Physica D 240 (2011) 265–281

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

# Internal modes of discrete solitons near the anti-continuum limit of the dNLS equation

Dmitry Pelinovsky *, Anton Sakovich

*Department of Mathematics, McMaster University, Hamilton ON, Canada, L8S 4K1*

ARTICLE INFO

*Article history:*
Received 23 April 2010
Received in revised form
12 August 2010
Accepted 8 September 2010
Available online 16 September 2010
Communicated by J. Bronski

*Keywords:*
Discrete NLS equation
Anti-continuum limit
Internal modes
Resolvent operator

ABSTRACT

Discrete solitons of the discrete nonlinear Schrödinger (dNLS) equation are compactly supported in the anti-continuum limit of the zero coupling between lattice sites. Eigenvalues of the linearization of the dNLS equation at the discrete soliton determine its spectral stability. Small eigenvalues bifurcating from the zero eigenvalue near the anti-continuum limit were characterized earlier for this model. Here we analyze the resolvent operator and prove that it is bounded in the neighborhood of the continuous spectrum if the discrete soliton is simply connected in the anti-continuum limit. This result rules out the existence of internal modes (neutrally stable eigenvalues of the discrete spectrum) near the anti-continuum limit.



## 1. Introduction

The *discrete nonlinear Schrödinger (dNLS) equation* is a mathematical model of many physical phenomena including the Bose–Einstein condensation in optical lattices, propagation of optical pulses in coupled waveguide arrays, and oscillations of molecules in DNAs [1]. *Discrete solitons* (stationary localized solutions) are used to interpret the results of physical experiments and to characterize the global dynamics of the dNLS equation with decaying initial data.

Discrete solitons are compactly supported in the *anti-continuum limit* of the zero coupling between lattice sites. Different families of discrete solitons can be uniquely characterized near the anti-continuum limit from a number of limiting configurations [2]. This is the main reason why the anti-continuum limit has been studied in detail after the pioneer works of Eilbeck et al. [3] and Aubry and Abramovici [4]. The existence of discrete solitons (also called *discrete breathers* in the context of the discrete Klein–Gordon equation) was rigorously justified with *implicit function theorem arguments* by MacKay and Aubry [5]. Their work on existence of discrete solitons led to further progress in understanding their stability properties as well as nonlinear dynamics of nonlinear lattices [6–8].

*Spectral stability* of discrete solitons is determined by the eigenvalues of the discrete spectrum of an associated linearized operator because its continuous spectrum is neutrally stable. Unstable eigenvalues can be fully characterized near the anti-continuum limit because they bifurcate from the zero eigenvalue of finite multiplicity and the zero eigenvalue is isolated from the continuous spectrum. Characterization of unstable eigenvalues for each family of discrete solitons bifurcating from a compact limiting solution was obtained by Pelinovsky et al. [9] with an application of the *Lyapunov–Schmidt reduction technique*. Besides the unstable eigenvalues, the same technique was used to characterize a number of neutrally stable eigenvalues of negative energy (also called eigenvalues of *negative Krein signature*) which bifurcate from the same zero eigenvalue. These isolated eigenvalues of negative energy may become unstable far from the anti-continuum limit because of collisions with eigenvalues of positive energy (also called *internal modes*) or with the continuous spectrum of the linearized operator. Isolated eigenvalues of negative energy may also induce nonlinear instability if their multiples belong to the continuous spectrum [10].

In the same anti-continuum limit, another bifurcation occurs beyond the applicability of the Lyapunov–Schmidt reduction technique: a pair of *semi-simple* nonzero eigenvalues of *infinite multiplicity* transforms into a pair of continuous spectral bands of small width. This transformation may produce additional eigenvalues of the discrete spectrum similar to what happens for the discrete kinks (which are

* Corresponding author.
*E-mail address:* dmpeli@math.mcmaster.ca (D. Pelinovsky).

0167-2789/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.physd.2010.09.002

non-compact solutions of the nonlinear lattice in the anti-continuum limit) [11]. No complex unstable eigenvalues may bifurcate from the semi-simple nonzero eigenvalues of infinite multiplicity because such eigenvalues are excluded by the count of unstable eigenvalues in [9]. Nevertheless, internal modes may in general be expected outside the continuous spectrum.

It is important to know the details of the existence of internal modes because of several reasons. First, these internal modes may collide with eigenvalues of negative energy to produce the Hamilton–Hopf instability bifurcations [9]. Second, analysis of asymptotic stability of discrete solitons depends on the number and location of the internal modes [12,13]. Third, the presence of internal modes may result in long-term quasi-periodic oscillations of discrete solitons [14].

In this paper, we address bifurcations of internal modes from semi-simple nonzero eigenvalues of infinite multiplicity. We continue the resolvent operator across the continuous spectrum and prove that it is bounded near the end points of the continuous spectrum if the discrete soliton is *simply connected* in the anti-continuum limit; see Definition 2. As a result, no internal modes exist in the neighborhood of the continuous spectrum. These results hold for *any* discrete soliton of the dNLS equation with *any* power nonlinearity near the anti-continuum limit.

There are multiple numerical evidences that no internal modes exist near the anti-continuum limit for the *fundamental* discrete soliton, which is supported at a single lattice site in the zero coupling limit. In particular, this fact is suggested by Fig. 1 in [15] and by Fig. 2.5 in [1]. Our article presents the first analytical proof of this phenomenon.

The paper is organized as follows. Section 2 reviews results on existence and stability of discrete solitons near the anti-continuum limit. Section 3 is devoted to analysis of the resolvent operator with the limiting compact potentials. Section 4 develops perturbative arguments for the full resolvent operator. Section 5 considers a case study for the resolvent operator associated with a non-simply connected two-site discrete soliton. The Appendix is devoted to the cubic dNLS equation, for which perturbation arguments are more delicate.

*Notations.* We denote the bi-infinite sequence $\{u_n\}_{n\in\mathbb{Z}}$ by $\mathbf{u}$. The $l^p$ space for sequences is denoted by $l^p(\mathbb{Z})$ and is equipped with the norm

$$\|\mathbf{u}\|_{l^p} := \left(\sum_{n\in\mathbb{Z}} |u_n|^p\right)^{1/p}, \quad p \geq 1.$$

The algebraically weighted space $l^p_s(\mathbb{Z})$ with $s \in \mathbb{R}$ is the $l^p(\mathbb{Z})$ space for the sequence

$$\{(1+n^2)^{s/2}u_n\}_{n\in\mathbb{Z}}.$$

A disk of radius $\delta > 0$ centered at the point $\lambda_0 \in \mathbb{C}$ on the complex plane is denoted by $B_\delta(\lambda_0) \subset \mathbb{C}$.

## 2. Review of results on discrete solitons

Consider the dNLS equation in the form

$$i\dot{u}_n + \epsilon(u_{n+1} - 2u_n + u_{n-1}) + |u_n|^{2p}u_n = 0, \quad n \in \mathbb{Z}, \tag{1}$$

where the dot denotes differentiation in $t \in \mathbb{R}$, $\{u_n(t)\}_{n\in\mathbb{Z}} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{C}$ is the set of amplitude functions, and parameters $\epsilon \in \mathbb{R}$ and $p \in \mathbb{N}$ define the coupling constant and the power of nonlinearity. The anti-continuum limit corresponds to $\epsilon = 0$, in which case the dNLS equation (1) becomes an infinite system of uncoupled differential equations.

Discrete solitons are defined in the form $u_n(t) = \phi_n e^{it}$, where the frequency is normalized thanks to the scaling symmetry of the power nonlinearity. By the standard arguments [16] based on the conserved quantity

$$\epsilon \neq 0: \quad \bar{\phi}_n\phi_{n+1} - \phi_n\bar{\phi}_{n+1} = \text{const in } n \in \mathbb{Z}, \tag{2}$$

it is known that if $\{\phi_n\}_{n\in\mathbb{Z}}$ decays to zero as $|n| \to \infty$, then $\{\phi_n\}_{n\in\mathbb{Z}}$ is a real-valued module to multiplication by $e^{i\theta}$ for any $\theta \in \mathbb{R}$. The real-valued stationary solutions are found from the second-order difference equation

$$(1 - \phi_n^{2p})\phi_n = \epsilon(\phi_{n+1} - 2\phi_n + \phi_{n-1}), \quad n \in \mathbb{Z}. \tag{3}$$

The algebraic system is uncoupled if $\epsilon = 0$.

Let us consider solutions of the difference equation (3) for $\boldsymbol{\phi} \in l^2(\mathbb{Z})$. If $\epsilon = 0$ and $p \in \mathbb{N}$, the limiting configuration of the discrete soliton is given by the compact solution

$$\epsilon = 0: \quad \boldsymbol{\phi}^{(0)} = \sum_{n\in U_+} \boldsymbol{\delta}_n - \sum_{n\in U_-} \boldsymbol{\delta}_n, \tag{4}$$

where $U_\pm$ are compact subsets of $\mathbb{Z}$ such that $U_+ \cap U_- = \varnothing$ and $\boldsymbol{\delta}_n$ is the standard unit vector in $l^2(\mathbb{Z})$ expressed via the Kronecker symbol by

$$(\boldsymbol{\delta}_n)_m = \delta_{n,m}, \quad m \in \mathbb{Z}.$$

We will denote the number of sites in $U_\pm$ by $|U_\pm|$. The following proposition gives a unique analytic continuation of the compact limiting solution (4) to a particular family of discrete solitons (see [5,16] for the proof).

**Proposition 1.** *Fix $U_+, U_- \subset \mathbb{Z}$ such that $U_+ \cap U_- = \varnothing$ and $|U_+| + |U_-| < \infty$. There exists $\epsilon_0 > 0$ such that the stationary dNLS equation (3) with $\epsilon \in (-\epsilon_0, \epsilon_0)$ admits a unique solution $\boldsymbol{\phi} \in l^2(\mathbb{Z})$ near $\boldsymbol{\phi}^{(0)} \in l^2(\mathbb{Z})$. The map $(-\epsilon_0, \epsilon_0) \ni \epsilon \mapsto \boldsymbol{\phi} \in l^2(\mathbb{Z})$ is analytic and*

$$\exists C > 0: \quad \|\boldsymbol{\phi} - \boldsymbol{\phi}^{(0)}\|_{l^2} \leq C|\epsilon|. \tag{5}$$

*Moreover, there are $\kappa > 0$ and $C > 0$ such that for any $\epsilon \in (-\epsilon_0, \epsilon_0)$*

$$|\phi_n| \leq C e^{-\kappa|n|}, \quad n \in \mathbb{Z}. \tag{6}$$

**Remark 1.** Thanks to the exponential decay (6), the solution $\boldsymbol{\phi} \in l^2(\mathbb{Z})$ of Proposition 1 belongs to $\boldsymbol{\phi} \in l^2_s(\mathbb{Z})$ for any $s \geq 0$.

By Proposition 1, the solution $\boldsymbol{\phi}$ for a given $\boldsymbol{\phi}^{(0)}$ can be expanded in the power series

$$\boldsymbol{\phi} = \boldsymbol{\phi}^{(0)} + \sum_{k=1}^{\infty} \epsilon^k \boldsymbol{\phi}^{(k)}, \quad \epsilon \in (-\epsilon_0, \epsilon_0), \tag{7}$$

where correction terms $\{\boldsymbol{\phi}^{(k)}\}_{k\in\mathbb{N}}$ are uniquely defined by a recursion formula.

Spectral stability of the discrete solitons is determined from the analysis of the spectral problem

$$L_+\mathbf{u} = -\lambda\mathbf{w}, \qquad L_-\mathbf{w} = \lambda\mathbf{u}, \tag{8}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter, $(\mathbf{u}, \mathbf{w}) \in l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$ is an eigenvector, and $L_\pm$ are discrete Schrödinger operators given by

$$\begin{cases} (L_+\mathbf{u})_n = -\epsilon(u_{n+1} - 2u_n + u_{n-1}) + (1 - (2p+1)\phi_n^{2p})u_n, \\ (L_-\mathbf{w})_n = -\epsilon(w_{n+1} - 2w_n + w_{n-1}) + (1 - \phi_n^{2p})w_n, \end{cases} \quad n \in \mathbb{Z}. \tag{9}$$

We recall the basic definitions and results from the stability analysis of the spectral problem (8).

