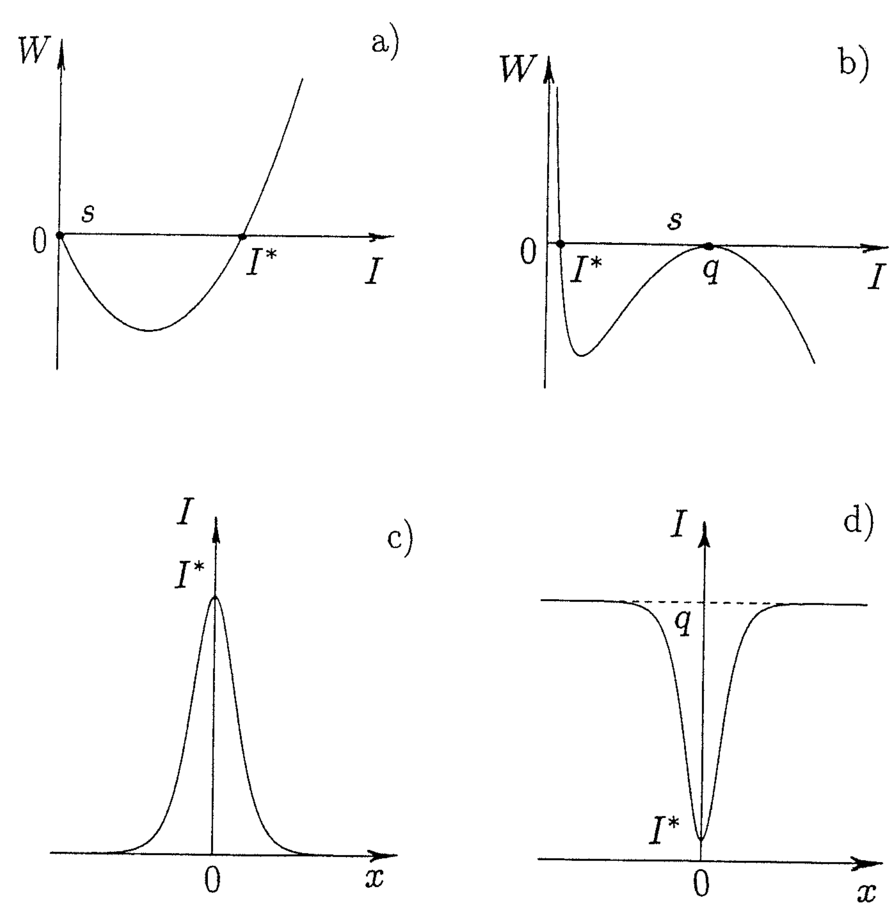

Fig.3. An equivalent potential $W(I)$, where $I = \Phi^2$, for the stationary-wave equation (2.16) with $F(I) = \sigma I$, where $\sigma = +1$ (a) and $\sigma = -1$ (b), as well as the bright soliton (c) and dark soliton (d) solutions corresponding to the separatrix trajectories 's'. Parameters I^* and q stand for soliton amplitude and intensity of the continuous-wave background, respectively.

We note that the last constant of motion is a Casimir functional, i.e. it does not affect the variational problem (2.6) written for the variable Ψ [i.e. (2.14)]. Nevertheless, the Lyapunov functional Λ can be expressed in the variables Φ and $\chi = \Theta_x$ and, then, variations of (2.17) with respect to these new variables produce (2.15) and (2.16).

Although the Lyapunov functional (2.17) involves three parameters Ω, V and ν both the bright and dark solitons are effectively *1-parameter* soliton solutions. Indeed, for bright solitons $\nu = 0$ and we apply the transformation

$$\Psi = \tilde{\Psi}(x - 2Vt, t) \exp[iVx + i(\omega - V^2)t] \tag{2.19a}$$

Then, the invariants and the Lyapunov functional transform as follows,

$$\begin{aligned} N &= N[\tilde{\Psi}] \\ P &= P[\tilde{\Psi}] - 2VN[\tilde{\Psi}] \\ H &= H[\tilde{\Psi}] - VP[\tilde{\Psi}] + V^2 N[\tilde{\Psi}] \\ \Lambda &= H[\tilde{\Psi}] + \omega N[\tilde{\Psi}] \end{aligned}$$

For dark solitons we apply a different transformation,

$$\Psi = \tilde{\Psi}(x - 2Vt, t) \exp[i(V - \nu)x + iF(q)t - i(V - \nu)^2 t] \tag{2.19b}$$

Then, neglecting the background (vacuum) values of the invariants (2.18) we find a new (renormalized) Lyapunov functional (see [53] for a physical motivation),

$$\Lambda = H^r[\tilde{\Psi}] + \nu P^r[\tilde{\Psi}]$$

where $H^r = H + F(q)N$ and $P^r = P + qS$. The introduction of the renormalized energy H^r and momentum P^r becomes possible because the parameter q is specified by the boundary conditions and can be considered as a constant.

Now we consider the linearized problem for the bright soliton solutions. After a substitution $\tilde{\Psi}(x,t) = \Phi(x) + [\delta u_1(x) + i\delta u_2(x)]\exp[\lambda t]$ into (2.19a) and then to (1.3), the linearized problem has the form

$$\mathcal{L}_1 \delta u_1 = -\lambda \delta u_2, \quad \mathcal{L}_0 \delta u_2 = \lambda \delta u_1 \tag{2.20a}$$

where $\mathcal{L}_0 = -\partial_x^2 - F(\Phi^2) + \omega$ and $\mathcal{L}_1 = \mathcal{L}_0 - 2\Phi^2 F'(\Phi^2)$. Using these operators we present the second variation $\delta^2\Lambda$ as a superposition of two quadratic forms $\delta^2\Lambda = \frac{1}{2}\langle \delta u_1, \mathcal{L}_1 \delta u_1 \rangle + \frac{1}{2}\langle \delta u_2, \mathcal{L}_0 \delta u_2 \rangle$. The second quadratic form is positive definite because the neutral mode $\delta u_2 \sim \Phi$ is even and nodeless (ground state). On the other hand, the operator $\mathcal{L}_1$ has a one-node neutral mode $\delta u_1 \sim \Phi_x$ and, according to the oscillation theorem, satisfies condition **B** (see Fig.1). Therefore, Theorem 2.2 can be applied and the stability of bright soliton solutions follows from concavity of the function $U = U(\omega) = \Lambda[\Phi]$ with respect to the renormalized parameter ω [5, 9, 36]. The latter condition is equivalent to the condition for the power $N_s = N_s(\omega) = N[\Phi]$ to be increasing. The same conclusion was found for the bright solitons in the cKG equation (1.4) [37]. Using (2.18c) we find the function $N_s(\omega)$ for the example of bright solitons shown in Fig.3(c), $N_s = 2\sqrt{\omega}$. Therefore, the bright solitons are stable in the framework of the NLS equation (1.3) for $F(I) = I$.

The linearized problem for dark solitons is more complicated. After the substitution $\tilde{\Psi}(x,t) = \Psi_s(x) + \delta\psi(x)\exp[\lambda t]$ into (2.19b) and (1.3) the linearized problem can be written in the complex representation

$$\mathcal{L}\delta\psi = i\lambda\delta\psi \tag{2.20b}$$

where $\mathcal{L} = -\partial_x^2 + 2i\nu\partial_x + F(q) - F(|\Psi_s|^2) - F'(|\Psi_s|^2)\left(|\Psi_s|^2 + \Psi_s^2(^*)\right)$. The second variation is given by $\delta^2\Lambda = \frac{1}{4}\left[\langle \delta\psi^*, \mathcal{L}\delta\psi \rangle + \langle \delta\psi, \mathcal{L}^*\delta\psi^* \rangle\right]$. The spectrum of the operator $\mathcal{L}$ in (2.20b) includes a neutral mode $\delta\psi \sim \Psi_{sx}$, a unique localized eigenfunction for negative eigenvalues and also a branch of continuous spectrum starting from the nonlocalized neutral mode, $\delta\psi \sim i\Psi_s$ (see [52] for the particular case $F(I) = -I$). In the case $\nu = 0$ the linearized problem (2.20b) reduces to (2.20a) with *real* and *odd* function Ψ_s (excepting some special cases discussed in [18, 40]). In this case, the operator $\mathcal{L}_1$ is positive definite, while the operator $\mathcal{L}_0$ possesses a discrete-spectrum mode with a negative eigenvalue and a continuous-spectrum branch starting from the zero eigenvalue. Thus, the linearized operator $\mathcal{L}$ belongs to an extension of condition **B** for which Theorem 2.2 cannot be directly applied. Nevertheless, our results reported in Example 2.3.2 indicate that the function $U = U(\nu) = \Lambda[\Psi_s]$ or, equivalently, the renormalized momentum $P_s^r = P_s^r(\nu) = P^r[\Psi_s]$ still determines the stability of dark solitons in agreement with the result of Theorem 2.2. For the dark solitons shown in Fig.3(d) we can evaluate the slope $dP_s^r/d\nu$ in a simple form,

$$\frac{dP_s^r}{d\nu} = 8\sqrt{c^2 - \nu^2} > 0$$

Therefore, the dark solitons are stable in the NLS equation (1.3) for $F(I) = -I$.

Example 2.2.3. Coupled solitons

The soliton solutions are given by the substitution,

$$\Psi_j = \Psi_{js}(x - 2Vt)\exp[i(V - \nu_j)x - i(V - \nu_j)^2 t + i\Omega_j t], \quad j = 1, 2$$

which reduces (1.5) to ordinary differential equations,

$$\Psi_{1sxx} - 2i\nu_1\Psi_{1sx} - \Omega_1\Psi_{1s} + 2\sigma_1\left(|\Psi_{1s}|^2 + \rho|\Psi_{2s}|^2\right)\Psi_{1s} = 0 \qquad (2.21a)$$

$$\Psi_{2sxx} - 2i\nu_2\Psi_{2sx} - \Omega_2\Psi_{2s} + 2\sigma_2\left(\rho|\Psi_{1s}|^2 + |\Psi_{2s}|^2\right)\Psi_{2s} = 0 \qquad (2.21b)$$

This system is not generally integrable and, moreover, it describes several families of soliton solutions which differ in the numbers of nodes for the soliton profiles. Only the nodeless (*fundamental*) soliton solutions are usually of interest because the others can be regarded as bound states of the fundamental solitons. Therefore, in this Chapter we confine ourselves only to the fundamental soliton solutions. Let us summarize the known results for different types of these solutions.

Case I: $\sigma_1 = \sigma_2 = +1$ (*bright-bright solitons*). These solitons have zero boundary conditions at infinity, i.e. $|\Psi_{js}|^2 \to 0$. Without loss of generality we can put $\nu_j = 0$, $\Omega_j = \omega_j$ $(j = 1, 2)$ and consider $\Psi_{js} = \Phi_j(x)$ as a real function. A two-parameter (ω_1, ω_2) family of fundamental solutions exists in the parameter interval $\min(s^{-2}, s^2) \le \omega_1/\omega_2 \le \max(s^{-2}, s^2)$, where $s = s(\rho)$ is the *positive* root of the quadratic equation $s(s+1) = 2\rho$ [43]. At the edges of this interval the coupled solitons degenerate to uncoupled bright solitons of each component while for the case $\omega_1 = \omega_2$ they become the symmetrical solitons with $\Phi_1 = \Phi_2$.

Case II: $\sigma_1 = -\sigma_2 = -1$ (*dark-bright solitons*). These solitons exist for the following boundary conditions, $|\Psi_{1s}|^2 \to q$ and $|\Psi_{2s}|^2 \to 0$ as $|x| \to \infty$. The set of parameters can be chosen in the form $\nu_1 = \nu_2 = V$, $\Omega_1 = -2q$ and $\Omega_2 = 2\rho q + \omega$. The fundamental soliton solutions have now two parameters V and ω. Two branches of these solutions were found numerically in [44] in the parameter domain, $0 < \omega < \omega^*(V)$ and $|V| < c = \sqrt{q}$, where the dependence $\omega = \omega^*(V)$ has been evaluated numerically.

Case III: $\sigma_1 = \sigma_2 = -1$ (*dark-dark solitons*). For these solutions we impose the boundary conditions $|\Psi_{js}|^2 \to q_j$ as $|x| \to \infty$ and put $\Omega_1 = -2(q_1 + \rho q_2)$ and $\Omega_2 = -2(\rho q_1 + q_2)$. Some particular fundamental solutions were found analytically in [45] but a general two-parameter (ν_1, ν_2) family of the dark-dark solitons has not yet been considered.

Thus, we conclude that coupled soliton solutions form *2-parameter* families. Although rigorous stability analysis has not been developed yet for these coupled solitons, we show that the asymptotic methods described in this Chapter can be successfully applied to cases I and II. Case III is more technically difficult and it is not considered here.

8.2.3 Bifurcation analysis for the onset of instability

The global criteria for soliton stability or instability have an important role in predictions of typical soliton dynamics. However, characteristic features of these dynamics should be extracted from an analysis of the spectrum of the linearized problem rather than from the global stability criteria. For example, in the case when the solitons are stable there may exist some localized eigenfunctions with imaginary values for the eigenvalue λ (the so-called *internal modes*) which make the dynamics of soliton interaction much more complicated [22]. The global criteria for soliton stability theory cannot predict the existence of these solutions to the linearized problem.

In this section we show that a bifurcation approach can be a regular tool to find stability and instability domains and to determine the existence and the parameters for both internal and unstable localized modes. The bifurcation theory is based on an asymptotic multi-scale expansion of the localized eigenfunction with respect to a small eigenvalue λ. This expansion is valid only under the assumption that the unstable eigenvalues can bifurcate from the origin of the complex plane. In the general case, this is not true. For example, in dissipative systems the unstable eigenvalues can pass the axis $\mathrm{Re}(\lambda) = 0$ with a nonzero imaginary part (see [14, 26, 56]). However, in Hamiltonian systems eigenvalues can appear only in the form of either symmetrical pairs on the real or imaginary axes or as a quartet in the complex plane. Therefore, in Hamiltonian systems the following four cases shown in Figs.4(a-d) are typical for the onset of soliton instability.