**Definition 1.** The eigenvalues of the spectral problem (8) with $\mathrm{Re}(\lambda) > 0$ (resp. $\mathrm{Re}(\lambda) = 0$) are called unstable (resp. neutrally stable). If $\lambda \in \mathrm{i}\mathbb{R}$ is a simple isolated eigenvalue, then the eigenvalue $\lambda$ is said to have a positive energy if $\langle L_+\mathbf{u}, \mathbf{u}\rangle_{l^2} > 0$ and a negative energy if $\langle L_+\mathbf{u}, \mathbf{u}\rangle_{l^2} < 0$.

**Remark 2.** If $\lambda \in \mathrm{i}\mathbb{R}$ is an isolated eigenvalue and $\langle L_+\mathbf{u}, \mathbf{u}\rangle_{l^2} = 0$, then $\lambda$ is not a simple eigenvalue. In this case, the concept of eigenvalues of positive and negative energies is defined by the diagonalization of the quadratic form $\langle L_+\mathbf{u}, \mathbf{u}\rangle_{l^2}$, where $\mathbf{u}$ belongs to the subspace of $l^2(\mathbb{Z})$ associated to the eigenvalue $\lambda$ of the spectral problem (8) and is invariant under the action of the corresponding linearized operator (see [17] for the relevant theory).

In the anti-continuum limit $\epsilon = 0$, the spectrum of $L_+$ (resp. $L_-$) includes a semi-simple eigenvalue $-2p$ (resp. 0) of multiplicity $N = |U_+| + |U_-| < \infty$ and a semi-simple eigenvalue 1 of multiplicity $|\mathbb{Z} \setminus \{U_+ \cup U_-\}| = \infty$. The spectral problem (8) has a pair of eigenvalues $\lambda = \pm\mathrm{i}$ of infinite multiplicity and the eigenvalue $\lambda = 0$ of geometric multiplicity $N$ and algebraic multiplicity $2N$. The following proposition describes the splitting of the zero eigenvalue near the anti-continuum limit for $\epsilon > 0$ (see [9] for the proof).

**Proposition 2.** *Fix $U_+, U_- \subset \mathbb{Z}$ such that $U_+ \cap U_- = \varnothing$ and $N := |U_+| + |U_-| < \infty$. Fix $\epsilon > 0$ sufficiently small and denote the number of sign differences of $\{\phi_n^{(0)}\}_{n\in U_+\cup U_-}$ by $n_0$.*

- *There are exactly $n_0$ negative and $N - 1 - n_0$ small positive eigenvalues of $L_-$ counting multiplicities and a simple zero eigenvalue.*
- *There are exactly $n_0$ pairs of small eigenvalues $\lambda \in \mathrm{i}\mathbb{R}$ and $N - 1 - n_0$ pairs of small eigenvalues $\lambda \in \mathbb{R}$ of the spectral problem (8) counting multiplicities and a double zero eigenvalue.*

Proposition 2 completes the characterization of unstable eigenvalues and neutrally stable eigenvalues of negative energy from negative eigenvalues of $L_+$ and $L_-$. In particular, we know from [17] that if $\mathrm{Ker}(L_+) = \{0\}$, $\mathrm{Ker}(L_-) = \mathrm{span}\{\boldsymbol{\phi}\}$, and $\langle L_+^{-1}\boldsymbol{\phi}, \boldsymbol{\phi}\rangle_{l^2} \neq 0$, then

$$\begin{cases} n(L_+) - p_0 = N_r^- + N_i^- + N_c, \\ n(L_-) = N_r^+ + N_i^- + N_c, \end{cases} \tag{10}$$

where $n(L_\pm)$ denotes the number of negative eigenvalues of $L_\pm$, $N_i^-$ denotes the number of eigenvalues $\lambda \in \mathrm{i}\mathbb{R}$ with negative energy, $N_c$ denotes the number of eigenvalues with $\mathrm{Re}(\lambda) > 0$ and $\mathrm{Im}(\lambda) > 0$, $N_r^\pm$ denotes the number of eigenvalues $\lambda \in \mathbb{R}$ with $\langle L_+\mathbf{u}, \mathbf{u}\rangle_{l^2} \gtrless 0$, and

$$p_0 = \begin{cases} 1 & \text{if } \langle L_+^{-1}\boldsymbol{\phi}, \boldsymbol{\phi}\rangle_{l^2} < 0, \\ 0 & \text{if } \langle L_+^{-1}\boldsymbol{\phi}, \boldsymbol{\phi}\rangle_{l^2} > 0. \end{cases}$$

To compute $p_0$, we extend the family of discrete solitons by parameter $\omega$ as solutions of

$$(\omega - \phi_n^{2p})\phi_n = \epsilon(\phi_{n+1} - 2\phi_n + \phi_{n-1}), \quad n \in \mathbb{Z}. \tag{11}$$

Differentiation of Eq. (11) in $\omega$ at $\omega = 1$ gives

$$\langle L_+^{-1}\boldsymbol{\phi}, \boldsymbol{\phi}\rangle_{l^2} = -\langle \partial_\omega \boldsymbol{\phi}|_{\omega=1}, \boldsymbol{\phi}\rangle_{l^2} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\omega}\|\boldsymbol{\phi}\|_{l^2}^2\Big|_{\omega=1} = -\frac{N}{2p} + \mathcal{O}(\epsilon),$$

where in the last equality we used Proposition 1 and the anti-continuum limit

$$\epsilon = 0: \quad \|\boldsymbol{\phi}(\omega)\|_{l^2}^2 = N\omega^{1/p}.$$

Therefore, $p_0 = 1$ for small $\epsilon > 0$.

By Proposition 2, we have $n(L_-) = n_0$ and $N_i^- \geq n_0$. Also, $n(L_+) = N$. Using the count (10), we have for small $\epsilon > 0$

$$N_r^+ = 0, \qquad N_r^- = N - 1 - n_0, \qquad N_i^- = n_0, \qquad N_c = 0. \tag{12}$$

Equality (12) shows that besides the small and zero eigenvalues described by Proposition 2, the spectral problem (8) may only have the continuous spectrum and the eigenvalues on $\mathrm{i}\mathbb{R}$ with positive energy. These eigenvalues of positive energy are called the *internal modes* and existence of such eigenvalues for small $\epsilon > 0$ is the main theme of this article.

## 3. The resolvent operator for the limiting configuration

Let us consider the truncated spectral problem (8) after $\boldsymbol{\phi}$ is replaced by $\boldsymbol{\phi}^{(0)}$. The resolvent operator is defined from the inhomogeneous system

$$\begin{cases} -\epsilon(u_{n+1} - 2u_n + u_{n-1}) + u_n - (2p+1) \sum_{m \in U_+ \cup U_-} \delta_{n,m} u_m + \lambda w_n = F_n, \\ -\epsilon(w_{n+1} - 2w_n + w_{n-1}) + w_n - \sum_{m \in U_+ \cup U_-} \delta_{n,m} w_m - \lambda u_n = G_n, \end{cases} \quad n \in \mathbb{N}, \tag{13}$$

where $\mathbf{F}, \mathbf{G} \in l^2(\mathbb{Z})$ are given. Since we are interested in the continuous spectrum and eigenvalues on $\mathrm{i}\mathbb{R}$, we set $\lambda = -\mathrm{i}\Omega$ and use new coordinates

$$\begin{cases} a_n := u_n + \mathrm{i}w_n, & b_n := u_n - \mathrm{i}w_n, \\ f_n := F_n + \mathrm{i}G_n, & g_n := F_n - \mathrm{i}G_n, \end{cases} \quad n \in \mathbb{Z}.$$

The inhomogeneous system (13) transforms to the equivalent form

$$\begin{cases} -\epsilon(a_{n+1} - 2a_n + a_{n-1}) + a_n - \sum_{m \in U_+ \cup U_-} \delta_{n,m}((1+p)a_m + pb_m) - \Omega a_n = f_n, \\ -\epsilon(b_{n+1} - 2b_n + b_{n-1}) + b_n - \sum_{m \in U_+ \cup U_-} \delta_{n,m}(pa_m + (1+p)b_m) + \Omega b_n = g_n, \end{cases} \tag{14}$$

which can be rewritten in the operator form

$$L \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} - \Omega \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ -\mathbf{g} \end{bmatrix}, \qquad L = \begin{bmatrix} -\epsilon\Delta + I - (1+p)V & -pV \\ pV & \epsilon\Delta - I + (1+p)V \end{bmatrix}, \tag{15}$$

where $\Delta : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is the discrete Laplacian operator

$$(\Delta u)_n := u_{n+1} - 2u_n + u_{n-1}, \quad n \in \mathbb{Z}$$

and $V : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is the associated compact potential

$$(Vu)_n = \sum_{m \in U_+ \cup U_-} \delta_{n,m} u_m, \quad n \in \mathbb{Z}.$$

Let $R_0(\lambda) : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ be a free resolvent of the discrete Schrödinger operator $-\Delta$ for $\lambda \notin \sigma(-\Delta) \equiv [0, 4]$. The free resolvent was studied recently by Komech et al. [18]. The free resolvent operator can be expressed in the Green function form

$$\forall \mathbf{f} \in l^2(\mathbb{Z}): \quad (R_0(\lambda)\mathbf{f})_n = \frac{1}{2\mathrm{i}\sin z(\lambda)} \sum_{m \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}z(\lambda)|n-m|} f_m, \tag{16}$$

where $z(\lambda)$ is a unique solution of the transcendental equation for $\lambda \notin [0, 4]$

$$2 - 2\cos z(\lambda) = \lambda, \quad \mathrm{Re}\, z(\lambda) \in [-\pi, \pi), \mathrm{Im}\, z(\lambda) < 0. \tag{17}$$

The limiting absorption principle (see, e.g., [19]) states that a bounded operator $R_0(\lambda) : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ for $\lambda \notin [0, 4]$ admits the limits

$$R_0^{\pm}(\omega) = \lim_{\mu \downarrow 0} R_0(\omega \pm \mathrm{i}\mu) : l^2_{\sigma}(\mathbb{Z}) \to l^2_{-\sigma}(\mathbb{Z}), \quad \sigma > \frac{1}{2}$$

for any fixed $\omega \in (0, 4)$.

The limiting free resolvent operators $R_0^{\pm}(\omega)$ can also be expressed in the Green function form

$$\forall \mathbf{f} \in l^1(\mathbb{Z}): \quad (R_0^{\pm}(\omega)\mathbf{f})_n = \frac{1}{2\mathrm{i}\sin\theta_{\pm}(\omega)} \sum_{m \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\theta_{\pm}(\omega)|n-m|} f_m, \tag{18}$$

where $\theta_{\pm}(\omega) = \pm\theta(\omega)$ and $\theta(\omega)$ is a unique solution of the transcendental equation for $\omega \in [0, 4]$

$$2 - 2\cos\theta(\omega) = \omega, \quad \mathrm{Re}\,\theta(\omega) \in [-\pi, 0], \mathrm{Im}\,\theta(\omega) = 0. \tag{19}$$

The limiting operators $R_0^{\pm}(\omega) : l^1(\mathbb{Z}) \to l^{\infty}(\mathbb{Z})$ are bounded for any fixed $\omega \in (0, 4)$ but diverge as $\omega \downarrow 0$ and $\omega \uparrow 4$. These divergences follow from the Puiseux expansion, e.g.,

$$\forall \mathbf{f} \in l^1_2(\mathbb{Z}): \quad (R_0^{\pm}(\omega)\mathbf{f})_n = \frac{1}{2\mathrm{i}\theta^{\pm}(\omega)} \sum_{m \in \mathbb{Z}} f_m - \frac{1}{2} \sum_{m \in \mathbb{Z}} |n-m| f_m + (\hat{R}_0^{\pm}(\omega)\mathbf{f})_n \quad \text{as} \quad \omega \downarrow 0, \tag{20}$$

where

$$\exists C > 0: \quad \|\hat{R}_0^{\pm}(\omega)\mathbf{f}\|_{l^{\infty}} \le C|\theta^{\pm}(\omega)| \, \|\mathbf{f}\|_{l^1_2}.$$

Divergences of $R_0^{\pm}(\omega)$ at the end points $\omega = 0$ and $\omega = 4$ indicate *resonances*, which may result in the bifurcation of new eigenvalues from the continuous spectrum on [0, 4] either for $\lambda < 0$ or $\lambda > 4$, when $-\Delta$ is perturbed by a small potential in $l^2(\mathbb{Z})$.

Let us denote the solution of the inhomogeneous system (15) by

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = R_L(\Omega) \begin{bmatrix} \mathbf{f} \\ -\mathbf{g} \end{bmatrix}, \qquad R_L(\Omega) = \begin{bmatrix} R_{11}(\Omega) & R_{12}(\Omega) \\ R_{21}(\Omega) & R_{22}(\Omega) \end{bmatrix}. \tag{21}$$

The following theorem represents the main result of this section. This theorem is valid for the simply connected sets $U_+ \cup U_-$, which are defined by the following definition.

**Definition 2.** We say that the set $U_+ \cup U_-$ is simply connected if no elements in $\mathbb{Z} \setminus \{U_+ \cup U_-\}$ are located between elements in $U_+ \cup U_-$.

**Theorem 1.** *Fix $U_+, U_- \subset \mathbb{Z}$ such that $U_+ \cap U_- = \varnothing$, $N := |U_+| + |U_-| < \infty$, and $U_+ \cup U_-$ is simply connected. There exist small $\epsilon_0 > 0$ and $\delta > 0$ such that for any fixed $\epsilon \in (0, \epsilon_0)$ the resolvent operator*

$$R_L(\Omega) : l^2(\mathbb{Z}) \times l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$$

*is bounded for any $\Omega \notin B_\delta(0) \cup [1, 1+4\epsilon] \cup [-1-4\epsilon, -1]$. Moreover, $R_L(\Omega)$ has exactly $2N$ poles (counting multiplicities) inside $B_\delta(0)$ and admits the limits*

$$R_L^{\pm}(\Omega) := \lim_{\mu \downarrow 0} R_L(\Omega \pm \mathrm{i}\mu)$$

*such that for any $\Omega \in [1, 1+4\epsilon] \cup [-1-4\epsilon, -1]$ and any $\epsilon \in (0, \epsilon_0)$, there is $C > 0$ such that*

$$\|R_L^{\pm}(\Omega)\|_{l_1^1 \times l_1^1 \to l^\infty \times l^\infty} \le C\epsilon^{-1}.$$

**Remark 3.** The other way to formulate the main theorem is to say that the end points of the continuous spectrum $\sigma_c(L) \equiv [1, 1+4\epsilon] \cup [-1-4\epsilon, -1]$ are not resonances and no eigenvalues of the linear operator $L$ may exist outside a small disk $B_\delta(0) \subset \mathbb{C}$. The $2N$ eigenvalues inside the small disk $B_\delta(0)$ are characterized in Proposition 2.

Solving the linear system (14) with the Green function (16), we obtain the exact solution for any $n \in \mathbb{Z}$

$$\begin{cases} a_n = \dfrac{1}{2\mathrm{i}\epsilon \sin z(\lambda_+)} \left( \displaystyle\sum_{m \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}z(\lambda_+)|n-m|} f_m + \sum_{m \in U_+ \cup U_-} \mathrm{e}^{-\mathrm{i}z(\lambda_+)|n-m|} ((1+p)a_m + pb_m) \right), \\ b_n = \dfrac{1}{2\mathrm{i}\epsilon \sin z(\lambda_-)} \left( \displaystyle\sum_{m \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}z(\lambda_-)|n-m|} g_m + \sum_{m \in U_+ \cup U_-} \mathrm{e}^{-\mathrm{i}z(\lambda_-)|n-m|} (pa_m + (1+p)b_m) \right), \end{cases} \tag{22}$$

where the map $\mathbb{C} \ni \lambda \mapsto z \in \mathbb{C}$ is defined by the transcendental equation (17) and

$$\lambda_\pm = \frac{\pm\Omega - 1}{\epsilon}.$$

The solution is closed if the set $\{(a_n, b_n)\}_{n \in U_+ \cup U_-}$ is found from the linear system of finitely many equations for any $n \in U_+ \cup U_-$

$$\begin{cases} 2\mathrm{i}\epsilon \sin z(\lambda_+) a_n - \displaystyle\sum_{m \in U_+ \cup U_-} \mathrm{e}^{-\mathrm{i}z(\lambda_+)|n-m|} ((1+p)a_m + pb_m) = \sum_{m \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}z(\lambda_+)|n-m|} f_m, \\ 2\mathrm{i}\epsilon \sin z(\lambda_-) b_n - \displaystyle\sum_{m \in U_+ \cup U_-} \mathrm{e}^{-\mathrm{i}z(\lambda_-)|n-m|} (pa_m + (1+p)b_m) = \sum_{m \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}z(\lambda_-)|n-m|} g_m. \end{cases} \tag{23}$$

Let us order lattice sites $n \in U_+ \cup U_-$ such that the first site is placed at $n = 0$, the second site is placed at $m_1$, the third site is placed at $m_1 + m_2$, and so on, the last site is placed at $m_1 + m_2 + \cdots + m_{N-1}$, where $N = |U_+| + |U_-|$ and all $m_j > 0$. If $U_+ \cup U_-$ is a simply connected set, then all $m_j = 1$.

Let $Q(q_1, q_2, \ldots, q_{N-1})$ be the matrix in $\mathbb{C}^{N \times N}$ defined by

$$Q(q_1, q_2, \ldots, q_{N-1}) := \begin{bmatrix} 1 & q_1 & q_1 q_2 & \cdots & q_1 q_2 \cdots q_{N-1} \\ q_1 & 1 & q_2 & \cdots & q_2 q_3 \cdots q_{N-1} \\ q_1 q_2 & q_2 & 1 & \cdots & q_3 \cdots q_{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_1 q_2 \cdots q_{N-1} & q_2 \cdots q_{N-1} & q_3 \cdots q_{N-1} & \cdots & 1 \end{bmatrix}. \tag{24}$$

Let $q_j^{\pm} = \mathrm{e}^{-\mathrm{i}m_j z(\lambda_\pm)}$ and $Q^{\pm}(\Omega, \epsilon) := Q(q_1^{\pm}, q_2^{\pm}, \ldots, q_{N-1}^{\pm})$. The coefficient matrix of the linear system (25) is given by

$$A(\Omega, \epsilon) := \begin{bmatrix} 2\mathrm{i}\epsilon \sin z(\lambda_+) I - (1+p) Q^+(\Omega, \epsilon) & -pQ^+(\Omega, \epsilon) \\ -pQ^-(\Omega, \epsilon) & 2\mathrm{i}\epsilon \sin z(\lambda_-) I - (1+p) Q^-(\Omega, \epsilon) \end{bmatrix}, \tag{25}$$

where $I$ is an identity matrix in $\mathbb{C}^{N \times N}$.

We split the proof of Theorem 1 into three subsections, where solutions of system (22) and (23) are studied for different values of $\Omega$.

### *3.1. Resolvent outside the continuous spectrum*

We consider the resolvent operator $R_L(\Omega)$ for a fixed small $\epsilon \in (0, \epsilon_0)$. The following lemma shows that $R_L(\Omega)$ is a bounded operator from $l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$ to $l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$ for all $\Omega \in \mathbb{C}$ except three disks of small radii centered at $\{0, 1, -1\}$.

**Lemma 1.** *There are $\epsilon_0 > 0$ and $\delta, \delta_\pm > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, the resolvent operator $R_L(\Omega) : l^2(\mathbb{Z}) \times l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$ is bounded for all $\Omega \in \mathbb{C} \setminus \{B_\delta(0) \cup B_{\delta_+}(1) \cup B_{\delta_-}(-1)\}$. Moreover, $R_L(\Omega)$ has exactly $2N$ poles (counting multiplicities) inside $B_\delta(0)$.*

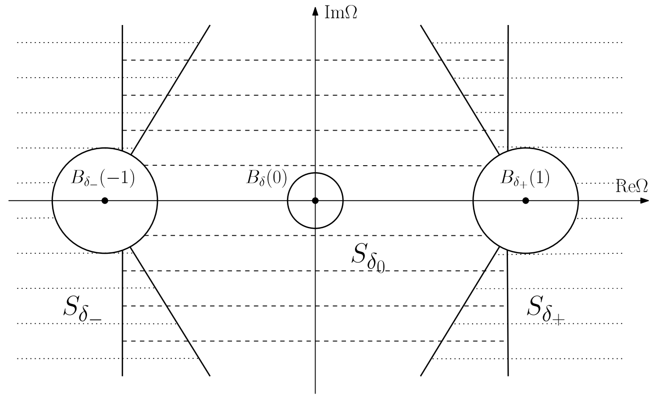


**Fig. 1.** Schematic display of various domains in the $\Omega$-plane.

**Proof.** From the property of the free resolvent operator $R_0(\lambda)$, we know that the Green function in the representation (22) is bounded and exponentially decaying as $|n| \to \infty$ for any $\Omega$ such that $\lambda_\pm \notin [0, 4]$. This gives $\Omega \notin \sigma_c(L) \equiv [1, 1+4\epsilon] \cup [-1-4\epsilon, -1]$. Therefore, $R_L(\Omega)$ is a bounded map from $l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$ to $l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$ for any $\Omega \notin \sigma_c(L)$ if and only if the system of linear equation (23) is uniquely solvable. We shall now consider the invertibility of the coefficient matrix $A(\Omega, \epsilon)$ of the linear system (23) in various domains in the $\Omega$-plane for small $\epsilon > 0$. Fig. 1 shows schematically the location of these domains on the $\Omega$-plane.

Fix $\delta_0 \in (0, 1)$. Let $\Omega$ belong to the vertical strip

$$S_{\delta_0} := \{\Omega \in \mathbb{C} : \mathrm{Re}(\Omega) \in [-\delta_0, \delta_0]\}.$$

Then $z(\lambda_\pm) = -\mathrm{i}\kappa_\pm$ are uniquely determined from the equation

$$\mathrm{e}^{\kappa_\pm} + \mathrm{e}^{-\kappa_\pm} - 2 = \frac{1 \mp \Omega}{\epsilon}, \quad \mathrm{Re}(\kappa_\pm) > 0, \mathrm{Im}(\kappa_\pm) \in [-\pi, \pi),$$

which admits the asymptotic expansion

$$\mathrm{e}^{\kappa_\pm} = \frac{1 \mp \Omega}{\epsilon} + 2 - \frac{\epsilon}{1 \mp \Omega} + \mathcal{O}(\epsilon^2) \quad \text{as } \epsilon \to 0$$

and

$$\mathrm{e}^{-\kappa_\pm} = \frac{\epsilon}{1 \mp \Omega} + \mathcal{O}(\epsilon^2) \quad \text{as } \epsilon \to 0.$$

Therefore, both $\epsilon \sinh(\kappa_\pm)$ and $Q^\pm(\Omega, \epsilon)$ are analytic in $\epsilon$ near $\epsilon = 0$ and

$$2\mathrm{i}\epsilon \sin z(\lambda_\pm) = 2\epsilon \sinh(\kappa_\pm) = 1 \mp \Omega + 2\epsilon + \mathcal{O}(\epsilon^2) \quad \text{as } \epsilon \to 0$$

and

$$Q^\pm(\Omega, \epsilon) = I + \mathcal{O}(\epsilon) \quad \text{as } \epsilon \to 0.$$

It now becomes clear that $A(\Omega, \epsilon)$ is analytic in $\Omega \in S_{\delta_0}$ and $\epsilon \in (-\epsilon_0, \epsilon_0)$ with the limit

$$A(\Omega, 0) = \begin{bmatrix} -(p+\Omega)I & -pI \\ -pI & -(p-\Omega)I \end{bmatrix}. \tag{26}$$

The matrix $A(\Omega, 0) \in \mathbb{C}^{2N \times 2N}$ is singular only for $\Omega = 0$. Thanks to the analyticity of $A(\Omega, \epsilon)$, the determinant $D(\Omega, \epsilon) = \det A(\Omega, \epsilon)$ is also analytic in these variables and

$$D(\Omega, \epsilon) = (-\Omega^2)^N + \mathcal{O}(\epsilon) \quad \text{as } \epsilon \to 0.$$

Therefore, there exist $2N$ zeros of $D(\Omega, \epsilon)$ for small $\epsilon \in (0, \epsilon_0)$ in a small disk $B_\delta(0)$ with $\delta = \mathcal{O}(\epsilon^{1/2N})$. By Cramer's rule, these zeros of $D(\Omega, \epsilon)$ give poles of $R_L(\Omega)$.