In the type I instability bifurcation the unstable eigenvalues λ arise on the real axis as a result of merging of two imaginary eigenvalues [see Fig.4(a)]. On the other hand, if the existence of localized eigenfunctions with imaginary eigenvalues is impossible, the unstable eigenvalues can still arise by emerging from the origin as in the type II instability bifurcation [see Fig.4(b)]. These two instabilities can be referred to as *translational*. In the other two types, III and IV [Fig.4(c,d)], the unstable eigenvalues arise similarly to the types I and II but they have nonzero imaginary values and form a quartet. These instabilities are referred to as *oscillatory*. Examples of the translational bifurcations were considered in [26, 49], while those of the oscillatory ones were discussed in [54, 55].

Then, on truncating the series (2.25), we obtain an approximation for the eigenvalue λ of this localized mode, $\lambda^2 = -M^{-1}N_s'(\omega)$. For $N_s'(\omega) > 0$ the pair of eigenvalues lies at the imaginary axis and the series (2.23) describes an internal oscillatory mode for the bright solitons. For $N_s'(\omega) < 0$ the eigenvalues are located at the real axis, which indicates the solitary wave instability according to the rigorous stability analysis. This case for soliton instability is type I and it is displayed in Fig.4(a).

Next, we consider the critical case $\omega = \omega_c$ when $N_s'(\omega) = 0$. In this case, we have a degenerate zero eigenvalue which leads to a weak secular instability [6]. In the time-dependent linearized problem arising after the substitution $\tilde{\Psi}(x,t) = \Phi(x) + \delta\psi(x,t)$ into (2.19a) and (1.3) this instability is associated with the (*implicit*) solution,

$$\delta\psi = iu_2^{(2)}(x) + t\frac{\partial\Phi}{\partial\omega} + \frac{i}{2}t^2\Phi \tag{2.27}$$

where $u_2^{(2)}(x)$ is given by (2.24).

Example 2.3.2. Dark solitons

Here we consider the spectral problem (2.20b) for the dark solitons to the gNLS equation (1.3) and put $\nu = V$ for convenience. This spectral problem possesses only one solvability condition, given by

$$i\lambda\int_{-\infty}^{+\infty}(\Psi_{sx}^*\delta\psi - \Psi_{sx}\delta\psi^*)\,dx = 0 \tag{2.28}$$

Let us look for solutions to (2.20b) in the form of an asymptotic series generated by the neutral mode $\delta\psi \sim \Psi_{sx}$,

$$\delta\psi = -2\Psi_{sx} + \sum_{n=1}^{\infty}\lambda^n\psi^{(n)} \tag{2.29}$$

The first-order correction can be also found explicitly,

$$\psi^{(1)} = \frac{\partial\Psi_s}{\partial V} + A(i\Psi_s)$$

Here we have added the secular neutral eigenfunction $\delta\psi \sim i\Psi_s$ with a factor A because the induced solution, $\sim \partial\Psi_s/\partial V$, has nonzero asymptotics as $x \to \pm\infty$. Then, at the leading order the constraint (2.28) determines the bifurcation criterion for the dark solitons,

$$\lambda^2\frac{dP_s^r}{dV} = 0$$

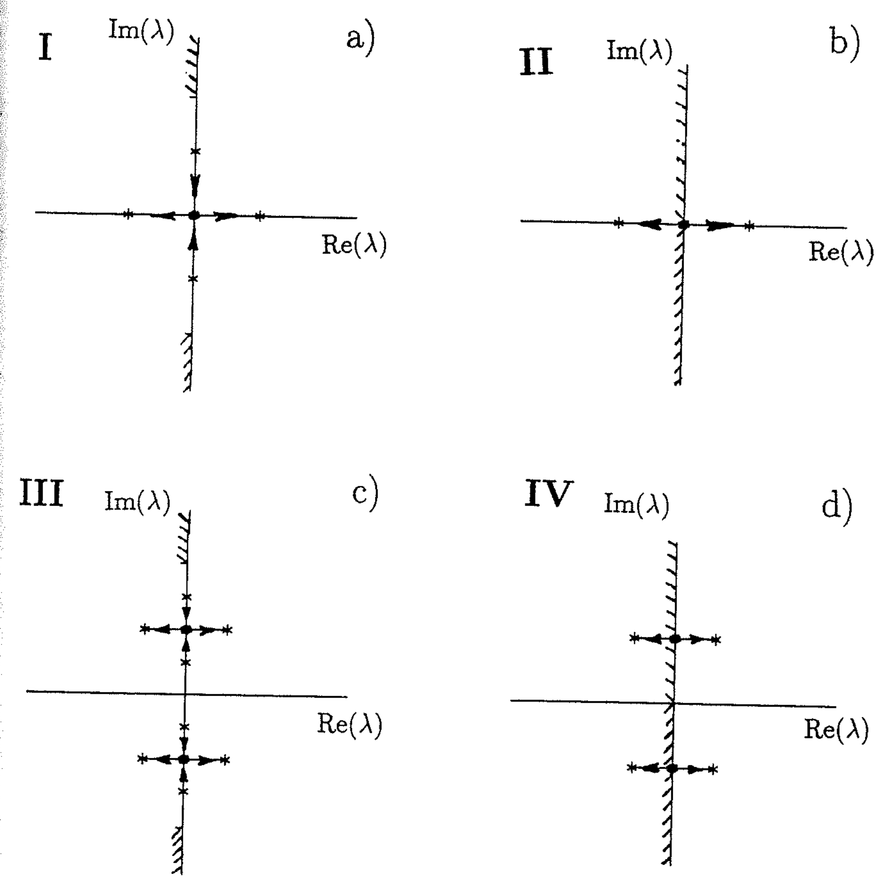

Fig.4(a)-(d). Four characteristic types of instability bifurcation of soliton solutions. Here λ is a complex eigenvalue for the linearized soliton problem, stars (*) and dashed (///) regions depict localized modes and branches of continuous spectrum, respectively.

where $P_s^r = P_s^r(V) = P^r[\Psi_s]$ [see (2.18b)]. Therefore, according to the variational problem for dark solitons, instabilities can arise only if $P_s^{r\prime}$ vanishes for a critical value $V = V_c$, where $P_s^{r\prime} = dP_s^r/dV$.

Next, we express the second-order correction implicitly,

$$\psi^{(2)} = \tilde{\psi}^{(2)}(x) - \frac{q}{2c^2}A\frac{\partial \Psi_s}{\partial q} + B\left(ix\Psi_s - \frac{\partial \Psi_s}{\partial V} + \frac{qV}{c^2}\frac{\partial \Psi_s}{\partial q}\right) \tag{2.30}$$

where c^2 is given in Assumption 2.4(ii) and $\tilde{\psi}^{(2)}(x)$ is an induced solution to the linearized problem (2.20b). It can be shown (see [18]) that the function $\tilde{\psi}^{(2)}$ grows linearly at infinity and, as a result, we have to add the secular neutral eigenfunction of the operator $\mathcal{L}$ which is described by the last term in (2.30) with the factor B. It is remarkable that the function $\tilde{\psi}^{(2)}$ does not affect the condition (2.28) up to the third-order terms while the nonlocalized eigenfunctions produce a nontrivial contribution to the third order. Therefore, assuming now that $P_s^{r\prime}(V) \sim O(\epsilon)$ [or, equivalently, $V - V_c \sim O(\epsilon)$] we find the expansion of (2.28) with respect to small λ [18],

$$\lambda^2\frac{dP_s^r}{dV} + \lambda^3\left[\left(\frac{dN_s}{dV} - \frac{qV}{2c^2}\frac{dS_s}{dV}\right)(A - 2VB) + \left(2V\frac{dN_s}{dV} - q\frac{dS_s}{dV}\right)B\right] + O(\lambda^4) = 0 \tag{2.31}$$

where $N_s = N_s(V) = N[\Psi_s]$ and $S_s = S_s(V) = S[\Psi_s]$ [see (2.18c,d)]. Here we have used the relations following from (2.7),

$$\begin{aligned} \frac{dP_s^r}{dV} + 2N_s &= q\frac{dS_s}{dV} - 2V\frac{dN_s}{dV} \\ \frac{\partial P_s^r}{\partial q} - S_s &= V\frac{dS_s}{dV} - \frac{2c^2}{q}\frac{dN_s}{dV} \end{aligned}$$

In order to close (2.31) we have to find the constants A and B. To do this, we consider the problem (2.20b) in the limit $x \to \pm\infty$, when

$$\Psi_s \to \Psi_{s\infty}^{\pm} = \sqrt{q}\exp\left[\pm\frac{i}{2}S_s\right]$$

In this limit, the problem (2.20b) reduces to the linear system with constant coefficients,

$$\mathcal{L}_\infty^{\pm}\delta\psi_\infty^{\pm} = i\lambda\delta\psi_\infty^{\pm} \tag{2.32}$$

where $\mathcal{L}_\infty^{\pm} = -\partial_x^2 + 2iV\partial_x + 2c^2[1 + \exp[\pm iS_s](^*)]$. The general solution to this problem is given by the superposition of four exponential functions, $\delta\psi_\infty^{\pm} = \sum_{n=1}^4 (u_n + iw_n)\exp[\mu_n x]$, where μ_n $(n = 1,2,3,4)$ are the roots of the equation,

$$(\mu^2 - 2c^2)^2 + (2V\mu - \lambda)^2 - 4c^4 = 0 \tag{2.33}$$

while the coefficients u_n and w_n are related by a simple expression which is not important for our analysis. Equation (2.33) admits four possible solutions for μ. However, two solutions are not small in the limit $\lambda \to 0$. They represent the exponentially growing and decaying modes which can be neglected in the limit $x \to \pm\infty$ subject to the condition that the eigenfunction $\delta\psi(x)$ is not exponentially diverging. The other two solutions of (2.33) are small in the limit $\lambda \to 0$ and they have the form $\mu \approx \mu_\pm\lambda$, where

$$\mu_\pm = \pm\frac{1}{2(c \pm V)} \tag{2.34}$$

so that $\mu_+ > 0$ and $\mu_- < 0$. We are interested in localized solutions to the problem (2.20b) in the limit $x \to \pm\infty$. Let the eigenvalue λ have a *positive* real part. Then, because of the neglect of the fast-varying modes with nonsmall μ, the localized solutions to (2.32) can be represented only by the unique form, $\delta\psi_\infty^{\pm} \sim \exp[\lambda\mu_\mp x] \sim [1 + \lambda\mu_\mp x + O(\lambda^2)]$.

Let us display the representation of $\delta\psi_\infty^{\pm}$ following from the asymptotic series (2.29) in the limit $x \to \pm\infty$ (see [18]),

$$\psi_\infty^{\pm(0)} = 0, \quad \psi_\infty^{\pm(1)} = \left(A \pm \frac{1}{2}\frac{dS_s}{dV}\right)i\Psi_{s\infty}^{\pm} \tag{2.35a}$$

and

$$\psi_\infty^{\pm(2)} = \left(B \mp \frac{1}{2(c^2 - V^2)}\left[\frac{c^2}{q}\frac{dN_s}{dV} + \frac{V}{2}\frac{dS_s}{dV}\right]\right)ix\Psi_{s\infty}^{\pm} \tag{2.35b}$$

The condition for the limiting asymptotic series to be convergent leads to the equation $\psi_\infty^{\pm(2)} = \mu_\mp x\psi_\infty^{\pm(1)}$ which determines the constants A and B as

$$A = \frac{c}{q}\frac{dN_s}{dV}, \quad B = -\frac{c}{2q(c^2 - V^2)}\left(V\frac{dN_s}{dV} + \frac{q}{2}\frac{dS_s}{dV}\right) \tag{2.36}$$

As a result, the expansion (2.31) has the explicit form,

$$\lambda^2\frac{dP_s}{dV} + \lambda^3 K + O(\lambda^4) = 0 \tag{2.37}$$

where

$$K = \left[\frac{c}{q}\left(\frac{dN_s}{dV}\right)^2 + \frac{q}{4c}\left(\frac{dS_s}{dV}\right)^2\right]\Bigg|_{V=V_c} > 0 \tag{2.38}$$

Therefore, the instability follows from (2.37) for $P_s^{r\prime}(V) < 0$. Further, we can also construct the symmetrical solution with *negative* real λ. Thus, inside the instability domain the eigenvalues λ are located at the real axis in the form of a symmetrical pair and they are given approximately by $\lambda \sim \mp K^{-1} P_s^{r\prime}(V)$. For the stable case $P_s^{r\prime}(V) > 0$ the asymptotic expansions do not produce a convergent eigenfunction to the linearized problem because the functions $\delta\psi_\infty^\pm$ are slowly divergent. This situation corresponds to a type II bifurcation [Fig.4(b)].

For the critical case $P_s^{r\prime}(V) = 0$, the time-dependent linearized problem arising after the substitution $\tilde{\Psi}(x,t) = \Psi_s(x) + \delta\psi(x,t)$ into (2.19b) and (1.3) produces an implicit solution which is, however, secularly divergent,

$$\delta\psi = \psi^{(2)}(x) + t\psi^{(1)}(x) + t^2 \Psi_{sx} \tag{2.39}$$

where $\psi^{(1)}(x)$ and $\psi^{(2)}(x)$ are found in the analysis above.

Thus, we conclude that there are two cardinally different types of the bifurcation technique whether the spectral problem generates weakly converging eigenfunctions or not. As a matter of fact, this difference is caused by the different structure of the continuous spectrum which is usually located along the imaginary axis of λ. The continuous spectrum is shown in Figs.4(a-d) by dashed regions. If the continuous spectrum is bounded away from the zero value of λ [see Fig.4(a)] the corrections to the asymptotic expansions converge exponentially. In the opposite case, when the continuous spectrum emerges from the zero value [see Fig.4(b)], the corrections are secular and the asymptotic series is weakly converging. As a result, the type I and II instability bifurcations have different characteristic features for the construction of a spectrum to the linearized problem. In one case, which is inherent to bright solitons in the gNLS and cKG equations, a pair of two imaginary eigenvalues located inside the gap for the continuous spectrum merge and bifurcate to the real axis [see Fig.4(a)]. In the other case, which is met for long-wave and dark solitons, a pair of real eigenvalues exist only inside the instability domain and it disappears at the critical value, being merged with a branch of the continuous spectrum [see Fig.4(b)].