Fix $\delta_+ \in (0, 1)$ and $\theta_+ \in \left(\frac{\pi}{2}, \pi\right)$. We now consider $\Omega$ in the domain

$$S_{\delta_+} := \{\Omega = 1 + r\mathrm{e}^{\mathrm{i}\theta},\ r > \delta_+,\ \theta \in (-\theta_+, \theta_+)\}.$$

In this domain, we have the same presentation for $z(\lambda_-) = -\mathrm{i}\kappa_-$ but a different presentation for $z(\lambda_+) = -\mathrm{i}\kappa_+ - \pi$. Now $\kappa_+$ is uniquely determined from the equation

$$\mathrm{e}^{\kappa_+} + \mathrm{e}^{-\kappa_+} + 2 = \frac{\Omega - 1}{\epsilon} = \frac{r}{\epsilon}\mathrm{e}^{\mathrm{i}\theta}, \quad \mathrm{Re}(\kappa_+) > 0, \mathrm{Im}(\kappa_+) \in [0, 2\pi),$$

which admits the asymptotic expansions

$$\mathrm{e}^{\kappa_+} = \frac{\Omega - 1}{\epsilon} - 2 - \frac{\epsilon}{\Omega - 1} + \mathcal{O}(\epsilon^2) \quad \text{as } \epsilon \to 0$$

and

$$2\mathrm{i}\epsilon \sin z(\lambda_+) = 1 - \Omega + 2\epsilon + \mathcal{O}(\epsilon^2) \quad \text{as } \epsilon \to 0.$$

Since $\mathrm{Re}(\kappa_+) \to \infty$ as $\epsilon \to 0$, $A(\Omega, 0)$ is the same as matrix (26) and it is invertible for $\Omega \in S_{\delta_+}$. Similar arguments can be developed for

$$S_{\delta_-} := \left\{\Omega = -1 + r\mathrm{e}^{\mathrm{i}\theta},\ r > \delta_-,\ \theta \in (\theta_-, 2\pi - \theta_-)\right\},$$

where $\delta_- \in (0, 1)$ and $\theta_- \in \left(0, \frac{\pi}{2}\right)$. Because there are choices of $\delta_0, \delta_\pm > 0$ such that

$$S_{\delta_0} \cup S_{\delta_+} \cup S_{\delta_-} = \mathbb{C} \setminus \{B_{\delta_+}(1) \cup B_{\delta_-}(-1)\},$$

we obtain the assertion of the lemma. □

**Remark 4.** The proof of Lemma 1 implies that poles of $R_L(\Omega)$ may have size $|\Omega| = \mathcal{O}(\epsilon^{1/2N})$. The results of the perturbation expansions (see [9] for details) imply that the eigenvalues bifurcating from 0 in the full spectral problem (8) have size $\mathcal{O}(\epsilon^{1/2})$. Moreover, the same perturbation expansion technique can be applied to show that eigenvalues of the truncated spectral problem (13) have the same size $\mathcal{O}(\epsilon^{1/2})$.

### *3.2. Resolvent inside the continuous spectrum*

We shall now consider the resolvent operator $R_L(\Omega)$ inside the continuous spectrum

$$\sigma_c(L) := [1, 1 + 4\epsilon] \cup [-1 - 4\epsilon, -1].$$

Thanks to the symmetry of system (22)–(23) in $\Omega$, we can consider only one branch of the continuous spectrum $[1, 1 + 4\epsilon]$. Therefore, we set $\Omega = 1 + \epsilon\omega$ with $\omega \in [0, 4]$ and define

$$z(\lambda_+) = z(\omega) \equiv \theta \quad \text{and} \quad z(\lambda_-) = z(-2\epsilon^{-1} - \omega) \equiv -\mathrm{i}\kappa.$$

It follows from (17) and (19) that $\theta \in [-\pi, 0]$ and $\kappa > 0$ are uniquely defined from equations

$$2 - 2\cos(\theta) = \omega, \qquad 2\epsilon(\cosh(\kappa) - 1) = 2 + \epsilon\omega, \quad \omega \in [0, 4]. \tag{27}$$

The choice of $\theta \in [-\pi, 0]$ corresponds to the limiting operator $R_0^+(\omega)$ of the free resolvent. Since $R_0^+(\omega) : l^2_\sigma(\mathbb{Z}) \to l^2_{-\sigma}(\mathbb{Z})$ is well defined for $\omega \in (0, 4)$ and $\sigma > \frac{1}{2}$, $R_L^+(1 + \epsilon\omega)$ is a bounded map from $l^2_\sigma(\mathbb{Z}) \times l^2_\sigma(\mathbb{Z})$ to $l^2_{-\sigma}(\mathbb{Z}) \times l^2_{-\sigma}(\mathbb{Z})$ for any $\omega \in (0, 4)$ and $\sigma > \frac{1}{2}$ if and only if there exists a unique solution of the linear system (23). On the other hand, the free resolvent is singular in the limits $\omega \downarrow 0$ and $\omega \uparrow 4$ and, therefore, we need to be careful in solving system (22)–(23) in this limit.

The main result of this section is given by the following theorem.

**Theorem 2.** *Let $m_1 = m_2 = \cdots m_{N-1} = 1$. There exists $\epsilon_0 > 0$ such that for any $\omega \in [0, 4]$ and any $\epsilon \in (0, \epsilon_0)$, there exist $C > 0$ such that*

$$\|R_L^+(1 + \epsilon\omega)\|_{l^1_1 \times l^1_1 \to l^\infty \times l^\infty} \le C\epsilon^{-1}, \tag{28}$$

*where the upper sign indicates that $\omega$ is parameterized by $\omega = 2 - 2\cos(\theta)$ for $\theta \in [-\pi, 0]$.*

To prove Theorem 2, we analyze the solutions of system (23) for $\omega \in [0, 4]$. Let us rewrite explicitly

$$q_j^+ = \mathrm{e}^{-\mathrm{i}m_j\theta} \quad \text{and} \quad q_j^- = \mathrm{e}^{-m_j\kappa}, \quad j \in \{1, 2, \dots, N-1\}.$$

The coefficient matrix (25) for $\Omega = 1 + \epsilon\omega$ with $\omega \in [0, 4]$ is rewritten in the form

$$A(\theta, \epsilon) \equiv \begin{bmatrix} 2\mathrm{i}\epsilon \sin(\theta) I - (1 + p) M(\theta) & -pM(\theta) \\ -pN(\kappa) & 2\epsilon \sinh(\kappa) I - (1 + p) N(\kappa) \end{bmatrix}, \tag{29}$$

where $M(\theta) \equiv Q(q_1^+, q_2^+, \dots, q_{N-1}^+)$ and $N(\kappa) \equiv Q(q_1^-, q_2^-, \dots, q_{N-1}^-)$. Note that $\theta$ and $M(\theta)$ are $\epsilon$-independent, whereas $N(\kappa)$ depends on $\epsilon$ via $\kappa$. The linear system (23) is now expressed in the matrix form

$$A(\theta, \epsilon)c = h(\theta, \epsilon), \tag{30}$$

where components of $c \in \mathbb{C}^{2N}$ and $h \in \mathbb{C}^{2N}$ are given by

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix}_{n \in U_+ \cup U_-} \quad \text{and} \quad \begin{Bmatrix} \sum_{m \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\theta|n-m|} f_m \\ \sum_{m \in \mathbb{Z}} \mathrm{e}^{-\kappa|n-m|} g_m \end{Bmatrix}_{n \in U_+ \cup U_-}.$$

Thanks to the asymptotic expansion

$$\mathrm{e}^\kappa = \frac{2}{\epsilon} + 2 + \omega - \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2) \quad \text{as } \epsilon \to 0,$$

we have

$$2\epsilon \sinh(\kappa) = 2 + (2 + \omega)\epsilon + \mathcal{O}(\epsilon^2) \quad \text{as } \epsilon \to 0.$$

Both $A(\theta, \epsilon)$ and $h(\theta, \epsilon)$ are analytic in $\theta \in [-\pi, 0]$ and $\epsilon \in (-\epsilon_0, \epsilon_0)$. The following lemma establishes the invertibility condition for matrix $A(\theta, \epsilon)$.

**Lemma 2.** *For any $\epsilon \in (0, \epsilon_0)$, matrix $A(\theta, \epsilon)$ has a zero eigenvalue of geometric and algebraic multiplicities $N-1$ for $\theta = -\pi$ and $\theta = 0$. If $m_1 = m_2 = \cdots = m_{N-1} = 1$, matrix $A(\theta, \epsilon)$ is invertible for any $\theta \in (-\pi, 0)$.*

**Proof.** We use the fact that matrix $A(\theta, \epsilon)$ is analytic in $\epsilon$ for small $\epsilon \in (-\epsilon_0, \epsilon_0)$. Therefore, it remains invertible if $A(\theta, 0)$ is invertible. To consider the limit $\epsilon \to 0$, we note that $\kappa \to \infty$ and $N(\kappa) \to I$ as $\epsilon \to 0$, so we have

$$A(\theta, 0) = \begin{bmatrix} -(1+p)M(\theta) & -pM(\theta) \\ -pI & (1-p)I \end{bmatrix}.$$

For any $p \in \mathbb{N}$, matrix $A(\theta, 0)$ is invertible if and only if matrix $M(\theta)$ is invertible. Let us then compute

$$D_N(q_1, q_2, \ldots, q_{N-1}) := \det Q(q_1, q_2, \ldots, q_{N-1}).$$

We note that $D_N(\pm 1, q_2, \ldots, q_{N-1}) = 0$ and $D_N(q_1, q_2, \ldots, q_{N-1})$ is a quadratic polynomial of $q_1$. Therefore,

$$D_N(q_1, q_2, \ldots, q_{N-1}) = (1 - q_1^2) D_N(0, q_2, \ldots, q_{N-1}) = (1 - q_1^2) D_{N-1}(q_2, \ldots, q_{N-1}).$$

Continuing the expansion recursively, we obtain the exact formula

$$D_N(q_1, q_2, \ldots, q_{N-1}) = (1 - q_1^2)(1 - q_2^2) \cdots (1 - q_{N-1}^2), \tag{31}$$

from which we conclude that $Q(q_1, q_2, \ldots, q_{N-1})$ is invertible if and only if all $q_j \neq \pm 1$. This implies that $M(\theta)$ is invertible if and only if all $e^{-im_j\theta} \neq \pm 1$, which is satisfied if all $m_j = 1$ and $\theta \in (-\pi, 0)$. The second assertion of the lemma is proved: for any $\epsilon \in [0, \epsilon_0)$, matrix $A(\theta, \epsilon)$ is invertible for $\theta \in (-\pi, 0)$ if all $m_j = 1$.

The first assertion of the lemma tells us that for any $\epsilon \in (0, \epsilon_0)$, matrices $A_+(\epsilon) := A(0, \epsilon)$ and $A_-(\epsilon) := A(-\pi, \epsilon)$ have a zero eigenvalue of geometric and algebraic multiplicities $N-1$. We write $A_\pm(\epsilon)$ explicitly in the form

$$A_\pm(\epsilon) = \begin{bmatrix} -(1+p)M_\pm & -pM_\pm \\ -pN(\kappa_\pm) & 2\epsilon \sinh(\kappa_\pm)I - (1+p)N(\kappa_\pm) \end{bmatrix},$$

where $\kappa_\pm > 0$ are uniquely defined by

$$2\epsilon(\cosh(\kappa_+) - 1) = 2, \qquad 2\epsilon(\cosh(\kappa_-) - 1) = 2 + 4\epsilon,$$

whereas matrices $M_\pm$ are given by

$$M_+ = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

and

$$M_- = \begin{bmatrix} 1 & (-1)^{m_1} & (-1)^{m_1+m_2} & \cdots & (-1)^{m_1+m_2+\cdots+m_{N-1}} \\ (-1)^{m_1} & 1 & (-1)^{m_2} & \cdots & (-1)^{m_2+\cdots+m_{N-1}} \\ (-1)^{m_1+m_2} & (-1)^{m_2} & 1 & \cdots & (-1)^{m_3+\cdots+m_{N-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{m_1+m_2+\cdots+m_{N-1}} & (-1)^{m_2+\cdots+m_{N-1}} & (-1)^{m_3+\cdots+m_{N-1}} & \cdots & 1 \end{bmatrix}.$$

It is clear that $\mathrm{Null}(M_+)$ and $\mathrm{Null}(M_-)$ are $(N-1)$ dimensional.

The first $N$ rows of $A_+(\epsilon)$ are identical to the first row, whereas the last $N$ rows of $A_+(\epsilon)$ are linearly independent at $\epsilon = 0$ and, by continuity, for small $\epsilon \in [0, \epsilon_0)$. Therefore, $\mathrm{Null}(A_+(\epsilon))$ is $(N-1)$ dimensional for any $\epsilon \in [0, \epsilon_0)$. Similarly, the second, third, and $N$th rows of $A_-(\epsilon)$ are identical to the first row multiplied by $(-1)^{m_1}$, $(-1)^{m_1+m_2}$, and $(-1)^{m_1+m_2+\cdots+m_{N-1}}$, respectively. The last $N$ rows of $A_-(\epsilon)$ are linearly independent for small $\epsilon \geq 0$. Therefore, $\mathrm{Null}(A_-(\epsilon))$ is $(N-1)$ dimensional for any $\epsilon \in [0, \epsilon_0)$.

It remains to prove that the zero eigenvalue of $A_\pm(\epsilon)$ is not degenerate (has equal geometric and algebraic multiplicities) for $\epsilon \in (0, \epsilon_0)$. It is clear from the explicit form of $A_\pm(0)$ and $M_\pm$ that

$$u \in \mathrm{Null}(A_\pm(0)) \Leftrightarrow u = \begin{bmatrix} (1-p)w \\ pw \end{bmatrix}, \quad w \in \mathrm{Null}(M_\pm). \tag{32}$$

To construct a generalized kernel, we consider the inhomogeneous equation

$$A_\pm(0)\tilde{u} = u, \quad u \in \mathrm{Null}(A_\pm(0)).$$

Then, we obtain for $w \in \mathrm{Null}(M_\pm)$,

$$\tilde{u} = \begin{bmatrix} (1-p)\tilde{w} - w \\ p\tilde{w} \end{bmatrix}, \qquad M_\pm \tilde{w} = (p-1)w.$$

If $p \neq 1$, then no $\tilde{w} \in \mathbb{C}^N$ exists because $M_\pm$ is symmetric. Therefore, for $p \neq 1$, the zero eigenvalue has equal geometric and algebraic multiplicities for the matrix $A_\pm(0)$ and, by continuity, for the matrix $A_\pm(\epsilon)$ for $\epsilon \in [0, \epsilon_0)$.

The case $p = 1$ needs a separate consideration since $\tilde{w} = 0$ and the zero eigenvalue of $A_\pm(0)$ has geometric multiplicity $N-1$ and algebraic multiplicity $2N-2$. This case is considered in the Appendix, where we show that the degeneracy is broken for any $\epsilon \neq 0$, so that $A_\pm(\epsilon)$ in the case $p = 1$ still has a zero eigenvalue of equal geometric and algebraic multiplicities $N-1$ for any $\epsilon \in (0, \epsilon_0)$. □

Because the coefficient matrix $A(\theta, \epsilon)$ is singular at $\theta = 0$ and $\theta = -\pi$, we shall consider the limiting behavior of solutions of the linear system (30) near these points. The following abstract lemma gives the sufficient condition that the unique solution $c$ of the linear system (30) for small $\theta \neq 0$ and fixed $\epsilon \in (0, \epsilon_0)$ remains bounded in the limit $\theta \to 0$. Because $\epsilon$ is fixed, we can drop this parameter from the notations of the lemma.

**Lemma 3.** *Assume that $A(\theta) \in \mathbb{C}^{2N\times 2N}$ and $h(\theta) \in \mathbb{C}^{2N}$ are analytic in $\theta \in (-\theta_0, \theta_0)$ for $\theta_0 > 0$ and consider solutions of*

$$A(\theta)c = h(\theta), \quad c \in \mathbb{C}^{2N}.$$

*Assume that $A(\theta)$ is invertible for $\theta \neq 0$ and singular for $\theta = 0$ and that the zero eigenvalue of $A(0)$ has equal geometric and algebraic multiplicities $n \leq 2N$. A unique solution $c$ for $\theta \neq 0$ is bounded as $\theta \to 0$ if*

$$h(0) \perp \mathrm{Null}(A^*(0)) \quad \text{and} \quad \mathrm{Null}(A'(0)|_{\mathrm{Null}(A(0))}) = \{0\}. \tag{33}$$

**Remark 5.** We denote the Hermite conjugate of a matrix $A_0 \in \mathbb{C}^{2N\times 2N}$ by $A_0^* = \overline{A_0^T}$. Let

$$\mathrm{Null}(A_0) = \mathrm{span}\{u_1, \ldots, u_n\} \quad \text{and} \quad \mathrm{Null}(A_0^*) = \mathrm{span}\{v_1, \ldots, v_n\}, \tag{34}$$

where $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ are mutually orthogonal bases, so that

$$\langle u_i, v_j\rangle_{\mathbb{C}^{2N}} = \delta_{i,j} \quad \text{for all } 1 \leq i, j \leq n. \tag{35}$$

The restriction of matrix $A_1 \in \mathbb{C}^{2N\times 2N}$ on $\mathrm{Null}(A_0)$ denoted by $A_1|_{\mathrm{Null}(A_0)}$ can be expressed by the matrix $P \in \mathbb{C}^{n\times n}$ with elements

$$P_{ij} = \langle v_j, A_1 u_i\rangle_{\mathbb{C}^{2N}} \quad \text{for all } 1 \leq i, j \leq n. \tag{36}$$

**Proof.** The proof of the lemma is achieved with the method of Lyapunov–Schmidt reductions. Using the analyticity of $A(\theta)$ and $h(\theta)$, let us expand

$$A(\theta) = A_0 + \theta A_1 + \theta^2 \tilde{A}(\theta), \qquad h(\theta) = h_0 + \theta h_1 + \theta^2 \tilde{h}(\theta),$$

where $A_0 = A(0)$, $A_1 = A'(0)$, $h_0 = h(0)$, $h_1 = h'(0)$, and $\tilde{A}(\theta)$ and $\tilde{h}(\theta)$ are bounded as $\theta \to 0$. Given the basis for $\mathrm{Null}(A_0)$ in (34), we consider the orthogonal decomposition of the solution

$$c = \sum_{j=1}^n a_j u_j + b, \quad b \perp \mathrm{Null}(A_0). \tag{37}$$

The linear system becomes

$$(A_0 + \theta A_1 + \theta^2 \tilde{A}(\theta))b + \theta \sum_{j=1}^n a_j (A_1 + \theta \tilde{A}(\theta)) u_j = h_0 + \theta h_1 + \theta^2 \tilde{h}(\theta). \tag{38}$$

Projections of system (38) to the basis for $\mathrm{Null}(A_0^*)$ in (34) give $n$ equations

$$\sum_{j=1}^n \left(P_{ij} + \theta \tilde{P}_{ij}(\theta)\right) a_j + \langle v_i, (A_1 + \theta \tilde{A}(\theta))b\rangle_{\mathbb{C}^{2N}} = \langle v_i, h_1 + \theta \tilde{h}(\theta)\rangle_{\mathbb{C}^{2N}}, \quad 1 \leq i \leq n, \tag{39}$$

where $P_{ij}$ is given in (36), $\tilde{P}_{ij}(\theta) = \langle v_i, \tilde{A}(\theta) u_j\rangle_{\mathbb{C}^{2N}}$ is bounded as $\theta \to 0$, and we have used the condition $h_0 \perp \mathrm{Null}(A_0^*)$.

Let $Q : \mathbb{C}^{2N} \to \mathrm{Ran}(A_0) \subset \mathbb{C}^{2N}$ and $Q^* : \mathbb{C}^{2N} \to \mathrm{Ran}(A_0^*) \subset \mathbb{C}^{2N}$ be the projection operators. Recall that $\mathrm{Ran}(A_0) \perp \mathrm{Null}(A_0^*)$ and $\mathrm{Ran}(A_0^*) \perp \mathrm{Null}(A_0)$. Projection of system (38) to $\mathrm{Ran}(A_0)$ gives an equation for $b$

$$Q(A_0 + \theta A_1 + \theta^2 \tilde{A}(\theta))Q^* b = Q(h_0 + \theta h_1 + \theta^2 \tilde{h}(\theta)) - \theta \sum_{j=1}^n a_j Q(A_1 + \theta \tilde{A}(\theta)) u_j. \tag{40}$$

Because $QA_0Q^*$ is invertible, there is a unique map $\mathbb{C}^n \ni (a_1, \ldots, a_n) \mapsto b \in \mathrm{Ran}(A_0^*)$ for any $\theta \in (-\theta_0, \theta_0)$ such that $b$ is a solution of system (40) and for any $\theta \in (-\theta_0, \theta_0)$, there is $C > 0$ such that

$$\|b - Q^* A_0^{-1} Q h_0\|_{\mathbb{C}^{2N}} \leq C\theta. \tag{41}$$

Since $\mathrm{Null}(A_1|_{\mathrm{Null}(A_0)}) = \{0\}$, matrix $P$ is invertible. For any $b$ from solution of system (40) satisfying bound (41), there exists a unique solution of system (39) for $(a_1, \ldots, a_n)$ for any $\theta \in (-\theta_0, \theta_0)$ such that

$$\exists C > 0: \quad \|a - P^{-1}(I - Q)(h_1 - A_1 Q^* A_0^{-1} Q h_0)\|_{\mathbb{C}^n} \leq C\theta. \tag{42}$$

For any $\theta \neq 0$, the solution of system $A(\theta)c = h(\theta)$ is unique. Therefore, the unique solution obtained from the decomposition (37) for any $\theta \in (-\theta_0, \theta_0)$ is equivalent to the unique solution of system $A(\theta)c = h(\theta)$ for $\theta \neq 0$. □

We shall check that the conditions (33) of Lemma 3 are satisfied for our particular matrix $A(\theta,\epsilon)$ and the right-hand side vector $h(\theta,\epsilon)$ for both end points $\theta=0$ and $\theta=-\pi$.