8.3 Asymptotic multi-scale expansion methods

8.3.1 General technique

In this part we show that the linear bifurcation analysis for translational instabilities can be further generalized to describe the nonlinear stability or instability development of the solitary waves. This can be done in the framework of soliton perturbation theory which is based on the assumption of a slow (adiabatic) evolution of the solitary wave shape so that only the (free) parameters of the soliton are varying. Indeed, we have seen in Section 8.2.3 that near the instability threshold the leading order of the unstable eigenfunction leads to just renormalizing translations of the solitary wave solutions along their phase and free parameters. Therefore, we can define the soliton orbit according to Definition 1.3 and include these leading-order terms in the varying profile of the soliton solutions. Then, we present small corrections to the solitary wave shape in the form of an asymptotic series and find from analysis of the corrections to this series a system of asymptotic equations describing the nonlinear dynamics of stable and unstable solitons. These nonlinear equations generalize the linear algebraic equation for the instability eigenvalue λ.

It is remarkable that the asymptotic equations for the soliton dynamics can also be found from analysis of the conserved quantities for the given evolution equation. As a result, the system of asymptotic equations may resemble the equations for motion for an equivalent finite-dimensional system of particles which was discussed in Section 8.2.1. However, for different types of translational instabilities we find different finite-dimensional systems. For the type I instability bifurcation the system of dynamical equations is conservative because radiation-induced effects are very weak compared to other (e.g. inertial) effects. On the other hand, for the type II instability bifurcation the radiation-induced effects are dominating and, in this case, the governing equations become completely dissipative. Thus, the two types of the translational instabilities should be treated differently.

In the rest of this Section, let us summarize the formal scheme for the asymptotic multi-scale expansion technique.

1. Assume the (free) soliton parameters are slowly varying functions of the time $T = \epsilon t$, where $\epsilon \ll 1$, and define the soliton orbit phases as integrals of the free parameters with respect to the time T.

2. Look for solutions of a given evolution equation in the form of an asymptotic series and impose the stationary soliton solution as a leading-order term of this series.

3. Find the first-order correction to the soliton shape from the inhomogeneous linearized problem under the bifurcation condition.

4(a). If the first-order correction is localized, find the finite-dimensional conservative system for the soliton parameters by means of an asymptotic expansion of the Lyapunov functional.

4(b). If the first-order correction is secular (e.g. contains a trailing shelf or

grows slowly in space), then
(i) introduce the (secular) eigenfunctions of the homogeneous linearized problem to the first-order correction;
(ii) solve the radiation problem outside the solitary wave core;
(iii) find the finite-dimensional dissipative system for the soliton parameters from the balance equations written for the densities of the conserved quantities.

8.3.2 'Conservative' equations for soliton instabilities

In this Section we consider the finite-dimensional conservative system describing variations of parameters of soliton solutions which are supposed to satisfy the variational problem (2.6). This system governs soliton instabilities in the vicinity of a type I bifurcation [see Fig.4(a)]. In the general case, this system can be represented through the conserved energy E given by

$$\frac{1}{\epsilon^2}\left[E - U(\omega)\right] = \frac{1}{2}\sum_{i,j} M_{ij}(\omega)\frac{d\omega_i}{dT}\frac{d\omega_j}{dT} \tag{3.1}$$

where an effective potential energy U is defined by

$$U = H_s(\omega) + \sum_{j=1}^{n}\omega_j\left[N_{js}(\omega) - N_j\right] \tag{3.2}$$

Here $T = \epsilon t$, $\epsilon \ll 1$, $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is a set of soliton parameters, N_j is a constant value of the integral invariant $N_j[u]$ associated with the parameter ω_j, E is a constant value for the Hamiltonian $H[u]$, and the subscripts 's' stand for these invariants evaluated at the soliton solutions. Below we give two examples of the derivation of the conservative system (3.1),(3.2) for $n = 1$ and $n = 2$ according to the general asymptotic scheme applied with the step **4(a)**.

Example 3.2.1. Bright solitons

Here we consider the cKG equation (1.4) which possesses bright soliton solutions according to the analysis described in Example 2.2.2. These solutions have two parameters V and Ω. However, the parameter V can be removed from the stability problem by means of a simple Lorentz transformation,

$$\xi = \frac{x - Vt}{\sqrt{1-V^2}}, \quad \tau = \sqrt{1-V^2}\,t, \quad |V| < 1 \tag{3.3}$$

which reduces (1.4) to the form,

$$(1-V^2)\Psi_{\tau\tau} - 2V\Psi_{\tau\xi} - \Psi_{\xi\xi} + \omega_0^2\Psi - F(|\Psi|^2)\Psi = 0 \tag{3.4}$$

The bright soliton solutions are given now by the substitution

$$\Psi = \Psi_s(\xi)\exp[i\omega\tau], \quad \Psi_s = \Phi(\xi)\exp[-iV\omega\xi]$$

where the real function Φ satisfies the equation,

$$\Phi_{\xi\xi} + \left[F(\Phi^2) + \omega^2 - \omega_0^2\right]\Phi = 0 \tag{3.5}$$

The bright-soliton solutions to this equation exist if the function $F(I)$ satisfies condition (ii) in Assumption 2.3 and $|\omega| < \omega_0$. We define the Lyapunov functional in the form,

$$\Lambda = H[\Psi, \Upsilon] + \omega N[\Psi, \Upsilon] \tag{3.6}$$

where H and N are given by

$$H = \frac{1}{2}\int_{-\infty}^{+\infty}\left[|\Psi_\xi|^2 + \omega_0^2|\Psi|^2 - \int_0^{|\Psi|^2} F(I)dI + (1-V^2)|\Upsilon|^2\right]d\xi \tag{3.6a}$$

$$N = \frac{i}{2}\int_{-\infty}^{+\infty}\left[(1-V^2)(\Psi^*\Upsilon - \Psi\Upsilon^*) - V(\Psi^*\Psi_\xi - \Psi\Psi_\xi^*)\right]d\xi \tag{3.6b}$$

and $\Upsilon = \Psi_\tau$. According to the general asymptotic scheme described in Section 8.3.1, we assume that the soliton parameter ω is a function of $T = \epsilon\tau$, where $\epsilon \ll 1$, and introduce the following asymptotic series,

$$\Psi = \left[\Phi(\xi;\omega) + \epsilon\psi^{(1)}(\xi;T) + O(\epsilon^2)\right]\exp[-iV\omega\xi + i\theta] \tag{3.7}$$

where $\theta = \epsilon^{-1}\int_0^T \omega(T')dT'$ represents the soliton orbit. Then it follows from (3.4) that the first-order correction $\psi^{(1)} = u_1^{(1)} + iu_2^{(1)}$ satisfies the inhomogeneous linearized problem,

$$\mathcal{L}_1 u_1^{(1)} = \left[2V\left(\frac{\partial\Phi}{\partial\omega}\right)_\xi - 2V\omega\xi\Phi\right]\frac{d\omega}{dT} \tag{3.8a}$$

$$\mathcal{L}_0 u_2^{(1)} = \left[-2\omega\frac{\partial\Phi}{\partial\omega} - (1+V^2)\Phi - 2V^2\xi\Phi_\xi\right]\frac{d\omega}{dT} \tag{3.8b}$$

where operators $\mathcal{L}_1$ and $\mathcal{L}_0$ are defined below (2.20a) by replacing ω to $(\omega_0^2 - \omega^2)$. Localized solutions to both equations (3.8a,b) exist under the following bifurcation equation,

$$\frac{dN_s}{d\omega} = 0 \tag{3.9}$$

where $N_s = N_s(\omega) = N[\Psi_s, \Upsilon_s]$ and $\Upsilon_s = i\omega\Psi_s$. Because a weak (secular) growth of the first-order correction $\psi^{(1)}$ is absent under the bifurcation condition (3.9) we proceed to use the general asymptotic scheme according to step **4(a)** and expand the Lyapunov functional (3.6) under the asymptotic series (3.7). After some calculations carried out with the help of the linearized equations (3.8a,b) we find the conservative model (3.1),(3.2) for $n = 1$ with the coefficient $M = M_{11}(\omega)$ given by

$$M = \int_{-\infty}^{+\infty} \left[\left(\frac{\partial \Phi}{\partial \omega} \right)^2 + \left(\frac{1}{\Phi} \int_0^{\xi} \frac{\partial(\omega\Phi^2)}{\partial \omega} d\xi' \right)^2 \right] d\xi > 0 \tag{3.10}$$

Applying the same analysis to the gNLS equation (1.3) (see [17] for details), we find the same model (3.1),(3.2) but with the integral invariants H and N given by (2.18a,c) and the coefficient M in the form [cf.(2.26)],

$$M = \int_{-\infty}^{+\infty} \left[\frac{1}{\Phi} \int_0^x \Phi \frac{\partial \Phi}{\partial \omega} dx' \right]^2 dx > 0 \tag{3.11}$$

where $\Phi(x)$ satisfies (2.16) under Assumption 2.3.

Example 3.2.2. Bright-bright solitons

Here we consider the coupled NLS equations (1.5) for $\sigma_1 = \sigma_2 = +1$ and impose zero boundary conditions at infinity for the functions Ψ_1 and Ψ_2. Then, the system (2.21) for bright-bright soliton solutions at $\nu_j = 0$ and $\Omega_j = \omega_j$ ($j = 1, 2$) is equivalent to the variational problem (2.6) for the following Lyapunov functional,

$$\Lambda = H[\Psi_1, \Psi_2] + \omega_1 N_1[\Psi_1] + \omega_2 N_2[\Psi_2] \tag{3.12}$$

where

$$H = \frac{1}{2} \int_{-\infty}^{+\infty} \left[|\Psi_{1x}|^2 + |\Psi_{2x}|^2 - |\Psi_1|^4 - 2\rho|\Psi_1|^2|\Psi_2|^2 - |\Psi_2|^4 \right] dx \tag{3.12a}$$

and

$$N_j = \frac{1}{2} \int_{-\infty}^{+\infty} |\Psi_j|^2 dx \tag{3.12b}$$

Using the same analysis as in Example 3.2.1, we represent solutions to (1.5) in the form of an asymptotic series,

$$\Psi_j = \left[\Phi_j(x; \omega_1, \omega_2) + \epsilon \psi_j^{(1)}(x; T) + O(\epsilon^2) \right] \exp[i\theta_j] \tag{3.13}$$

where $\epsilon \ll 1$, $T = \epsilon t$, $\omega_j = \omega_j(T)$, and $\theta_j = \epsilon^{-1} \int_0^T \omega_j(T') dT'$. The functions $\Phi_j(x)$ represent the bright-bright soliton solutions, while $\psi_j^{(1)}$ can be found from a linearized problem. The fundamental soliton solutions are described by the *nodeless even* functions Φ_j (see Example 2.2.3: Case I). Using this fact and the system (1.5), we present the first-order corrections $\psi_j^{(1)} = iu_{j2}^{(1)}$ explicitly [cf. (2.24)],

$$u_{j2}^{(1)} = -\Phi_j \int_0^x \frac{dx'}{\Phi_j^2} \int_0^{x'} \Phi_j \frac{d\Phi_j}{dT} dx'' \tag{3.14}$$

where

$$\frac{d\Phi_j}{dT} = \sum_{i=1}^{2} \frac{\partial \Phi_j}{\partial \omega_i} \frac{d\omega_i}{dT}$$

It follows from (3.14) that the first-order corrections are localized under the following bifurcation condition,

$$\frac{\partial N_{1s}}{\partial \omega_1} \frac{\partial N_{2s}}{\partial \omega_2} - \frac{\partial N_{1s}}{\partial \omega_2} \frac{\partial N_{2s}}{\partial \omega_1} = 0 \tag{3.15}$$

where $N_{js} = N_{js}(\omega_1, \omega_2) = N_j[\Phi_j]$ for $j = 1, 2$. Near this instability threshold, the soliton dynamics can be described by the conservative model (3.1),(3.2) for $n = 2$ with the coefficients M_{ij} given by

$$M_{ij} = \sum_{k=1}^{2} \int_{-\infty}^{+\infty} \left[\frac{1}{\Phi_k^2} \left(\int_0^x \Phi_k \frac{\partial \Phi_k}{\partial \omega_i} dx' \right) \left(\int_0^x \Phi_k \frac{\partial \Phi_k}{\partial \omega_j} dx' \right) \right] dx \tag{3.16}$$

We would like to point out that the same system can also be obtained for the coupled NLS equations (1.5) with generalized nonlinear functions.

For both the examples presented above we have found that the quadratic form generated by the coefficients M_{ij} is positive definite [see (3.10),(3.11), and (3.16)] so that $\Lambda - \Lambda[u_s] \geq 0$. This result corresponds to Theorem 5.2 proved in [10] for this class of stability problems. Further, the values of N_j and E are constant in time. Thus, the parameters ω_j resemble the generalized coordinates q_j in a conservative finite-dimensional system with the energy (2.1) while the Lyapunov functional produces an effective potential energy $U(\omega)$ in this system according to (3.2).

Let us apply Theorem 2.1 to the stability of soliton solutions in the framework of this finite-dimensional system. Denote the parameters of a soliton solution as $\omega_j = \omega_{j0}$. Then, the potential energy $U(\omega)$ is extremal at this point if the constants N_j are related to the values $\omega = \omega_0$ according to $N_j = N_{js}(\omega_0)$.

Expanding $U(\omega)$ up to the quadratic terms and using the relations (2.7) we find the potential energy quadratic form,

$$\delta^2 U = \frac{1}{2}\sum_{i,j} \frac{\partial N_{is}}{\partial \omega_j}\Big|_{\omega=\omega_0} \delta\omega_i \delta\omega_j \tag{3.17}$$

where $\delta\omega_j = \omega_j - \omega_{j0}$.

First, we consider the case $n = 1$. In this case, it follows from (3.17) that the soliton solution corresponds to a local minimum of $U(\omega)$ if

$$\frac{dN_s}{d\omega}\Big|_{\omega=\omega_0} > 0 \tag{3.18}$$

This result completely corresponds to the general stability theorem (see Theorem 2.2). When the first derivative of N_s vanishes, a type I bifurcation occurs [see Fig.4(a)] and the soliton becomes a local maximum of $U(\omega)$.