**Lemma 4.** *Let $h_+(\epsilon):=h(0,\epsilon)$ and $h_-(\epsilon):=h(-\pi,\epsilon)$. For any $\epsilon\in(0,\epsilon_0)$, it is true that*

$$h_\pm(\epsilon)\perp \mathrm{Null}(A^*_\pm(\epsilon)) \quad \textit{and} \quad \mathrm{Null}(\partial_\theta A_\pm(\epsilon)|_{\mathrm{Null}(A_\pm(\epsilon))})=\{0\}. \tag{43}$$

**Proof.** It is sufficient to develop the proof for $\theta=0$. The proof for $\theta=-\pi$ is similar.

Recall that the first $N$ rows of $A(0,\epsilon)$ are identical to the first row. Since components of $h(0,\epsilon)$ are given by

$$\left\{\begin{array}{c}\displaystyle\sum_{m\in\mathbb{Z}} f_m\\ \displaystyle\sum_{m\in\mathbb{Z}} \mathrm{e}^{-\kappa|n-m|}g_m\end{array}\right\}_{n\in U_+\cup U_-},$$

the first $N$ entries of $h(0,\epsilon)$ are also identical so that $h(0,\epsilon)\in\mathrm{Ran}(A(0,\epsilon))\perp\mathrm{Null}(A^*(0,\epsilon))$ for any $\epsilon\in(0,\epsilon_0)$. Therefore, the first condition (43) is satisfied.

Next, we compute $A_1(\epsilon)=\partial_\theta A(\theta,\epsilon)|_{\theta=0}$. We know that

$$2\epsilon(\cosh(\kappa)-1)=2+\epsilon(2-2\cos(\theta))\Rightarrow\frac{\mathrm{d}\kappa}{\mathrm{d}\theta}=\frac{\sin(\theta)}{\sinh(\kappa)},$$

therefore,

$$A_1(\epsilon)\equiv\mathrm{i}\begin{bmatrix}2\epsilon I+(1+p)R & pR\\ 0 & 0\end{bmatrix}, \tag{44}$$

where

$$R=\begin{bmatrix}0 & m_1 & m_1+m_2 & \cdots & m_1+m_2+\cdots+m_{N-1}\\ m_1 & 0 & m_2 & \cdots & m_2+m_3+\cdots+m_{N-1}\\ m_1+m_2 & m_2 & 0 & \cdots & m_3+\cdots+m_{N-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ m_1+m_2+\cdots+m_{N-1} & m_2+\cdots+m_{N-1} & m_3+\cdots+m_{N-1} & \cdots & 0\end{bmatrix}.$$

Let $P(\epsilon)$ be the matrix in $\mathbb{C}^{(N-1)\times(N-1)}$ which represents the restriction $A_1(\epsilon)|_{\mathrm{Null}(A_0(\epsilon))}$. The existence of $a\in\mathrm{Null}(P(\epsilon))\subset\mathbb{C}^{N-1}$ is equivalent to the existence of $u\in\mathrm{Null}(A_0(\epsilon))\subset\mathbb{C}^{2N}$ such that $A_1(\epsilon)u\in\mathrm{Ran}(A_0(\epsilon))\perp\mathrm{Null}(A_0^*(\epsilon))$. In other words, we need to find $u\in\mathrm{Null}(A_0(\epsilon))$ such that the first $N$ entries of $A_1(\epsilon)u$ are identical (the other $N$ entries of $A_1(\epsilon)u$ are zeros).

By continuity in $\epsilon\in[0,\epsilon_0)$, the second condition (43) is satisfied if it is satisfied for $\epsilon=0$. Therefore, it is sufficient to check the existence of $u\in\mathrm{Null}(A_0(0))$ such that the first $N$ entries of $A_1(0)u$ are identical.

It follows from relations (32) and (44) that the existence of $u\in\mathrm{Null}(A_0(0))$ such that the first $N$ entries of $A_1(0)u$ are identical is equivalent to the existence of $w\in\mathrm{Null}(M_+)\subset\mathbb{C}^N$ such that all entries of $Rw$ are identical.

If $w=[w_1,w_2,\ldots,w_N]^T\in\mathrm{Null}(M_+)$, then

$$w_1+w_2+\cdots+w_N=0. \tag{45}$$

The condition $(Rw)_1=(Rw)_2$ gives

$$m_1(w_2+\cdots+w_N)=m_1w_1.$$

Constraint (45) implies that if $m_1\neq0$, then $w_1=0$ and $w_2+\cdots+w_N=0$. Continuing by induction for condition $(Rw)_j=(Rw)_{j+1}$, where $j\in\{1,2,\ldots,N-1\}$, we obtain that if $m_j\neq0$, then $w_j=0$ for all $j\in\{1,2,\ldots,N-1\}$. In view of constraint (45), we have $w_N=0$ that is $w=0\in\mathbb{C}^N$. As a result, we have proved that $\mathrm{Null}(A_1(0)|_{\mathrm{Null}(A_0(0))})=\{0\}$. By continuity in $\epsilon\in[0,\epsilon_0)$, $\mathrm{Null}(A_1(\epsilon)|_{\mathrm{Null}(A_0(\epsilon))})=\{0\}$ for small $\epsilon\neq0$, which gives the second condition (43) for $\theta=0$. □

**Remark 6.** Lemma 4 is proved without assuming that all $m_j=1$.

**Proof of Theorem 2.** By Lemma 4, assumptions of Lemma 3 are satisfied and the unique solution of system (30) for $\theta\in(-\pi,0)$ is continued to the unique bounded limit $c_0=\lim_{\theta\to0}c$. From the first $N$ equations of system (23), we infer that

$$\theta=0:\sum_{m\in U_+\cup U_-}((1+p)a_m+pb_m)=-\sum_{m\in\mathbb{Z}}f_m.$$

As a result, the simple pole singularity at $\theta=0$ $(z(\lambda_+)=0)$ in the Green function representation (22) with the Puiseux expansion (20) is canceled. Similarly, the simple pole singularity at $\theta=-\pi$ is canceled. On the other hand, the representation (22) contains $\epsilon$ in the denominator, which does not cancel out generally. As a result, Lemma 2 for all $m_j=1$ and Lemma 4 give that for any $\omega\in[0,4]$ and any $\epsilon\in(0,\epsilon_0)$, there exists $C>0$ such that

$$\|\mathbf{a}\|_{l^\infty}\le C\epsilon^{-1}.$$

This gives bound (28) and hence Theorem 2. □

### 3.3. Matching conditions for the resolvent operator

To complete the proof of Theorem 1, we need to prove that no singularities of linear system (23) are located inside the disks $B_{\delta_+}(1)$ and $B_{\delta_-}(-1)$ for $\epsilon$-independent $\delta_\pm > 0$. It is again sufficient to consider the disk $B_{\delta_+}(1)$ because of the symmetry in the $\Omega$-plane.

The free resolvent operator $R_0^+(\lambda) : l^2_\sigma(\mathbb{Z}) \to l^2_{-\sigma}(\mathbb{Z})$ with $\sigma > \frac{1}{2}$ is extended meromorphically in variable $\theta(\lambda)$ for $\lambda \in \mathbb{C}^+ \setminus [0, 4]$ with simple poles at $\theta = 0$ ($\lambda = 0$) and $\theta = -\pi$ ($\lambda = 4$). By Theorem 2, the resolvent operator $R_L^+(1 + \epsilon\omega) : l^1_1(\mathbb{Z}) \times l^1_1(\mathbb{Z}) \to l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$ is bounded for $\omega \in [0, 4]$ and the pole singularities are canceled. As a result, the resolvent operator $R_L^+(1 + \epsilon\lambda)$ can be extended as a bounded operator from $l^2_\sigma(\mathbb{Z}) \times l^2_\sigma(\mathbb{Z})$ to $l^2_{-\sigma}(\mathbb{Z}) \times l^2_{-\sigma}(\mathbb{Z})$ with $\sigma > \frac{1}{2}$ for any $\lambda \in \mathbb{C}^+ \setminus [0, 4]$. We need to show that no singularities of the resolvent operator $R_L(1 + \epsilon\lambda)$ exist in the upper semi-annulus

$$D_{\delta_+} = \left\{\lambda \in \mathbb{C}^+ : \gamma_+ < |\lambda| < \delta_+\epsilon^{-1}\right\},$$

where $\gamma_+ > 4$ and $\delta_+ \in (0, 1)$. A similar analysis can also be used to show that the resolvent operator $R_L^-(1 + \epsilon\lambda)$ can be extended as a bounded operator in the lower semi-disk in $B_{\delta_+}(1)$.

**Lemma 5.** *For any $\epsilon \in (0, \epsilon_0)$ and all $\lambda \in D_{\delta_+}$, the resolvent operator $R_L(1 + \epsilon\lambda)$ is a bounded operator from $l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$ to $l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$.*

**Proof.** Since the continuous spectrum does not touch the boundaries of $D_{\delta_+}$, the statement is true if and only if there exists a unique solution of linear system (23).

Let us denote $z(\lambda_+) = z(\lambda)$ and $z(\lambda_-) = -i\kappa(\lambda)$, where $z(\lambda)$ is found from the transcendental equation (17) and $\kappa(\lambda)$ with $\mathrm{Re}(\kappa(\lambda)) > 0$ admits the asymptotic expansion for $\lambda \in D_{\delta_+}$

$$e^{\kappa(\lambda)} = \frac{2 + \epsilon\lambda}{\epsilon} + 2 - \frac{\epsilon}{2 + \epsilon\lambda} + \mathcal{O}(\epsilon^2) \quad \text{as } \epsilon \to 0.$$

As earlier, we denote $q_j^+ = e^{-im_j z(\lambda)}$ and $q_j^- = e^{-m_j\kappa(\lambda)}$ for $j \in \{1, 2, \ldots, N - 1\}$.

We write the coefficient matrix (25) for $\Omega = 1 + \epsilon\lambda$ in the form

$$A(\lambda, \epsilon) \equiv \begin{bmatrix} -\epsilon\sqrt{\lambda(\lambda - 4)}I - (1 + p)M(\lambda) & -pM(\lambda) \\ -pN(\kappa) & \sqrt{(2 + \epsilon\lambda)^2 + 4\epsilon(2 + \epsilon\lambda)}I - (1 + p)N(\kappa) \end{bmatrix}, \tag{46}$$

where $M(\lambda) \equiv Q(q_1^+, q_2^+, \ldots, q_{N-1}^+)$, $N(\kappa(\lambda)) \equiv Q(q_1^-, q_2^-, \ldots, q_{N-1}^-)$, and the appropriate branches of $\sin z(\lambda)$ and $\sinh(\kappa(\lambda))$ are chosen in the domain $D_{\delta_+}$.

Let $|\lambda| = \mathcal{O}(\epsilon^{-r})$ as $\epsilon \to 0$ for $r \in [0, 1)$. Then, we have

$$A(\lambda, \epsilon) \to \begin{bmatrix} -(1 + p)M(\lambda) & -pM(\lambda) \\ -pI & (1 - p)I \end{bmatrix} \quad \text{as } \epsilon \to 0, \tag{47}$$

where $M(\lambda) \to I$ as $\epsilon \to 0$ if $r \in (0, 1)$ and $M(\lambda) \nrightarrow I$ as $\epsilon \to 0$ if $r = 0$. The limiting matrix (47) is not singular if $\gamma_+ > 4$. Hence $A(\lambda, \epsilon)$ is not singular for small $\epsilon \geq 0$ if $|\lambda| = \mathcal{O}(\epsilon^{-r})$ with $r \in [0, 1)$.

Let $|\lambda| = \mathcal{O}(\epsilon^{-r})$ as $\epsilon \to 0$ for $r \in (0, 1]$. Then, we have

$$A(\lambda, \epsilon) \to \begin{bmatrix} -(1 + \epsilon\lambda + p)I & -pI \\ -pI & (1 + \epsilon\lambda - p)I \end{bmatrix} \quad \text{as } \epsilon \to 0. \tag{48}$$

Again, the limiting matrix is not singular if $\epsilon\lambda \neq -1$ (that is $\delta_+ < 1$) and hence $A(\lambda, \epsilon)$ is not singular for small $\epsilon \geq 0$ if $|\lambda| = \mathcal{O}(\epsilon^{-r})$ with $r \in (0, 1]$.