Let us consider now the case $n = 2$. Then, the soliton corresponds to a local minimum of $U(\omega_1, \omega_2)$ if the quadratic form (3.17) is positive definite. The latter condition is equivalent to

$$\frac{\partial N_{js}}{\partial \omega_j}\Big|_{\omega=\omega_0} > 0, \quad j = 1, 2 \tag{3.19a}$$

and

$$\left[\frac{\partial N_{1s}}{\partial \omega_1}\frac{\partial N_{2s}}{\partial \omega_2} - \frac{\partial N_{1s}}{\partial \omega_2}\frac{\partial N_{2s}}{\partial \omega_1}\right]\Big|_{\omega=\omega_0} > 0 \tag{3.19b}$$

The first condition (3.19a) implies that the two-component solitons are stable in the framework of each partial component, while the second one (3.19b) determines the stability of the solitons under the action of cross-component interactions. When the second condition is violated, a bifurcation (3.15) takes place which generates a pair of real eigenvalues to the linearized problem. In this case, the soliton realizes a saddle point of the potential energy $U(\omega_1, \omega_2)$. Finally, if the first condition is violated for all components but the second one is valid, two pairs of eigenvalues lie on the real axis and the soliton realizes a local maximum of $U(\omega_1, \omega_2)$.

Obviously the analysis presented above can be generalized for an arbitrary value of n according to Theorem 2.1. We note that Theorem 2.1 is valid if the quadratic form standing on the right-hand side of (3.1) is positive definite. In this case, the instability of the soliton solutions can arise only according to the type I (translational) bifurcation [see Fig.4(a)]. Indeed, it was proved [10-12] that only this translational instability is possible for the examples presented above. We suppose that for other examples where the (kinetic) quadratic form is not positive definite, the type III (oscillatory) bifurcation [see Fig.4(c)] might occur even in the region where Theorem 2.2 predicts stability of soliton solutions, i.e. conditions (3.18) or (3.19) are satisfied (see [54]). In those cases, the criterion of soliton stability as well as the underlying finite-dimensional conservative model should be somehow modified.

8.3.3 'Dissipative' equations for soliton instabilities

In this Section we consider the dissipative finite-dimensional equations which can be derived for the type II bifurcation [see Fig.4(b)]. As earlier, we suppose that the stationary soliton solutions satisfy the variational problem (2.6) but, for convenience, we change the notation ω_j for the soliton parameters to V_j, and N_j for the associated integral invariants to P_j. The set of dissipative equations is given in terms of the energy $E(T)$ of the system,

$$E(T) = U(V;T) = H_s(V) + \sum_{j=1}^{n} V_j \left[P_{js}(V) - P_j(T)\right] \tag{3.20}$$

and by balance equations for the varying quantities $P_j(T)$ and $E(T)$,

$$\frac{1}{\epsilon}\frac{dP_j}{dT} = 2F_j(V, \dot{V}) \quad \text{for } j = 1, 2, ..., n \tag{3.21a}$$

$$\frac{1}{\epsilon}\frac{dE}{dT} = -2\left[F(V, \dot{V}) + \sum_{j=1}^{n} V_j F_j(V, \dot{V})\right] \tag{3.21b}$$

Here the set of the soliton parameters $V = (V_1, V_2, ..., V_n)$ depends on time $T = \epsilon t$, where $\epsilon \ll 1$, the subscript 's' denotes the integral invariants evaluated at the stationary soliton solutions, and the functions F and F_j are quadratic with respect to dV/dT,

$$F_m = \frac{1}{2}\sum_{i,j} K_{ij}^m(V)\frac{dV_i}{dT}\frac{dV_j}{dT} \tag{3.22a}$$

$$F = \frac{1}{2}\sum_{i,j} K_{ij}(V)\frac{dV_i}{dT}\frac{dV_j}{dT} \tag{3.22b}$$

This system resembles the dissipative system (2.1),(2.3) for the motion of particles with generalized coordinates V_j in the potential $U(V;T)$ under the action of Rayleigh's dissipative function F. However, the inertial effects are beyond the leading-order approximation because the kinetic energy terms are of $O(\epsilon^2)$ while the system (3.20)-(3.22) appears in the order of $O(\epsilon)$. Using the Lagrange equations (2.4) with $L = -\epsilon^{-1}U$ we obtain dynamical equations in the form,

$$\frac{1}{\epsilon}\left[P_j(T) - P_{js}(V)\right] = \sum_{i=1}^{n} K_{ij}(V)\frac{dV_i}{dT} \quad \text{for } j = 1, 2, ..., n \tag{3.23}$$

Because the local values of the quantitues $P_j(T)$ and $E(T)$ are not conservative the system of dynamical equations (3.23) is not closed and should be considered together with the balance equations (3.21a). This fact causes our system of equations to differ from the closed dissipative model considered in Section 8.2.1.

The finite-dimensional dissipative system (3.20)-(3.23) can be derived for the type II bifurcation using the general asymptotic scheme with the step **4(b)** induced due to nonlocalized corrections to the asymptotic series. The basic element of this step is a solution of the radiation problem which is related to the existence of *nonzero* Casimir functionals evaluated at the soliton solutions. In the two examples given below we explain how to solve the radiation problem for some particular cases and derive the dissipative system (3.20)-(3.23) for $n = 1$ and $n = 2$.

Example 3.3.1. Long-wave solitons

Here we consider the gBs equation (1.2) which possesses long-wave soliton solutions in the form $u = u_s(x - Vt)$. These solutions satisfy (2.9) for $\alpha = 1$, $\beta = 0$, and V replaced by $(c_0^2 - V^2)$. Therefore, the long-wave solitons to the gBs equation exist if the nonlinear function $f(u)$ satisfies the condition (ii) of Assumption 2.2 and $|V| < c_0$. Let us define the Lyapunov functional in the form $\Lambda = H[u,w] + VP[u,w]$, where the Hamiltonian $H[u,w]$ and momentum $P[u,w]$ are given by

$$H = \frac{1}{2}\int_{-\infty}^{+\infty}\left[u_x^2 - 2\int_0^u f(u)du + c_0^2u^2 + w^2\right]dx \tag{3.24a}$$

$$P = \int_{-\infty}^{+\infty} uw dx \tag{3.24b}$$

and $w_x = u_t$. Besides these two invariants, the gBs equation (1.2) has two additional (Casimir) functionals $M_u[u]$ and $M_w[w]$,

$$M_u = \int_{-\infty}^{+\infty} u dx, \quad M_w = -\int_{-\infty}^{+\infty} w dx \tag{3.24c}$$

Now we apply the general asymptotic scheme described in Section 8.3.1 and consider the soliton parameter V to depend on $T = \epsilon t$, where $\epsilon \ll 1$. Furthermore, we introduce the asymptotic series,

$$u = u_s(\xi; V) + \epsilon u_1(\xi; T) + O(\epsilon^2) \tag{3.25a}$$

$$w = w_s(\xi; V) + \epsilon w_1(\xi; T) + O(\epsilon^2) \tag{3.25b}$$

where $\xi = x - \epsilon^{-1}\int_0^T V(T')dT'$ and $w_s = -Vu_s$. Using (1.2) we find the linearized system for the first-order corrections u_1 and w_1,

$$w_{1\xi} = -Vu_{1\xi} + \left(\frac{\partial u_s}{\partial V}\right)\frac{dV}{dT} \tag{3.26a}$$

$$(\mathcal{L}u_1)_\xi = -\left(2V\frac{\partial u_s}{\partial V} + u_s\right)\frac{dV}{dT} \tag{3.26b}$$

where the operator $\mathcal{L}$ is given below (2.11) with $\alpha = 1$, $\beta = 0$, and V is replaced there by $(c_0^2 - V^2)$. First, we note that a bounded solution to these equations exists under the following bifurcation condition,

$$\frac{dP_s}{dV} = 0 \tag{3.27}$$

where $P_s = P_s(V) = P[u_s, w_s]$. However, the first-order corrections still contain secular terms (a trailing shelf) in the limit $\xi \to \pm\infty$,

$$u_1 \to U^{\pm} = C_u \mp \frac{1}{2(c_0^2 - V^2)}\left(V\frac{dM_{us}}{dV} + \frac{dM_{ws}}{dV}\right)\frac{dV}{dT} \tag{3.28a}$$

$$w_1 \to W^{\pm} = C_w - VC_u \pm \frac{1}{2(c_0^2 - V^2)}\left(c_0^2\frac{dM_{us}}{dV} + V\frac{dM_{ws}}{dV}\right)\frac{dV}{dT} \tag{3.28b}$$

where $M_{us} = M_u[u_s]$, $M_{ws} = M_w[w_s]$, and C_u and C_w are integration constants.

Now we consider the radiation problem for the gBs equation (1.2) in the asymptotic limit $\xi \to \pm\infty$ and $u \to \epsilon U^{\pm}(X,T)$, $w \to \epsilon W^{\pm}(X,T)$, where $X = \epsilon x$. It follows from (1.2) that $U^{\pm}$ and $W^{\pm}$ are related in the leading order as follows,

$$U_T^{\pm} = W_X^{\pm}, \quad W_T^{\pm} = c_0^2 U_X^{\pm} \tag{3.29}$$

These equations describe two counter-propagating waves which move to the left and to the right with the limiting long-wave velocity c_0. The long-wave solitons have velocities smaller than c_0, i.e. $|V| < c_0$, and, therefore, both waves are excited by an evolving soliton according to $U^{\pm} = U^{\pm}(X \mp c_0 T)$, $W^{\pm} = W^{\pm}(X \mp c_0 T)$. Then, the radiation conditions are given by

$$W^{\pm} = \mp c_0 U^{\pm} \tag{3.30}$$

These equations enable us to specify the constants C_u and C_w as

$$C_u = -\frac{1}{2c_0(c_0^2 - V^2)}\left[c_0^2 \frac{dM_{us}}{dV} + V\frac{dM_{ws}}{dV}\right]\frac{dV}{dT} \tag{3.31a}$$

$$C_w = \frac{1}{2c_0}\frac{dM_{ws}}{dV}\frac{dV}{dT} \tag{3.31b}$$

Finally, we find from the balance equations for the energy H and momentum P of the gBs equation (1.2) the finite-dimensional dissipative system (3.20)-(3.23) for $n = 1$ with the coefficients $K = K_{11}$ and $K_1 = K_{11}^1$ in the form,

$$K = \frac{1}{2}\left[c_0\left(\frac{dM_{us}}{dV}\right)^2 + \frac{1}{c_0}\left(\frac{dM_{ws}}{dV}\right)^2\right] > 0 \tag{3.32a}$$

$$K_1 = \frac{1}{2(c_0^2 - V^2)}\left[Vc_0\left(\frac{dM_{us}}{dV}\right)^2 + 2c_0\frac{dM_{us}}{dV}\frac{dM_{ws}}{dV} + \frac{V}{c_0}\left(\frac{dM_{ws}}{dV}\right)^2\right] \tag{3.32b}$$

Applying the same analysis to the gKdV equation (1.1) (see [19] for details) we find the same dissipative model but with the integral quantities H and P given by (2.10) and the coefficients,

$$K = \frac{1}{2}\left(\frac{dM_{us}}{dV}\right)^2 > 0, \quad K_1 = -\frac{1}{2V}\left(\frac{dM_{us}}{dV}\right)^2 < 0 \tag{3.33}$$

where M_{us} is the same as above [see (3.24c)] and we have confined ourselves to the case of positive V [see Assumption 2.2(i)]. We note that for the gKdV equation there exists only one Casimir functional M_u and only one component for the radiation field $U^{\pm}(X,T)$. Besides, for this particular case, the energy E is a constant of motion because $F + VF_1 = 0$ [see (3.21b)].

Example 3.3.2. Dark-bright solitons

Here we consider the coupled NLS equations (1.5) for $\sigma_1 = -\sigma_2 = -1$ and impose the following boundary conditions, $|\Psi_1|^2 \to q$ and $|\Psi_2|^2 \to 0$ as $|x| \to \infty$. The dark-bright soliton solutions satisfy (2.21) for $\nu_1 = \nu_2 = V$, $\Omega_1 = -2q$,

and $\Omega_2 = 2\rho q + \omega$. These solitons are stationary points for the variational problem (2.6) with the Lyapunov functional

$$\Lambda = H^r[\Psi_1, \Psi_2] + VP^r[\Psi_1, \Psi_2] + \omega N_2[\Psi_2] \tag{3.34}$$

where $H^r = H - 2qN_1 + 2\rho q N_2$, $P^r = P + qS_1$,

$$H = \frac{1}{2}\int_{-\infty}^{+\infty}\left[|\Psi_{1x}|^2 - |\Psi_{2x}|^2 + |\Psi_1|^4 + 2\rho|\Psi_1|^2|\Psi_2|^2 + |\Psi_2|^4 - q^2\right]dx \tag{3.34a}$$

$$N_1 = \frac{1}{2}\int_{-\infty}^{+\infty}\left[|\Psi_1|^2 - q\right]dx \tag{3.34b}$$

$$N_2 = -\frac{1}{2}\int_{-\infty}^{+\infty}|\Psi_2|^2 dx \tag{3.34c}$$

$$P = \frac{i}{2}\int_{-\infty}^{+\infty}\left[(\Psi_1^*\Psi_{1x} - \Psi_1\Psi_{1x}^*) - (\Psi_2^*\Psi_{2x} - \Psi_2\Psi_{2x}^*)\right]dx \tag{3.34d}$$

and

$$S_1 = -\frac{i}{2}\int_{-\infty}^{+\infty}\left[\frac{1}{\Psi_1}\Psi_{1x} - \frac{1}{\Psi_1^*}\Psi_{1x}^*\right]dx \tag{3.34e}$$

Using the same analysis as in Example 3.3.1 we represent solutions to (1.5) in the form of an asymptotic series,

$$\Psi_j = \left[\Psi_{js}(\xi; V, \omega) + \epsilon\psi_j^{(1)}(\xi; T) + O(\epsilon^2)\right]\exp[i\theta_j] \tag{3.35}$$

where

$$\xi = x - 2\epsilon^{-1}\int_0^T V(T')dT', \quad \theta_1 = -2qt + C_\theta, \quad \text{and} \quad \theta_2 = 2\rho q t + \epsilon^{-1}\int_0^T \omega(T')dT'$$

Here $C_\theta = C_\theta(T)$ is a varying constant, the functions Ψ_{js} represent the stationary (fundamental) soliton solutions while the first-order corrections $\psi_j^{(1)}$ can be found from the linearized problem in the form [cf. (2.30)],

$$\psi_j^{(1)} = \tilde{\psi}_j^{(1)}(x) + \frac{dC_\theta}{dT}\left[-\frac{q}{2c^2}\frac{\partial\Psi_{js}}{\partial q} + \rho\frac{\partial\Psi_{js}}{\partial\omega}\right]$$
$$+ C_q\left[i\xi\Psi_{js} - \frac{\partial\Psi_{js}}{\partial V} + \frac{qV}{c^2}\frac{\partial\Psi_{js}}{\partial q} - 2V(1+\rho)\frac{\partial\Psi_{js}}{\partial\omega}\right] \tag{3.36}$$

where $C_q = C_q(T)$ and $c = \sqrt{q}$. The induced solution $\tilde{\psi}_1^{(1)}$ is not exponentially divergent under the following bifurcation condition,

$$\frac{\partial P_s^r}{\partial V}\frac{\partial N_{2s}}{\partial\omega} - \frac{\partial P_s^r}{\partial\omega}\frac{\partial N_{2s}}{\partial V} = 0 \tag{3.37}$$

where $P_s^r(V,\omega)$ and $N_{2s}(V,\omega)$ stand for the functionals (3.34c,d) evaluated at the soliton solutions. However, the function $\tilde{\psi}_1^{(1)}$ is secularly (linearily) growing and we need to consider the radiation problem in the limit $\xi \to \pm\infty$ and

$$\Psi_1 \to \left[\sqrt{q} + \epsilon Q^{\pm}(X,T)\right] \exp[-2iqt + i\Theta^{\pm}(X,T)]$$
$$\Psi_2 \to 0$$

where $X = \epsilon x$ and $Q^{\pm}$ $\Theta^{\pm}$ and, additionally, $\Theta_X^{\pm}$ are generated at the moving soliton according to the limiting relations [cf. (2.35)],

$$Q^{\pm} = \frac{\sqrt{q}}{4c^2}\left(2VC_q - \frac{dC_\theta}{dT}\right) \mp \frac{1}{4\sqrt{q}(c^2-V^2)}\left[V\frac{dN_{1s}}{dT} + \frac{q}{2}\frac{dS_{1s}}{dT}\right] \tag{3.38a}$$

$$\Theta^{\pm} = C_\theta \pm \frac{1}{2}S_{1s} \tag{3.38b}$$

and

$$\Theta_X^{\pm} = C_q \mp \frac{1}{2(c^2-V^2)}\left[\frac{c^2}{q}\frac{dN_{1s}}{dT} + \frac{V}{2}\frac{dS_{1s}}{dT}\right] \tag{3.38c}$$

Here $N_{1s}(V,\omega)$ and $S_{1s}(V,\omega)$ are functionals (3.34b,e) evaluated at the soliton solutions and the derivative dN_{1s}/dT [and, similarly, dS_{1s}/dT] is defined by

$$\frac{dN_{1s}}{dT} = \frac{\partial N_{1s}}{\partial V}\frac{dV}{dT} + \frac{\partial N_{1s}}{\partial \omega}\frac{d\omega}{dT}$$

In the asymptotic limit $\xi \to \pm\infty$, the coupled NLS equations (1.5) reduce to the radiation problem,

$$Q^{\pm} = -\frac{\sqrt{q}}{4c^2}\Theta_T^{\pm} \tag{3.39a}$$

and

$$\Theta_{TT}^{\pm} - 4c^2\Theta_{XX}^{\pm} = 0 \tag{3.39b}$$

The dark-bright solitons propagate with the velocities V smaller than the limiting speed c, i.e. $|V| < c$ [see Example 2.2.3: Case II]. Therefore, a dark soliton generates two counter-propagating waves according to $Q^{\pm} = Q^{\pm}(X \mp 2cT)$, $\Theta^{\pm} = \Theta^{\pm}(X \mp 2cT)$, and the conditions for the radiation fields are

$$Q^{\pm} = \pm\frac{\sqrt{q}}{2c}\Theta_X^{\pm} \tag{3.40}$$

Using formulas (3.38) and (3.40) we find the constants $C_\theta(T)$ and $C_q(T)$ as follows [cf. (2.36)],

$$C_\theta = \frac{c}{q}N_{1s} \tag{3.41a}$$

$$C_q = -\frac{c}{2q(c^2-V^2)}\left[V\frac{dN_{1s}}{dT} + \frac{q}{2}\frac{dS_{1s}}{dT}\right] \tag{3.41b}$$

Finally, analyzing the balance equations for the quantities P^r and N_2 [see (3.34c,d)], we obtain the dissipative system (3.20)-(3.23) for $n = 2$, $V_1 = V$, $V_2 = \omega$, $P_1 = P^r$, $P_2 = N_2$, and the coefficients K_{ij}, K_{ij}^1, and K_{ij}^2 are given by

$$K_{ij} = \frac{c}{q}\frac{\partial N_{1s}}{\partial V_i}\frac{\partial N_{1s}}{\partial V_j} + \frac{q}{4c}\frac{\partial S_{1s}}{\partial V_i}\frac{\partial S_{1s}}{\partial V_j} \tag{3.42a}$$

$$K_{ij}^1 = \frac{1}{c^2-V^2}\left[\frac{cV}{q}\frac{\partial N_{1s}}{\partial V_i}\frac{\partial N_{1s}}{\partial V_j} + c\frac{\partial N_{1s}}{\partial V_i}\frac{\partial S_{1s}}{\partial V_j} + \frac{qV}{4c}\frac{\partial S_{1s}}{\partial V_i}\frac{\partial S_{1s}}{\partial V_j}\right] \tag{3.42b}$$

and

$$K_{ij}^2 = 0 \tag{3.42c}$$

The last identity implies that the value of $P_2 = N_2$ is conservative. We mention that the same form of dissipative equations can be obtained for the dark-bright solitons in the coupled NLS equations (1.5) with generalized nonlinear functions.

We have found for both the examples presented above that the quadratic form for the dissipative function F (3.22b) generated by the coefficients K_{ij} is positive definite [see (3.32a),(3.33), and (3.42a)]. Under this condition, the stability of soliton solutions for the type II bifurcation is again determined by concavity of the potential energy function U given by (3.20) in the parameter space V similarly to the analysis presented for the type I bifurcation. Indeed, in the linear approximation, variations of P_j in time can be neglected and the function $U(V)$ has the quadratic form (3.17) with the replacement of N_{js} by P_{js} and ω_j by V_j. Then, the stability of soliton solutions is described by Theorem 2.1.

However, in those cases where the quadratic form generated by K_{ij} is not positive definite the simple criterion for the soliton stability does not work. For example, for some modifications of the gBs equations (see [26, 55]) the type IV (oscillatory) bifurcation [see Fig.4(d)] of the long-wave solitons can occur in the parameter region where the potential energy quadratic form (3.17) is still positive definite. We leave the detailed study of this special oscillatory bifurcation for further work.

8.4 Scenarios of instability-induced dynamics

8.4.1 Weakly nonlinear approximation and solutions

In this part we analyse the nonlinear dynamics of soliton instabilities described by the conservative and dissipative finite-dimensional systems which were derived in Sections 8.3.2 and 8.3.3 for translational stability bifurcations. In both cases we confine ourselves to the simple case $n = 1$ when the solitons have only one effective parameter.

For the type I instability bifurcation at $n = 1$ the conservative system (3.1),(3.2) can be written in the form of the energy integral,

$$\frac{1}{\epsilon^2}[E - U(\omega)] = \frac{1}{2}M(\omega)\left(\frac{d\omega}{dT}\right)^2 \tag{4.1}$$

where

$$U = H_s(\omega) + \omega\,[N_s(\omega) - N] \tag{4.2}$$

N is a constant value of the integral invariant $N[u]$ associated with the parameter ω, E is a constant value for the Hamiltonian $H[u]$, $N_s = N_s(\omega) = N[u_s]$, $H_s = H_s(\omega) = H[u_s]$, and we assume that the coefficient $M(\omega)$ is positive. Equation (4.1) coincides with the energy conservation law for an effective particle of mass M with the coordinate ω in the potential field $U(\omega)$. The potential energy $U(\omega)$ is determined by the soliton invariant $N_s(\omega)$ according to the following differential equation [see (2.7a) for a proof],

$$\frac{dU}{d\omega} = N_s(\omega) - N \tag{4.3}$$

To determine the stability and instability domains for a given evolution equation, we evaluate the function $N_s(\omega)$ and analyse the turning points of this function, $\omega = \omega_c$, where the first derivative $N_s'(\omega_c)$ vanishes. In these points the type I (translational) bifurcation occurs according to Theorem 2.2. In this Section we consider the general case when the function $N_s(\omega)$ is not a constant. The special case, when this function is constant, is considered in the next Section.

Some general types for the function $N_s(\omega)$ are displayed in Figs.5(a)-8(a). The corresponding phase planes $(\omega, \dot{\omega})$, where $\dot{\omega} = d\omega/dT$, deduced from (4.1) are presented in Figs.5(b)-8(b). Let us discuss these cases separately.

When the second derivative N_s at the turning point $\omega = \omega_c$ does not vanish, the asymptotic model (4.1) can be simplified further in the small-amplitude approximation when the soliton parameter ω is close to ω_c. This approximation

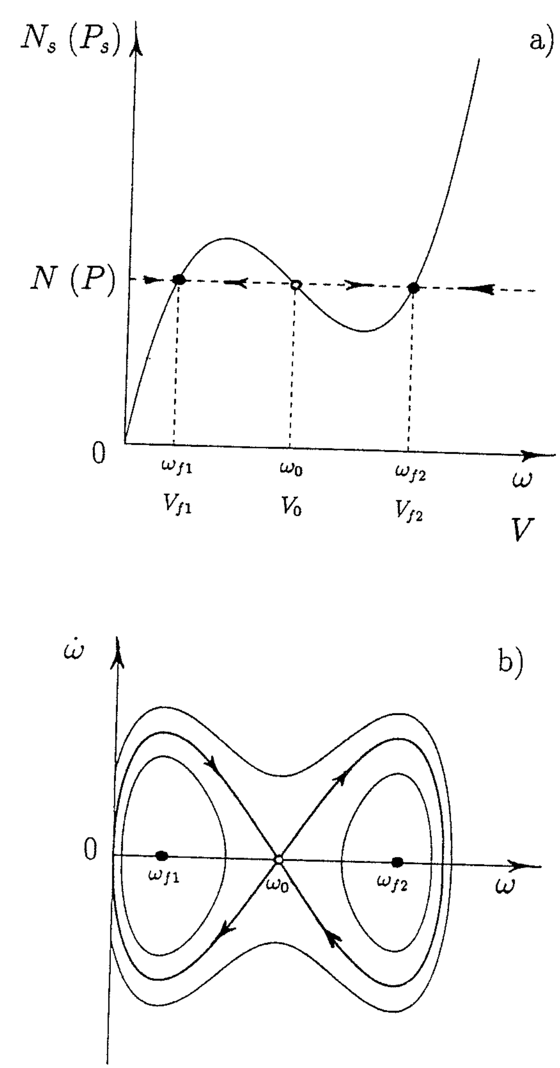

Fig.5. A typical two-branch curve for the functions $N_s(\omega)$ or $P_s(V)$ with $N_s''(\omega_c) > 0$ (a) and the corresponding phase plane $(\omega, \dot{\omega})$ for the conservative dynamical system described by (4.1) (b). Here ω_0 (ω_f) stands for the parameter ω of an unstable (stable) soliton corresponding to the value of N, while ω_c denotes the critical (bifurcation) value. Parameters V_0, V_f, and V_c designate the corresponding values for the soliton parameter V. The arrows in (a) display soliton dynamics according to the dissipative system (4.10).

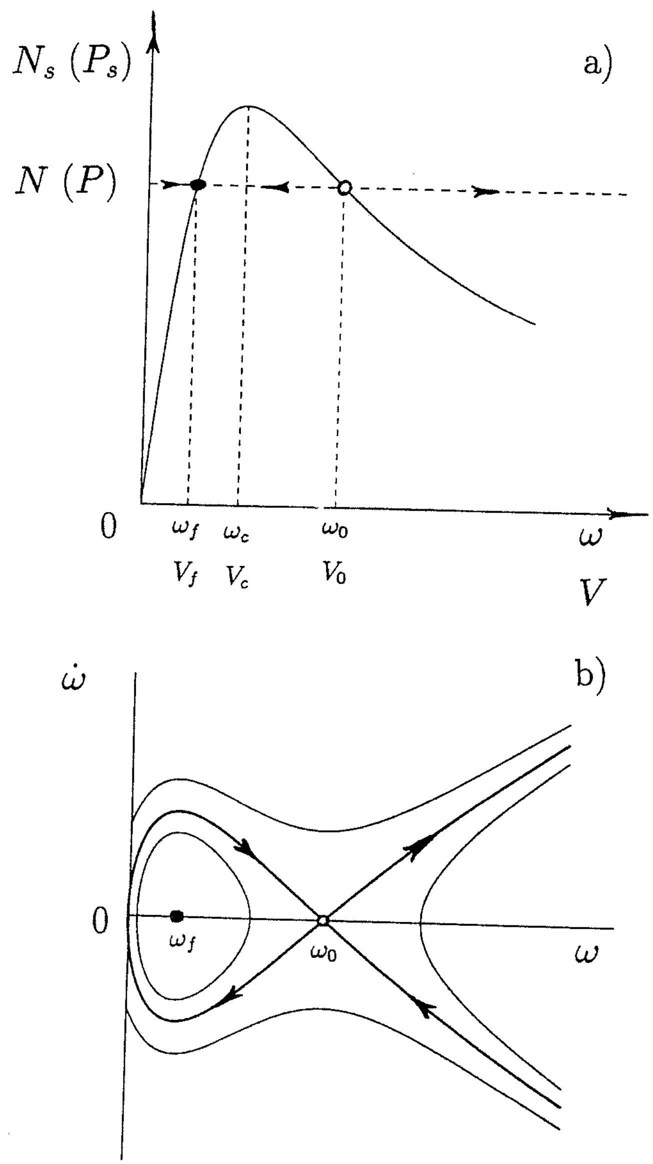

Fig.6. A typical two-branch curve for the functions $N_s(\omega)$ or $P_s(V)$ with $N_s''(\omega_c) < 0$(a) and the corresponding phase plane $(\omega, \dot{\omega})$ for (4.1) (b). Notation is the same as in Fig.5.