Since the above asymptotic scaling overlap at $r \in (0, 1)$, the matrix $A(\lambda, \epsilon)$ is not singular in the domain $D_{\delta_+}$ for small $\epsilon > 0$. □

Theorem 1 is proved with Lemma 1, Theorem 2, and Lemma 5.

## 4. Perturbation arguments for the full resolvent

Let us now consider the full spectral problem (8). Thanks to Proposition 1 and expansion (7), we can represent $\phi_n^{2p}$ by

$$\phi_n^{2p} = \sum_{m \in U_+ \cup U_-} \delta_{n,m}(1 + \epsilon\chi_m) + \epsilon^2 W_n,$$

where $\{\chi_m\}_{m \in U_+ \cup U_-}$ is a set of numerical coefficients and $\{W_n\}_{n \in \mathbb{Z}} \in l^2(\mathbb{Z})$ is a new potential such that $\|\mathbf{W}\|_{l^2} = \mathcal{O}(1)$ as $\epsilon \to 0$.

In variables $\{(a_n, b_n)\}_{n \in \mathbb{Z}}$, the resolvent problem can be rewritten in the operator form

$$(\tilde{L} + \epsilon^2\tilde{W}) \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} - \Omega \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ -\mathbf{g} \end{bmatrix}, \tag{49}$$

where

$$\tilde{L} = \begin{bmatrix} -\epsilon\Delta + I - (1 + p)\tilde{V} & -p\tilde{V} \\ p\tilde{V} & \epsilon\Delta - I + (1 + p)\tilde{V}, \end{bmatrix}, \qquad \tilde{W} = \begin{bmatrix} -(1 + p)W & -pW \\ pW & (1 + p)W, \end{bmatrix},$$

and $\tilde{V}$ is the associated compact potential such that

$$(\tilde{V}u)_n = \sum_{m\in U_+\cup U_-} \delta_{n,m}(1+\epsilon\chi_m)u_m, \quad n\in\mathbb{Z}.$$

Let us denote the solution of the inhomogeneous system (49) by

$$\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} = R(\Omega)\begin{bmatrix}\mathbf{f}\\ -\mathbf{g}\end{bmatrix}, \tag{50}$$

where $R(\Omega)$ is the resolvent operator of the full spectral problem (8). The following theorem represents the main result of our paper.

**Theorem 3.** *Fix $U_+, U_- \subset \mathbb{Z}$ such that $U_+\cap U_- = \varnothing$, $N := |U_+| + |U_-| < \infty$, and $U_+\cup U_-$ is simply connected. For any integer $p \geq 2$, there are $\epsilon_0 > 0$ and $\delta > 0$ such that for any fixed $\epsilon \in (0, \epsilon_0)$ the resolvent operator*

$$R(\Omega) : l^2(\mathbb{Z})\times l^2(\mathbb{Z}) \to l^2(\mathbb{Z})\times l^2(\mathbb{Z})$$

*is bounded for any $\Omega \notin B_\delta(0)\cup[1, 1+4\epsilon]\cup[-1-4\epsilon, -1]$. Moreover, $R(\Omega)$ has exactly $2N$ poles (counting multiplicities) inside $B_\delta(0)$ and admits the limits*

$$R^\pm(\Omega) := \lim_{\mu\downarrow 0} R(\Omega \pm \mathrm{i}\mu)$$

*such that for any $\Omega \in [1, 1+4\epsilon]\cup[-1-4\epsilon, -1]$ and any $\epsilon \in (0, \epsilon_0)$, there is $C > 0$ such that*

$$\|R^\pm(\Omega)\|_{l^1_1\times l^1_1\to l^\infty\times l^\infty} \leq C\epsilon^{-1}.$$

**Proof.** Let $R_{\tilde{L}}(\Omega)$ be the resolvent operator for the inverse operator $(\tilde{L}-\Omega I)^{-1}$ associated with the compactly supported potential $\tilde{V}$. We shall prove that Theorem 1 remains valid for the resolvent operator $R_{\tilde{L}}(\Omega)$. Assuming it, the rest of the proof of Theorem 3 relies on the perturbation arguments and the resolvent identities

$$R(\Omega) = R_{\tilde{L}}(\Omega)(I+\epsilon^2\tilde{W}R_{\tilde{L}}(\Omega))^{-1} = (I+\epsilon^2 R_{\tilde{L}}(\Omega)\tilde{W})^{-1}R_{\tilde{L}}(\Omega).$$

Indeed, outside the continuous spectrum located at

$$\sigma_c(\tilde{L}+\epsilon^2\tilde{W}) = \sigma_c(\tilde{L}) = \sigma_c(L) \equiv [-1-4\epsilon, -1]\cup[1, 1+4\epsilon],$$

the resolvent operator $R_{\tilde{L}}(\Omega)$ is only singular inside the disk $B_{\delta_0}(0)$, where the perturbation theory of isolated eigenvalues apply. Inside the continuous spectrum, $R_{\tilde{L}}(\Omega)$ is extended as a bounded operator from $l^1_1(\mathbb{Z})\times l^1(\mathbb{Z})$ to $l^\infty(\mathbb{Z})\times l^\infty(\mathbb{Z})$ such that for any $\Omega \in [1, 1+4\epsilon]$ and any $\epsilon \in (0, \epsilon_0)$, there is $C > 0$ such that

$$\exists C > 0:\quad \|R^\pm_{\tilde{L}}(\Omega)\|_{l^1_1\times l^1_1\to l^\infty\times l^\infty} \leq C\epsilon^{-1}. \tag{51}$$

Since $\tilde{W}$ is a bounded $(\Omega,\epsilon)$-independent operator from $l^\infty(\mathbb{Z})\times l^\infty(\mathbb{Z})$ to $l^1_1(\mathbb{Z})\times l^1_1(\mathbb{Z})$ (note here that $\boldsymbol{\phi} \in l^2_s(\mathbb{Z})$ for any $s \geq 0$, see Remark 1), bound (51) implies that

$$\exists C > 0:\quad \|\epsilon^2\tilde{W}R_{\tilde{L}}(\Omega)\|_{l^1_1\times l^1_1\to l^1_1\times l^1_1} \leq C\epsilon,$$

so that $(I+\epsilon^2\tilde{W}R_{\tilde{L}}(\Omega))$ is an invertible bounded operator from $l^1_1(\mathbb{Z})\times l^1_1(\mathbb{Z})$ to $l^1_1(\mathbb{Z})\times l^1_1(\mathbb{Z})$ for small $\epsilon > 0$.

We only need to extend Theorem 1 to the resolvent operator $R_{\tilde{L}}(\Omega)$. The Green function representation (22) and the linear system (23) are now written with the factor $(1+\epsilon\chi_m)$ in the sum over $m \in U_+\cup U_-$. This implies that the coefficient matrix $A(\Omega,\epsilon)$ is now written as

$$\tilde{A}(\Omega,\epsilon) := \begin{bmatrix} 2\mathrm{i}\epsilon\sin z(\lambda_+)I - (1+p)Q^+(\Omega,\epsilon)(I+\epsilon D) & -pQ^+(\Omega,\epsilon)(I+\epsilon D)\\ -pQ^-(\Omega,\epsilon)(I+\epsilon D) & 2\mathrm{i}\epsilon\sin z(\lambda_-)I - (1+p)Q^-(\Omega,\epsilon)(I+\epsilon D)\end{bmatrix},$$

where $D$ is a diagonal matrix of elements $\{\chi_m\}_{m\in U_+\cup U_-}$. If $p \geq 2$, Lemmas 1, 2, 4 and 5 remain valid with new coefficient matrix $\tilde{A}(\Omega,\epsilon)$ as these lemmas were proved from the limit $\epsilon = 0$ (perturbation theory of the Appendix is only required for $p = 1$), where $\tilde{A}(\Omega,0) = A(\Omega,0)$. Therefore, Theorem 1 holds for the resolvent operator $R_{\tilde{L}}(\Omega)$ if $p \geq 2$. □

**Corollary 1.** *The result of Theorem 3 holds for $p = 1$ if $N = 1$.*

**Proof.** If $N = 1$ (which is the case of fundamental discrete soliton), the $2\times 2$ coefficient matrix

$$\tilde{A}(\Omega,\epsilon) = \begin{bmatrix} 2\mathrm{i}\epsilon\sin z(\lambda_+) - (1+p)(1+\epsilon\chi_0) & -p(1+\epsilon\chi_0)\\ -p(1+\epsilon\chi_0) & 2\mathrm{i}\epsilon\sin z(\lambda_-) - (1+p)(1+\epsilon\chi_0)\end{bmatrix},$$

is only singular in $B_\delta(0)$ for small $\epsilon > 0$, where a double pole of $R_{\tilde{L}}(\Omega)$ and $R(\Omega)$ resides. □

Unfortunately, in the cubic case $p = 1$, we cannot generally extend the result of Theorem 3 to multi-site discrete solitons with $N \geq 2$ because the perturbation theory for $\tilde{A}(\Omega, \epsilon)$ near the end points of the continuous spectra $\Omega = \pm 1$ and $\Omega = \pm(1 + 4\epsilon)$ draws no conclusion in a general case. For instance, reworking the perturbative arguments of the Appendix, we obtain the necessary condition for $\mathrm{Null}(A_\pm(\epsilon))^2 > \mathrm{Null}(A_\pm(\epsilon))$ in the form

$$\epsilon(2I - J - 2D)w + \mathcal{O}(\epsilon^2) \perp w \in \mathrm{Null}(M_+),$$

where $I$ is the identity matrix in $\mathbb{R}^N$, $J$ is the two-diagonal matrix (55) from the Appendix, and $D$ is a diagonal matrix of $\{\chi_m\}_{m \in U_+ \cup U_-}$. Because $(2I - J - 2D)$ is no longer positive definite, the degenerate cases with $\mathrm{Null}(A_\pm(\epsilon))^2 > \mathrm{Null}(A_\pm(\epsilon))$ are possible.

To illustrate this possibility, we set $N = 3$ and consider three distinct simply connected discrete solitons associated with the sets

$$\text{(a) } U_+ = \{0, 1, 2\}; \qquad \text{(b) } U_+ = \{0, 1\}, U_- = \{2\}; \qquad \text{(c) } U_+ = \{0, 2\}, U_- = \{1\}.$$

Computations of the power expansions (7) give

$$\text{(a) } \chi_m = \begin{cases} 1, & m = 0, \\ 0, & m = 1, \\ 1, & m = 2, \end{cases} \qquad \text{(b) } \chi_m = \begin{cases} 1, & m = 0, \\ 2, & m = 1, \\ 3, & m = 2, \end{cases} \qquad \text{(c) } \chi_m = \begin{cases} 3, & m = 0, \\ 4, & m = 1, \\ 3, & m = 2. \end{cases}$$

As a result, the matrix $C \equiv 2I - J - 2D$ is obtained in the form

$$\text{(a) } C = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad \text{(b) } C = \begin{bmatrix} 0 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -4 \end{bmatrix}, \qquad \text{(c) } C = \begin{bmatrix} -4 & -1 & 0 \\ -1 & -6 & -1 \\ 0 & -1 & -4 \end{bmatrix}.$$

We have

$$\mathrm{Null}(M_+) = \mathrm{span}\{w_1, w_2\}, \qquad w_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad w_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

from which we compute the matrix of projections $P_{ij} = \langle C w_i, w_j \rangle_{\mathbb{C}^3}$ in the form

$$\text{(a) } P = \begin{bmatrix} 0 & 0 \\ 0 & \frac{8}{3} \end{bmatrix}, \qquad \text{(b) } P = \begin{bmatrix} -2 & \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & -\frac{2}{3} \end{bmatrix}, \qquad \text{(c) } P = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}.$$

The projection matrices in cases (a) and (b) are singular. In order to show that $\mathrm{Null}(A_\pm(\epsilon))^2 = \mathrm{Null}(A_\pm(\epsilon))$ for $\epsilon \in (0, \epsilon_0)$, we need to extend the perturbation arguments of the Appendix to the order $\mathcal{O}(\epsilon^2)$. Although it is quite possible that the non-degeneracy condition $\mathrm{Null}(A_\pm(\epsilon))^2 = \mathrm{Null}(A_\pm(\epsilon))$ is still satisfied for simply connected multi-site discrete solitons for $p = 1$, we do not include computations of the higher-order perturbation theory in this paper.