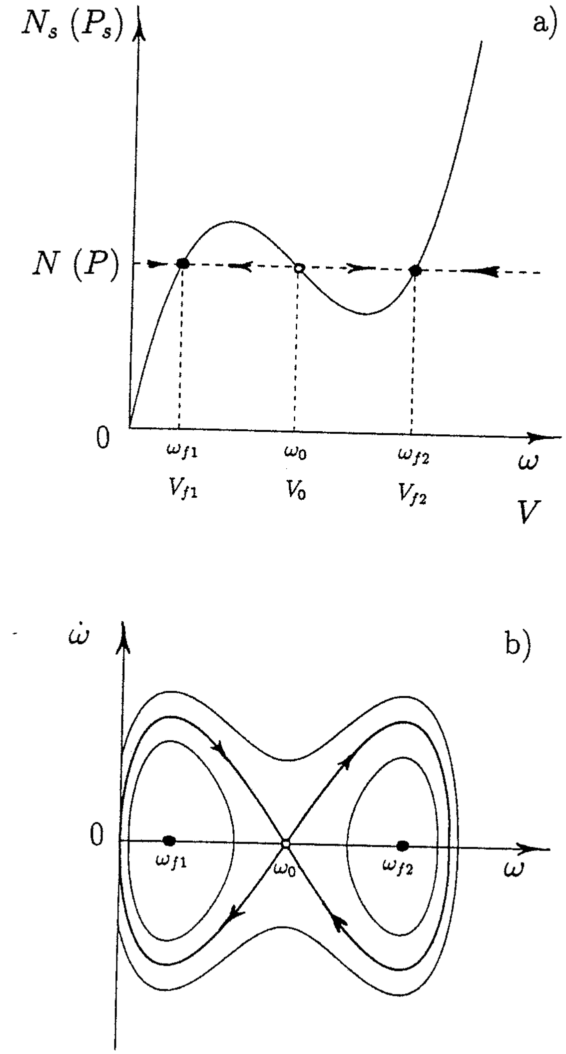

Fig.7. A typical three-branch curve for the functions $N_s(\omega)$ or $P_s(V)$ with $N_s'''(\omega_c) > 0$ (a) and the corresponding phase plane $(\omega, \dot{\omega})$ for (4.1) (b). Notation is the same as in Fig.5 but ω_{f1} and ω_{f2} designate two stable soliton solutions.

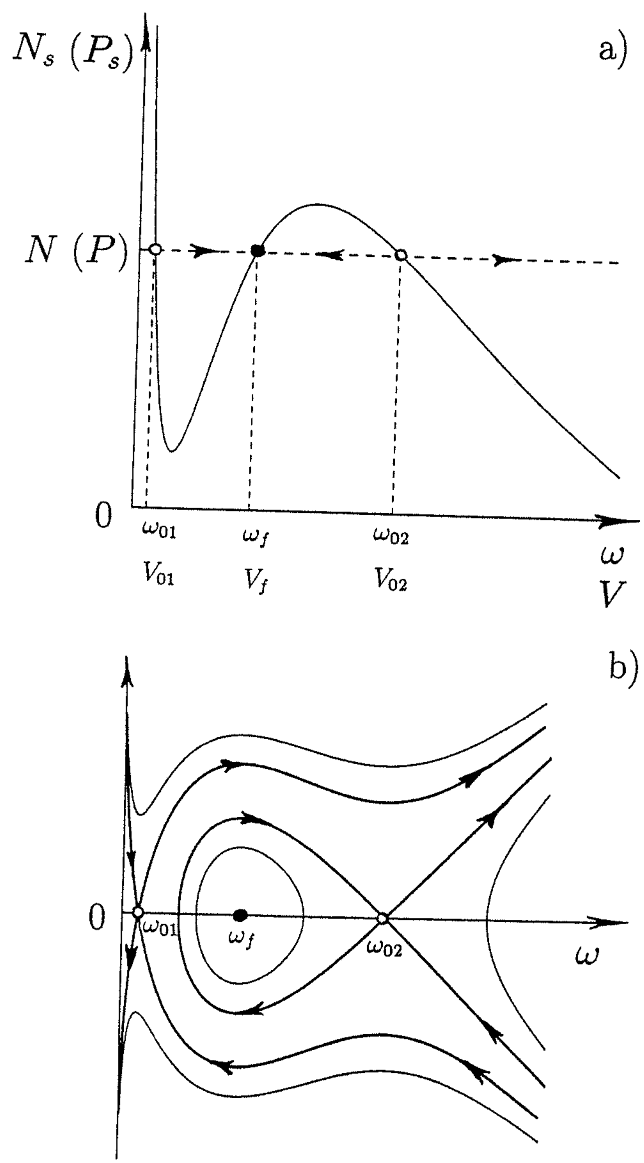

Fig.8. A typical three-branch curve for the functions $N_s(\omega)$ or $P_s(V)$ with $N_s'''(\omega_c) < 0$ (a) and the corresponding phase plane $(\omega, \dot{\omega})$ for (4.1) (b). Notation is the same as in Fig.5 but ω_{01} and ω_{02} designate two unstable soliton solutions.

is based on the existence of the small parameter ϵ in (4.1) which allows us to define the small-amplitude asymptotic expansion,

$$\omega = \omega_0 + \epsilon^2 w(T), \quad E = H_s(\omega_0) + \epsilon^6 \hat{E} \tag{4.4}$$

Here and henceforth ω_0 corresponds to a certain value of the soliton parameter chosen on the branch of the function $N_s(\omega)$ with negative slope near the turning point ω_c so that $(\omega_0 - \omega_c) \sim O(\epsilon^2)$ and we specify $N = N_s(\omega_0)$. Then, in the asymptotic limit $\epsilon \to 0$, equation (4.1) reduces to the form

$$\hat{E} = \frac{1}{2} M(\omega_c) \left(\frac{dw}{dT} \right)^2 + \frac{1}{2\epsilon^2} N_s'(\omega_0) w^2 + \frac{1}{6} N_s''(\omega_c) w^3 \tag{4.5}$$

The equilibrium state $\omega = \omega_0$ (or, equivalently, $w = 0$) corresponding to an unstable soliton (for which $N_s'(\omega_0) < 0$) is a saddle point of the dynamical system (4.5). For $\hat{E} = 0$ we find from (4.5) an analytical solution for the separatrix loop connecting the saddle equilibrium state. This analytical solution has the form

$$w = -\frac{3N_s'(\omega_0)}{\epsilon^2 N_s''(\omega_c)} \operatorname{sech}^2 \left[\frac{1}{2} \lambda T \right] \tag{4.6}$$

where λ is the growth rate of the linear instability ($\lambda > 0$) given by

$$\lambda^2 = -\frac{N_s'(\omega_0)}{\epsilon^2 M(\omega_c)} > 0$$

The separatrix loop described by (4.6) separates two different dynamical regimes of soliton instability. These separatrix trajectories are shown in Figs.5(b)-8(b) by bold lines.

For the case when $N_s''(\omega_c) > 0$ [see Figs.5(a,b)], the perturbations inside the separatrix loop oscillate around the other equilibrium state $\omega = \omega_f > \omega_0$ located on the stable branch of the function $N_s(\omega)$ with the same value of $N = N_s(\omega_0) = N_s(\omega_f)$. Outside the separatrix loop, the perturbations result in the vanishing of the soliton parameter ω in finite time. For the other case, when $N_s''(\omega_c) < 0$ [see Figs.6(a,b)], the perturbations bounded by the separatrix loop also oscillate near the stable equilibrium state $\omega = \omega_f < \omega_0$, while the unbounded perturbations lead to infinite growth of the soliton parameter ω in finite time.

If the parameter $N_s''(\omega_c)$ vanishes the quadratic approximation of $N_s(\omega)$ is no longer valid. In this case, a more complicated (cubic) bifurcation takes place and the dependence of $N_s(\omega)$ has three branches of soliton solutions.

To describe this bifurcation we modify the small-amplitude expansion (4.4) according to

$$\omega = \omega_0 + \epsilon w(T), \quad E = H_s(\omega_0) + \epsilon^4 \hat{E} \tag{4.7}$$

and reduce the energy integral (4.1) to the form,

$$\hat{E} = \frac{1}{2} M(\omega_c) \left(\frac{dw}{dT}\right)^2 + \frac{1}{2\epsilon^2} N_s'(\omega_0) w^2 + \frac{1}{6\epsilon} N_s''(\omega_0) w^3 + \frac{1}{24} N_s'''(\omega_c) w^4 \tag{4.8}$$

The separatrix loop solutions of (4.8) for $\hat{E} = 0$ can also be found analytically

$$w = -\frac{6N_s'(\omega_0)}{\epsilon\,[N_s''(\omega_0) + \sigma \cosh[\lambda T]]} \tag{4.9}$$

where λ is defined below (4.6) and σ satisfies the equation,

$$\sigma^2 = [N_s''(\omega_0)]^2 - 3N_s'(\omega_0) N_s'''(\omega_c)$$

If the intermediate branch of unstable solitons for the function $N_s(\omega)$ is bounded by two branches of stable solitons, i.e. $N_s'''(\omega_c) > 0$ [see Figs.7(a,b)], the soliton instability leads only to periodic oscillations of the soliton parameter ω. These oscillations occur near only one stable equilibrium state $\omega = \omega_{f1} < \omega_0$ or $\omega = \omega_{f2} > \omega_0$ if the initial perturbation lies inside the regions bounded by one of the separatrix loops given by (4.9) for a certain sign of σ. In the opposite case, when the perturbation lies outside both the separatrix loops, the soliton parameter oscillations surround both the stable states. We would like to point out here that, if the coefficients $N_s'(\omega_0)$ and $N_s''(\omega_0)$ both vanish, the soliton solution is unstable in the linearized and energetic senses (see Definitions 1.1 and 1.2) while the nonlinear stability still takes place for the case $N_s'''(\omega_c) > 0$, according to Definition 1.3. This corresponds to a bounded regime for the dynamics of soliton perturbations.

Finally, when the intermediate branch of stable solitons for the function $N_s(\omega)$ is bounded by two branches of the unstable solitons, i.e. $N_s'''(\omega_c) < 0$ [see Figs.8(a,b)], there is only one separatrix loop described by (4.9) for sign(σ) = sign($N_s''(\omega_0)$) which connects one of the unstable equilibrium states, either $\omega = \omega_{01} < \omega_f$ or $\omega = \omega_{02} > \omega_f$. Inside this separatrix loop, the soliton dynamics is oscillatory while, outside the separatrix, both an increase and a decrease of the soliton parameter ω are possible depending on initial conditions. Obviously, the analysis presented above can be extended for more complicated bifurcations when the function $N_s(\omega)$ has more than three branches of soliton solutions.

Thus, there are generally three types of nonlinear soliton dynamics in the vicinity of the type I instability bifurcation: (i) *periodic soliton oscillations* around the nearest stable soliton state; (ii) a *vanishing* of the soliton parameter ω; or (iii) an *infinite growth* of this parameter. Only the first type is bounded and can be completely described by the asymptotic (conservative) equation (4.1). The other two types are unbounded in the framework of the asymptotic analysis and they lead to a cardinal transformation of the nonlinear wave field. Analysis of particular models (see, e.g., [16, 17]) reveals that this transformation may result in decay and disappearence of soliton solutions into linear small-amplitude wave packets or in the formation of singularities in the amplitude of the wave profile.

Now we consider the type II instability bifurcation for the case $n = 1$ which is described by the dissipative equation (3.23) which can be written in the form

$$\frac{1}{\epsilon}[P(T) - P_s(V)] = K(V)\frac{dV}{dT} \tag{4.10}$$

We assume again that the coefficient K is positive. Here variations of $P(T)$ in time can be neglected in the small-amplitude approximation. Then, equation (4.10) describes the dynamics of an effective dissipative inertialess particle in the potential given by the function $P_s(V)$. Let us consider again four typical profiles of this function presented in Figs.5(a)-8(a).

If the function $P_s(V)$ has two branches connecting at a turning point $V = V_c$, where $P_s'(V_c) = 0$, we apply a small-amplitude expansion, $V = V_0 + \epsilon v(T)$, and reduce (4.10) to the form,

$$K(V_c)\frac{dv}{dT} + \frac{1}{\epsilon}P_s'(V_0)v + \frac{1}{2}P_s''(V_c)v^2 = 0 \tag{4.11}$$

Here again we introduce the value V_0 for the parameter of a stationary (unstable) soliton solution located at the branch of $P_s(V)$ with negative slope near the turning point so that $V_0 - V_c \sim O(\epsilon)$ and specify $P = P_s(V_0)$. Equation (4.11) describes a monotonic transformation of the soliton solution under the action of an initial perturbation away from the unstable equilibrium stable $V = V_0$ (or, equivalently, $v = 0$). This transformation is shown in Figs.5(a) and 6(a) by arrows and is described by the general solution to (4.11),

$$v = \frac{v_0 v_f}{v_0 + (v_f - v_0)\exp[-\lambda T]} \tag{4.12}$$

where $v_0 = v(0)$,

$$v_f = -\frac{2P_s'(V_0)}{\epsilon P_s''(V_c)}, \quad \text{and} \quad \lambda = -\frac{P_s'(V_0)}{\epsilon K(V_c)}$$

If the sign of v_0 coincides with the sign of v_f, the soliton dynamics results in the formation of a stationary soliton solution with the parameter $V_f = V_0 + \epsilon v_f$ which corresponds to a stable soliton with the same value of $P = P_s(V_0) = P_s(V_f)$. In the opposite case, the soliton dynamics is unbounded and it leads to a zero or an infinite value of V in finite time [see Figs.5(a) and 6(a)].

If the function $P_s(V)$ has three branches of soliton solutions we apply a different approximation, $V = V_0 + \sqrt{\epsilon} v(T)$, and reduce (4.10) to the cubic equation,

$$K(V_c)\frac{dv}{dT} + \frac{1}{\epsilon}P_s'(V_0)v + \frac{1}{2\sqrt{\epsilon}}P_s''(V_0)v^2 + \frac{1}{6}P_s'''(V_c)v^3 = 0 \qquad (4.13)$$

If $P_s'''(V_c) > 0$ [see Fig.7(a)], the unstable soliton monotonically transforms to one of the stable solitons depending on the initial perturbation. In the opposite case, when $P_s'''(V_c) < 0$ [see Fig.8(a)], the soliton dynamics results in formation of a stable soliton only in the interval between unstable branches of $P_s(V)$.

Thus, the nonlinear soliton dynamics near the type II instability bifurcation leads also to three scenarios: (i) *monotonic transition* to a stable soliton; (ii) a *vanishing* of the soliton parameter V; or (iii) an *infinite growth* of this parameter (see also examples in [18, 19]). We conclude that the last two (unbounded) scenarios are essentially the same as for the type I bifurcation while the first (bounded) scenario is different.

This difference is explained by considering the balance between the inertial and dissipative properties of the soliton dynamics. For the type I bifurcation, the inertial effects are dominating, while the radiation-induced dissipative effects are not excited at the leading-order approximation in our asymptotic approach. As a result, the stable solitons near the instability threshold always have a non-trivial internal (oscillating) mode which leads to *oscillatory* soliton dynamics under the action of small perturbations. On the other hand, for the type II bifurcation, the inertial effects can be neglected at the leading-order approximation, while the radiative dissipation leads to *monotonic* soliton dynamics.

Only for the long-term soliton dynamics or far from the instability threshold might the balance between inertial and dissipative effects be changed. For example, on larger time scales the oscillating solitons might generate small-amplitude radiative waves which result in the gradual decrease of amplitude of the oscillations. Furthermore, the internal mode might disappear far from the instability threshold. Nevertheless, the main features of the soliton dynamics described by these finite-dimensional asymptotic models near the instability bifurcations remain qualitatively similar for all soliton solutions of the given type.

In the rest of this Section we consider an example, where the asymptotic solutions following from the dissipative equation (4.10) can be compared with *exact* solutions of a nonlinear evolution equation.

Example 4.1.1. The integrable Boussinesq equation

We apply the general results described in this Section to the gBs equation (1.2) with the nonlinear function $f(u) = u^2$. The long-wave soliton solutions are given explicitly by

$$u_s = \frac{3}{2}(c_0^2 - V^2)\mathrm{sech}^2\left[\frac{1}{2}\sqrt{c_0^2 - V^2}(x - Vt)\right] \qquad (4.14)$$

where $|V| < c_0$. Then, we evaluate the dependence $P_s(V)$ from (3.24b) [see Example 3.3.1],

$$P_s = -6V(c_0^2 - V^2)^{3/2} \qquad (4.15)$$

This dependence is shown in Fig.9(a). The turning points are located at $V = \pm V_c = \pm\frac{1}{2}c_0$. The long-wave solitons are unstable for $|V| < V_c$ and the nonlinear dynamics of this instability is described by the asymptotic equation (4.11). Using formulas (3.24c), (3.32a), and (4.14) we evaluate the coefficients of this equation for the case $V_0 > 0$ [see Fig.9(a)],

$$K(V_c) = 12c_0, \quad P_s'(V_0) = -12\sqrt{3}c_0(V_c^2 - V_0^2), \quad P_s''(V_c) = 12\sqrt{3}c_0^2 \qquad (4.16)$$

Let us consider the general solution (4.12) of the asymptotic equation (4.11) for $v_0 = v(0) > 0$. In this case, the analytical solution (4.12) describes a monotonic transition of the unstable soliton with the velocity $V = V_0$ to the stable one with the velocity $V = V_f$ given by

$$V_f = V_c + (V_c - V_0) \qquad (4.17a)$$

With the help of (3.28a) and (3.31a) we are able to find the radiation fields $U^\pm$ generated by the long-wave soliton evolving according to (4.12). It turns out that $U^+ = 0$, while the spatial structure of U^- is given by

$$U^- = \frac{2\lambda^2}{3c_0^2}\,\mathrm{sech}^2\left[\frac{\lambda}{3c_0}(x + c_0 t) + \phi\right] \qquad (4.18)$$

where λ is the growth rate of the soliton instability,

$$\lambda = \sqrt{3}(V_c^2 - V_0^2)$$

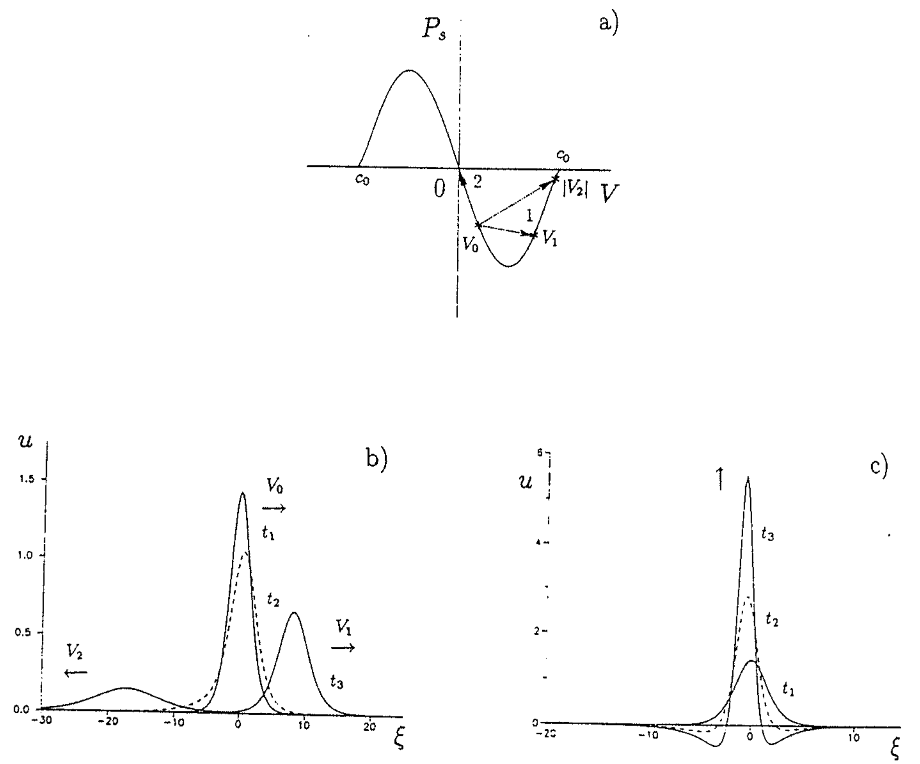

Fig.9. The momentum $P_s(V)$ (a) for the gBs equation (1.2) with $f(u) = u^2$ given by (4.15) and typical instability-induced soliton dynamics described by (4.19) for $c_0 = 1$, $V_0 = 0.2$, and the cases $\mu = 1$ (b) and $\mu = -1$ (c). Arrows 1 in (a) correspond to the splitting of an unstable soliton with $V = V_0$ into two stable solitons with $V = V_1$ and $V = V_2$ shown in (b) for $t_1 = -30$, $t_2 = 0$, and $t_3 = 15$. The arrow 2 in (a) corresponds to the collapse of the unstable soliton shown in (c) for $t_1 = -30$, $t_2 = 0$, and $t_3 = 1$.

and

$$\phi = -\frac{1}{2}\log\left[\frac{v_f - v_0}{v_0}\right]$$

The profile (4.18) completely corresponds to the stationary soliton solution (4.14) with the velocity $V = V_r$, where

$$V_r = -c_0 + \frac{2}{3c_0}(V_c - V_0)^2 \qquad (4.17b)$$

Thus, we conclude that for $v_0 > 0$ an unstable soliton with the velocity V_0 splits into two stable solitons with the velocities V_f and V_r without the generation of linear dispersive waves. In the opposite case, when $v_0 = v(0) < 0$, the analytical solution (4.12) describes the formation of zero values of the soliton velocity in finite time which is accompanied by singularities in the structure of the radiation field U^-. This dynamics implies a collapse of the soliton shape due to the soliton instability.

The gBs equation (1.2) with the nonlinear function $f(u) = u^2$ belongs to the class of integrable equations (see, e.g., [1]) and possesses a broad set of *exact* solutions. It is important that this set includes also particular solutions describing the nonlinear dynamics of soliton instability which can be compared with the results of our *asymptotic* theory. The exact solutions for soliton instabilities were first constructed by Spector *et al.* [57], and they have the form

$$u = 6\frac{\partial^2}{\partial x^2}\log\left(1 + \exp[k_0\xi] + \mu\exp\left[\left(\frac{1}{2}k_0 + k'\right)\xi + \lambda t\right]\right) \qquad (4.19)$$

where $\xi = x - V_0 t$, $k_0 = \sqrt{c_0^2 - V_0^2}$, $k' = -\frac{\sqrt{3}}{2}V_0$, and λ is given below (4.18). As above, the parameter λ corresponds to the growth rate of the linear instability of the long-wave solitons. Indeed, expanding (4.19) for $\mu \to 0$ as $u = u_s + \delta u\exp[\lambda t]$, we obtain an exact solution (see [57]),

$$\delta u \sim \frac{d^2}{d\xi^2}\left(\frac{\exp\left[-\frac{\sqrt{3}}{2}V_0\xi\right]}{\cosh\left[\frac{1}{2}\sqrt{c_0^2 - V_0^2}\xi\right]}\right) \qquad (4.20)$$

to the linearized problem [cf.(3.26b)],

$$\mathcal{L}(\delta u)_{\xi\xi} = -2\lambda V_0(\delta u)_\xi + \lambda^2\delta u \qquad (4.21)$$

where the linearized operator $\mathcal{L}$ is given by

$$\mathcal{L} = -\partial_\xi^2 - 2u_s + c_0^2 - V_0^2$$

We note that the linear instability mode (4.20) is localized only in the instability domain for $|V_0| < V_c$. In the stability domain, where $|V_0| > V_c$, this exact solution diverges exponentially. This is a general feature for the type II instability bifurcation (see also Example 2.3.2).

The soliton dynamics described by the exact solution (4.19) for $\mu = \pm 1$, $c_0 = 1$, and $V_0 = 0.2$ is shown in Figs.9(b,c). Obviously the instability of a long-wave soliton with the parameter $V = V_0$ for $\mu = +1$ [see Fig.9(b)] leads to the splitting of the unstable soliton into two stable solitons propagating in opposite directions [see arrows 1 in Fig.9(a)]. The velocities V_1 and V_2 of these new solitons can be found from (4.19) as

$$V_1 = \frac{1}{2}\left[\sqrt{3(c_0^2 - V_0^2)} - V_0\right] \sim V_f$$

and

$$V_2 = \frac{1}{2}\left[-\sqrt{3(c_0^2 - V_0^2)} - V_0\right] \sim V_r$$

For $\mu = -1$ the soliton instability results in formation of the singularity in the profile of the unstable soliton in finite time $t \approx 1.73$ for $x \approx -0.754$ [see Fig.9(c)]. An initial stage of this unbounded process corresponds to the vanishing of the soliton velocity V and growth of the soliton amplitude [see arrow 2 in Fig.9(a)]. Thus, comparing results predicted by the exact solution (4.19) and by the analytic formulas (4.12),(4.17), and (4.18), we conclude that the nonlinear instability-induced soliton dynamics in the integrable Boussinesq equation is asymptotically described by the dissipative finite-dimensional system with good accuracy.

8.4.2 Strongly nonlinear case: critical collapse

In the general analysis presented in the previous Section we omitted the special case when the functions $N_s(\omega)$ or $P_s(V)$ are constant for arbitrary values of ω or V. This special (*critical*) case exists at the edge between the stability and instability domains for those evolution equations which have some scaling invariance (see, e.g., [8]). The weakly nonlinear approximations based on expansions of the functions $N_s(\omega)$ and $P_s(V)$ into Taylor series are invalid for this critical, strongly nonlinear case and we have to investigate the general equations (4.1) or (4.10).

First, let us consider the conservative case. We assume that $N_s(\omega) = N_c = const$ and $H_s(\omega) = H_c = const$ and re-scale the constants, $E - H_c = \epsilon^2 \Delta E$, $N - N_c = \epsilon^2 \Delta N$. Then, the asymptotic equation (4.1) transforms to

$$\Delta E + \omega \Delta N = \frac{1}{2} M(\omega) \left(\frac{d\omega}{dT}\right)^2 \tag{4.22}$$

Because of scaling invariance of the given evolution equation, the coefficient $M(\omega)$ has to possess the form,

$$M = \frac{m}{\omega^\mu} \tag{4.23}$$

where the constant m is supposed to be positive and μ is the scaling power. Using this representation, we are able to investigate behaviour of solutions to (4.22) in the general case. First, we note that this equation does not describe any bounded soliton dynamics and the *positive* parameter ω either grows to infinity or vanishes. The first scenario is usually related with the phenomenon of critical collapse (see, e.g., [2, 58] and references therein), while the second one results in spreading and decaying of the solitons into small-amplitude dispersive wave packets. Therefore, henceforth we refer to infinite growth of the soliton parameter ω as a *soliton collapse*. For critical collapse a number of general results have been rigorously proved in [58] and we reproduce these results in the framework of the asymptotic equation (4.22).

(i) For $\Delta N < 0$ (*subcritical* initial conditions) collapse is not possible and the soliton pulse always decays.

(ii) For $\Delta N = 0$ (*critical* initial conditions) the collapse occurs if the initial disturbance leads to an increase of ω, i.e. $\dot{\omega}(0) > 0$, where $\dot{\omega} = d\omega/dT$. In this case, the collapsing spike remains self-similar to the soliton solution with the parameter ω varying according to the scaling law,

$$\omega = \omega_0 \left(\frac{t_0}{t_0 - t}\right)^\nu \tag{4.24}$$

where $\omega_0 = \omega(0)$,

$$\nu = \frac{2}{\mu - 2}, \quad \text{and} \quad t_0 = \sqrt{\frac{m\nu^2 \omega_0^{2-\mu}}{2\Delta E}}$$

(iii) For $\Delta N > 0$ (*supercritical* initial conditions) the soliton pulse is always collapsing if $\Delta E < 0$. In the opposite case, when $\Delta E \geq 0$, collapse occurs only if $\dot{\omega}(0) > 0$. The formation of singularities in the supercritical case is also described by (4.24) in the limit $t \to t_0$ and $\omega \to \infty$ but with the different ν and t_0 given by

$$\nu = \frac{2}{\mu - 1}, \quad \text{and} \quad t_0 = \sqrt{\frac{m\nu^2 \omega_0^{1-\mu}}{2\Delta N}}$$

Example 4.2.1. Critical collapse of bright solitons [17]

We consider the gNLS equation (1.3) for the nonlinear function $F(I) = 3I^2$, where $I = |\Psi|^2$. In this case, the bright soliton solutions are expressed through the function Φ (see Example 2.2.2), which has the explicit form

$$\Phi = \omega^{1/4}\mathrm{sech}^{1/2}[2\sqrt{\omega}x] \tag{4.25}$$

Using this explicit formula we find the soliton invariants (2.18a,c), $N_s(\omega) = N_c = \pi/4$ and $H_s(\omega) = H_c = 0$. Therefore, this gNLS equation represents the critical, strongly nonlinear case for soliton instability and we apply the asymptotic equation (4.22) with the coefficient $M(\omega)$ given by (3.11). After the integration we find this coefficient in the form (4.23) with the constant $m = \pi^3/512$ and the scaling power $\mu = 3$. For this scaling power, a general solution of (4.22) can be found in the analytic form,

$$\omega = -\frac{2m\Delta E}{2m\Delta N - (\Delta E)^2(t-t_0)^2} \tag{4.26}$$

where t_0 is the integration constant. We find from (4.26) that the singularities appear for $\Delta N \geq 0$ in a time instant $t = \min(t_0 - \tau, t_0 + \tau)$, where $\tau = \sqrt{2m\Delta N}|\Delta E|^{-1}$. The scaling power ν of singularity formation is given by $\nu = 2$ for $\Delta N = 0$ and $\nu = 1$ for $\Delta N > 0$ according to the general analysis presented above.

The gNLS equation (1.3) with the nonlinear function $F(I) = 3I^2$ has a family of *exact self-similar solutions* in the form,

$$\Psi = [\omega(t)]^{1/4}\phi(X)\exp[i\theta(X,t)] \tag{4.27}$$

where

$$\theta = \int_0^t \omega(t')dt' - \frac{1}{8\omega^2}\left(\frac{d\omega}{dt}\right)X^2$$

$X = \sqrt{\omega(t)}x$, the dependence $\omega(t)$ is given by (4.26), and the real function $\phi(X)$ satisfies the differential equation,

$$\phi_{XX} - \phi + 3\phi^5 + \frac{\Delta N}{8m}X^2\phi = 0 \tag{4.28}$$

If $\Delta N = 0$, the latter equation has a localized solution in the form of the stationary soliton (4.25) for $\omega = 1$. In this case, the critical collapse is absolutely radiationless and it is described by the *exact self-similar* solution to the gNLS equation (see [2]). We have reconstructed this exact solution in the framework of the conservative *asymptotic* equation derived with the help of a formal small parameter ϵ.

For positive but small ΔN the last term in (4.28) leads to nonlocalized (oscillatory-type) behaviour which is important only for large X (and, therefore, t), $|X| \sim 2\sqrt{2m}/\sqrt{\Delta N}$. This implies that radiation is generated for large time intervals outside the soliton core and this radiation is not described by a naive asymptotic expansion based on smallness of the last term in (4.28). Therefore, in the supercritical case $\Delta N > 0$ the solution (4.26) of the asymptotic equation (or, equivalently, the exact self-similar solution) describes only an initial stage of collapse of the bright solitons. To describe this collapse at a later stage, a modification of the asymptotic multi-scale technique has been proposed in a number of papers (see [59] and references therein). It was shown that the generation of a radiation field at $\Delta N > 0$ modifies the scaling law $\nu = 1$ of the singularity formations by the double-logaritmic factor,

$$\omega \sim \frac{\log\log(t_0 - t)^{-1}}{(t_0 - t)} \quad \text{as} \quad t \to t_0 \tag{4.29}$$

Now we consider the critical collapse for the dissipative asymptotic equation (4.10) with $P_s(V) = P_c = const$. This equation is not closed because the function $P(T)$ cannot be considered as a constant in the strongly nonlinear approximation. Thus, we have to add to this equation the balance equations (3.21a) and (3.22a) for $n = 1$ and $K_1 = K_{11}^1$. Then, the governing equation can be written in the form,

$$K(V)\frac{d^2V}{dT^2} + \left[\frac{dK}{dV} - K_1(V)\right]\left(\frac{dV}{dT}\right)^2 = 0 \tag{4.30}$$

where the coefficients $K(V)$ and $K_1(V)$ are supposed to have the form,

$$K = \frac{k}{V^\mu}, \quad K_1 = \frac{k_1}{V^{\mu+1}} \tag{4.31}$$

with the positive constant k. These asymptotic equations can be solved explicitly and the general solution is

$$V = V_0\left(\frac{t_0}{t_0 - t}\right)^\nu \tag{4.32}$$

where $V_0 = V(0)$ and t_0 are constants of integration and

$$\nu = \frac{k}{(\mu - 1)k + k_1}$$

Therefore, similarly to the conservative case, the parameter V either increases or decreases infinitely, and bounded regimes of the soliton instability are prohibited. We call the infinite growth of V the collapse of solitons. It follows from (4.32) that the collapse occurs if $\dot{V}(0) > 0$ or $P(0) > P_c$. In the opposite case, when $\dot{V}(0) < 0$ and $P(0) < P_c$ the solitons decay. For the critical case $P(0) = P_c$ the dissipative asymptotic equations do not describe nontrivial dynamics of the solitons.

Example 4.2.2. Critical collapse of long-wave solitons [19]

We consider the gKdV equation (1.1) for $\alpha = 1$, $\beta = 0$, and the nonlinear function $f(u) = 3u^5$. The long-wave soliton solutions $u = u_s(\xi)$, where $\xi = x - Vt$, are given by the same formula (4.25) but with the replacement $x \to \xi$ and $\omega \to V$. The dependence $P_s(V)$ follows from (2.10b) as $P_s(V) = P_c = \pi/4$ and we apply the asymptotic equation (4.30) with the coefficients K and K_1 given by (3.33). After simple calculations they can be presented in the form (4.31) with the parameters $\mu = 5/2$,

$$k = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^4}{256\pi}$$

and $k_1 = -k$, where $\Gamma(z)$ is the Gamma function. Then, the scaling law of the long-wave soliton collapse is described by the analytical solution (4.32) with $\nu = 2$. The critical collapse of the long-wave solitons is accompanied by generation of the radiation field U^- behind the soliton which moves as $t \to t_0$ to infinity. The profile of the radiation field is given by

$$U^- = \frac{u_0}{(x + x_0)^{3/2}} \quad \text{for} \quad x > 0$$

where u_0 and x_0 are positive constants. We would like to point out that, because of the radiation, the dissipative asymptotic system provides the value of the scaling law ν ($\nu = 2$) to be different from that of the self-similar solutions, where $\nu = 2/3$ (see discussion in [19]).

8.5 Note on regular soliton perturbation theory

In this last Section we show that soliton stability problems can often be studied by means of the regular soliton perturbation theory which has already been reviewed in much detail (see, e.g., [23]). This application of the regular perturbation theory is based on a perturbation of the original nonlinear evolution equation (2.5) to the form

$$u_t = JH_0'[u] + \epsilon R[u] \tag{5.1}$$

where $H_0[u]$ represents the energy functional of the unperturbed soliton equation and $R[u]$ is the functional for the external perturbation multiplied by the small parameter ϵ. This perturbed form of the evolution equation is very convenient if we know the soliton solutions and their stability properties for the unperturbed system, ($\epsilon = 0$). Thus, we can study the dynamical (stability) properties of these solitons under the action of the external perturbation using multi-scale asymptotic expansions valid for $\epsilon \ll 1$. To simplify the analysis, we suppose here that the soliton solutions of (5.1) for $\epsilon = 0$ have only one parameter V so that $u = u_s(x - Vt; V)$. Then, the asymptotic expansion has the form

$$u = u_s(\xi; V) + \epsilon u_1(\xi; T) + O(\epsilon^2) \tag{5.2}$$

where $V = V(T)$, $T = \epsilon t$, and $\xi = x - \epsilon^{-1} \int_0^T V(T')dT'$. The first-order perturbation u_1 satisfies the linearized inhomogeneous problem,

$$J\Lambda_0''[u_s]u_1 = \frac{\partial u_s}{\partial V}\frac{dV}{dT} - R[u_s] \tag{5.3}$$

Here $\Lambda_0[u] = H_0[u] + VP[u]$ is the Lyapunov functional of the unperturbed system and $P[u]$ is the integral invariant associated with the parameter V. Using the solvability condition to (5.3) we can find the adiabatic equation for the varying parameter V in the abstract form,

$$\frac{dP_s}{dV}\frac{dV}{dT} = G(V) = \langle u_s, R[u_s]\rangle \tag{5.4}$$

where the notation $\langle u, w\rangle$ stands for a proper inner product (see, e.g., (2.12) or (5.9b) below). Besides the adiabatic response of an evolving soliton due to the external perturbation described by (5.4), there may also be nonadiabatic effects which can be studied after finding the first-order correction u_1 from (5.3). However, to analyse the perturbation-induced soliton dynamics it is often sufficient to consider (5.4) without reconstruction of the profiles of the first-order correction.

It follows from (5.4) that the equilibrium state $V = V_0$ for which $G(V_0) = 0$ corresponds to stationary soliton solutions in the full equation (5.1) which are constructed approximately here to the first order of the perturbation theory.

Moreover, we can investigate the stability properties of this stationary soliton solution to the same order. Suppose that a soliton solution with $V = V_0$ is stable in the unperturbed system (5.1) and the stability is determined by the simple criterion,

$$\left.\frac{dP_s}{dV}\right|_{V=V_0} > 0 \tag{5.5}$$

Then, we find from (5.4) that the stability criterion for the soliton solution in the perturbed system is

$$\left.\frac{dG}{dV}\right|_{V=V_0} < 0 \tag{5.6}$$

Example 5.1. Perturbed KdV equation

Let us consider the perturbed KdV equation in the form (see [23])

$$u_t + 6uu_x + u_{xxx} = \epsilon R[u] = \epsilon\left(\gamma_1 u + \gamma_2 u_{xx}\right) \tag{5.7}$$

The perturbation terms in this equation describe linear active or dissipative effects such that for positive values of the coefficients γ_1 and γ_2 the first term leads to wave growth while the second leads to wave decay. The soliton solution $u = u_s(x - Vt)$ of the unperturbed KdV equation ($\epsilon = 0$) is given by

$$u_s = \frac{1}{2} V \operatorname{sech}^2\left[\frac{1}{2}\sqrt{V}x\right] \tag{5.8}$$

The compatability condition (5.4) holds with the functions $P_s(V)$ and $G(V)$ given by

$$P_s = \frac{1}{2}\int_{-\infty}^{+\infty} u_s^2 dx = \frac{1}{3}V^{3/2} \tag{5.9a}$$

$$G = \int_{-\infty}^{+\infty} u_s R[u_s] dx = \frac{2}{3}\gamma_1 V^{3/2} - \frac{2}{15}\gamma_2 V^{5/2} \tag{5.9b}$$

Using these formulas we can rewrite (5.4) in the form of the adiabatic equation,

$$\frac{dV}{dT} = \frac{4}{3}\gamma_1 V - \frac{4}{15}\gamma_2 V^2 \tag{5.10}$$

It follows from this equation that there is only one equilibrium state $V = V_0 = 5\gamma_1/\gamma_2$, where $G(V_0) = 0$. This unique soliton solution realizes a balance between active and dissipative effects. Using the criterion (5.5) and (5.6) we find that

$$\left.\frac{dP_s}{dV}\right|_{V=V_0} = \frac{1}{2}V_0^{1/2} > 0, \quad \left.\frac{dG}{dV}\right|_{V=V_0} = -\frac{2}{3}\gamma_1 V_0^{1/2} < 0$$

and, therefore, this soliton is stable. Indeed, the adiabatic equation (5.10) describes the monotonic transformation of any initial soliton profile with a given value of the parameter V to the stable stationary soliton solution with $V = V_0$. This transformation is described by the analytical solutions (4.12) with the parameters, $\lambda = 4\gamma_1/3$, $v_f = V_0$, and $v_0 = V(0) > 0$.

When the slope dP_s/dV approaches a zero value, the adiabatic equation (5.4) breaks down, as was first found in [24]. In this case, we have to take into account the internal, instability-induced soliton dynamics to the same order of perturbation theory as the external, perturbation-induced soliton dynamics. In other words, we have to extend the adiabatic equation (5.4) to the next-order approximation described in Section 8.3.3 and reorder the perturbation term as $R = \epsilon R[u]$. Then, the modified adiabatic equation has the form,

$$\frac{1}{\epsilon}\frac{dP_s}{dV}\frac{dV}{dT} + K(V)\frac{d^2V}{dT^2} + \left[\frac{dK}{dV} - K_1(V)\right]\left(\frac{dV}{dT}\right)^2 = G(V) \tag{5.11}$$

where the coefficients K and K_1 are introduced in (3.22). If the coefficient K is positive the stability of the stationary solitons in the perturbed equation (5.1) is given again by conditions (5.5) and (5.6). However, when the coefficient K vanishes, the modified asymptotic equation (5.11) breaks down as well and it is necessary to extend this equation to still higher orders of the asymptotic equations. It seems that this procedure can be continued further to describe not only simple but also 'higher-order' soliton instabilities.

Finally, we would like to point out that the adiabatic equation (5.4) *cannot be used* if the slope dP_s/dV does not vanish but is *strictly negative*. At first glance it might seem that, in this case, both terms are again of the same order in the regular perturbation theory while the conditions for soliton stability (5.6) and other features of the soliton dynamics simply change their 'sign'. However, the fact that the slope dP_s/dV is negative means that the soliton solutions in the unperturbed problem are unstable with respect to small perturbations and this strong instability destroys these solitons *before* they start to evolve under the external perturbation. As a result, the regular soliton perturbation theory leads to *wrong* conclusions when the solitons, which are supposed to be slowly evolving, are actually unstable within an unperturbed problem.

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