## 5. Case study for a non-simply connected two-site soliton

We explain now why the resolvent operator associated with non-simply connected multi-site discrete solitons have singularities in the anti-continuum limit. These singularities appear in Lemma 2 because the determinant $D_N(q_1, q_2, \ldots, q_{N-1})$ given by (31) has zeros for some $\theta \in (-\pi, 0)$.

Let us consider a case study of a two-site soliton with $n_1 = 0$ and $n_2 = m \geq 2$. For clarity of presentation, we only consider $p \geq 2$. The power series expansions (7) give

$$m \geq 3: \quad \phi_n^{2p} = (\delta_{n,0} + \delta_{n,m})(1 + 2\epsilon - 2\epsilon^2) + \epsilon^3 W_n, \quad n \in \mathbb{Z}, \tag{52}$$

and

$$m = 2: \quad \phi_n^{2p} = (\delta_{n,0} + \delta_{n,m})(1 + 2\epsilon - 3\epsilon^2) + \epsilon^3 W_n, \quad n \in \mathbb{Z}, \tag{53}$$

where $\{W_n\}_{n \in \mathbb{Z}} \in l^2(\mathbb{Z})$ is a new potential such that $\|\mathbf{W}\|_{l^2} = \mathcal{O}(1)$ as $\epsilon \to 0$.

Let us consider the coefficient matrix $A(\theta, \epsilon)$ at the continuous spectrum $[1, 1 + 4\epsilon]$ defined by (29). We have explicitly

$$M(\theta) = \begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{i}m\theta} \\ \mathrm{e}^{-\mathrm{i}m\theta} & 1 \end{bmatrix}, \qquad N(\kappa) = \begin{bmatrix} 1 & \mathrm{e}^{-2\kappa} \\ \mathrm{e}^{-2\kappa} & 1 \end{bmatrix}.$$

Note that $\det M(\theta) = 1 - \mathrm{e}^{-2\mathrm{i}m\theta}$. Besides the end points $\theta = -\pi$ and $\theta = 0$, the matrix $M(\theta)$ (and, therefore, the limiting matrix $A(\theta, 0)$) is singular at the intermediate points $\theta_j = -\frac{\pi j}{m}$ for $j = 1, 2, \ldots, m - 1$.

If $m = 2$, there is only one intermediate-point singularity of $A(\theta, 0)$ at $\theta = -\frac{\pi}{2}$. We have $\dim \mathrm{Null}\, A\left(-\frac{\pi}{2}, 0\right) = 1$ and

$$\mathrm{Null}\, A^*\left(-\frac{\pi}{2}, 0\right) = \mathrm{span}\{e_1\}, \qquad e_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The first two entries of the right-hand side vector $h(\theta, \epsilon)$ in the linear system (30) are given explicitly by

$$h_1(\theta, \epsilon) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\theta|n|} f_n, \qquad h_2(\theta, \epsilon) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\theta|n-2|} f_n.$$

The constraint $\langle e_1, h\left(-\frac{\pi}{2}, 0\right)\rangle_{\mathbb{C}^4} = 0$ of Lemma 3 gives $h_1\left(-\frac{\pi}{2}, 0\right) = -h_2\left(-\frac{\pi}{2}, 0\right)$ and it is equivalent to the constraint $f_1 = 0$. If $f \in l^1(\mathbb{Z})$ with $f_1 \neq 0$, then the solution of the linear system (23) and hence the resolvent operator (22) has a singularity at $\Omega = 1 + 2\epsilon$ $\left(\theta = -\frac{\pi}{2}\right)$ as $\epsilon \to 0$. This singularity indicates a resonance at the mid-point of the continuous spectrum in the anti-continuum limit.

We would like to show that the resonance does not actually occur at the continuous spectrum if $\epsilon > 0$ and does not lead to (unstable) eigenvalues of the continuous spectrum. To do so, we use the perturbation theory up to the quadratic order in $\epsilon$.

Expanding the solutions of the transcendental equation

$$2\epsilon(\cosh(\kappa) - 1) = 2 + \epsilon\omega, \quad \omega = 2 - 2\cos(\theta),$$

we obtain

$$e^{-\kappa} = \frac{1}{2}\epsilon - \frac{2+\omega}{4}\epsilon^2 + \mathcal{O}(\epsilon^3) \quad \text{as } \epsilon \to 0$$

and

$$2\epsilon\sinh(\kappa) = 2 + (2+\omega)\epsilon - \epsilon^2 + \mathcal{O}(\epsilon^3) \quad \text{as } \epsilon \to 0.$$

Using expansion (53) for $m = 2$, we obtain the extended coefficient matrix $\tilde{A}(\theta, \epsilon)$ in the form

$$\tilde{A}(\theta, \epsilon) := \begin{bmatrix} 2i\epsilon\sin(\theta)I - (1+p)\nu(\epsilon)M(\theta) & -p\nu(\epsilon)M(\theta) \\ -p\nu(\epsilon)N(\kappa) & 2\epsilon\sinh(\kappa)I - (1+p)\nu(\epsilon)N(\kappa) \end{bmatrix},$$

where $\nu(\epsilon) = 1 + 2\epsilon - 3\epsilon^2 + \mathcal{O}(\epsilon^3)$. Using MATHEMATICA, we expand the roots of $\det\tilde{A}(\theta, \epsilon) = 0$ near $\theta = -\frac{\pi}{2}$ and $\epsilon = 0$ to obtain

$$\theta = -\frac{\pi}{2} + (p-1)\epsilon + 2(1-p)\epsilon^2 + i(p-1)^2\epsilon^2 + \mathcal{O}(\epsilon^3) \quad \text{as } \epsilon \to 0. \tag{54}$$

Since $\mathrm{Im}(\theta) > 0$ for small $\epsilon > 0$ and $z(\lambda_+) = \theta$, the solution of the linear system (30) is singular at the point $z(\lambda_+)$, which does not belong to the domain $\mathrm{Im}\, z(\lambda_+) < 0$ and hence violates the condition (17).

The singularity of the solution of the linear system (30) is still located near the continuous spectrum for small $\epsilon > 0$ and, therefore, the resolvent operator $R(\Omega)$ becomes large near the points $\Omega = \pm(1+2\epsilon)$ (although, it is always a bounded operator from $l^2_\sigma(\mathbb{Z}) \times l^2_\sigma(\mathbb{Z})$ to $l^2_{-\sigma}(\mathbb{Z}) \times l^2_{-\sigma}(\mathbb{Z})$ for small $\epsilon > 0$ and fixed $\sigma > \frac{1}{2}$). Since $\sin(\theta)$ is nonzero for $\theta = -\frac{\pi}{2}$, the norm of $R(\Omega)$ is proportional to the 2-norm of inverse matrix $\tilde{A}^{-1}(\theta, \epsilon)$.

Fig. 2 illustrates the singularities of the resolvent operator $R(\Omega)$ by plotting pseudospectra of the coefficient matrix $A(\Omega, \epsilon)$ in the complex $\Omega$-plane for $p = 2$ and $\epsilon = 0.05$. The subplots (a) and (b) for $m = 1$ show that the matrix is singular at the edges of the continuous spectrum $\Omega = \pm 1$ and $\Omega = \pm(1+2\epsilon)$, and at four points on the imaginary axis, the latter being attributed to the splitting of zero eigenvalue in the anti-continuum limit. The subplots (c) and (d) for $m = 2$ and $m = 3$, respectively show that in addition to singularities at the edges of continuous spectrum there are also $m-1$ local maxima at its intermediate points. This local maxima correspond to the minima of $\det A(\Omega, \epsilon)$. We also notice the wedges on the level sets as they cross the continuous spectrum occurring due to the jump discontinuities in $z(\lambda_+)$ and $A(\Omega, \epsilon)$ because the resolvent operator $R(\Omega)$ is discontinuous across the continuous spectrum.

Fig. 3 further illustrates what exactly happens at the continuous spectrum. On the left, we plot $\left\|A(\Omega, \epsilon)^{-1}\right\|_2$ versus $\theta \in (-\pi, 0)$ for the case $m = 2$. On the right, we show that the height of the local maxima near $\theta = -\pi/2$ is proportional to $\epsilon^{-2}$ as prescribed by formula (54).

Fig. 4 gives an illustration for pseudospectra of the resolvent operator $R(\Omega)$. Recall that on the continuous spectrum, $R(\Omega)$ is a bounded operator from $l^2_\sigma(\mathbb{Z}) \times l^2_\sigma(\mathbb{Z})$ to $l^2_{-\sigma}(\mathbb{Z}) \times l^2_{-\sigma}(\mathbb{Z})$ for fixed $\sigma > \frac{1}{2}$. To incorporate the weighted $l^2$ spaces, we consider the renormalized resolvent operator

$$\tilde{R}_L(\Omega) = (\tilde{L} - \Omega\tilde{I}_2)^{-1} : l^2(\mathbb{Z}) \times l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) \times l^2(\mathbb{Z}),$$

where $\tilde{L}$ is derived from $L$ by replacing operators $I$, $\Delta$ and $V$ with $\tilde{I}$, $\tilde{\Delta}$ and $\tilde{V}$, and $\tilde{I}_2 = \mathrm{diag}\{\tilde{I}, \tilde{I}\}$. Here

$$\tilde{I}_{n,m} = \kappa_n^2\delta_{n,m}, \qquad \tilde{V}_{n,m} = \tilde{I}_{n,m}\sum_{j\in U_+\cup U_-}\delta_{n,j},$$

$$\tilde{\Delta}_{n,n} = -2\kappa_n^2, \qquad \tilde{\Delta}_{n,n+1} = \tilde{\Delta}_{n+1,n} = \kappa_n\kappa_{n+1},$$

and $\kappa_n = (1+n^2)^{\sigma/2}$. The lattice problem is considered for $2K+1$ grid points and the corresponding matrix representation of operators $\tilde{L}$ and $\tilde{I}_2$ is constructed subject to the Dirichlet boundary conditions.

The level sets for the $(2K+1) \times (2K+1)$ matrix approximation of the resolvent $\tilde{R}(\Omega)$ are plotted on Fig. 4. The subplots of Fig. 4 correspond to the subplots of Fig. 2. We observe that the norm of $\tilde{R}(\Omega)$ has the same global behavior as for the norm of $A(\Omega, \epsilon)^{-1}$. However, the resolvent operator $\tilde{R}(\Omega)$ has no singularities at the edges $\Omega = \pm 1$ and $\Omega = \pm(1+4\epsilon)$ because these singularities are canceled according to Lemma 4 (which remains true for any $m \geq 1$, see Remark 6).

Although no arguments exist to exclude resonances at the mid-point of the continuous spectrum for the linearized dNLS equation (8), the case study of a two-site discrete soliton suggests that the resonances do not happen at the continuous spectrum for small but finite values of $\epsilon > 0$. Moreover, the resonances do not bifurcate to the isolated eigenvalues of the continuous spectrum because isolated eigenvalues near the continuous spectrum would violate the count of unstable eigenvalues (12). Therefore, the only scenario for these resonances is to move to the resonant poles on the wrong sheets $\mathrm{Im}(z(\lambda_\pm)) > 0$ of the definition of $z(\lambda_\pm)$.

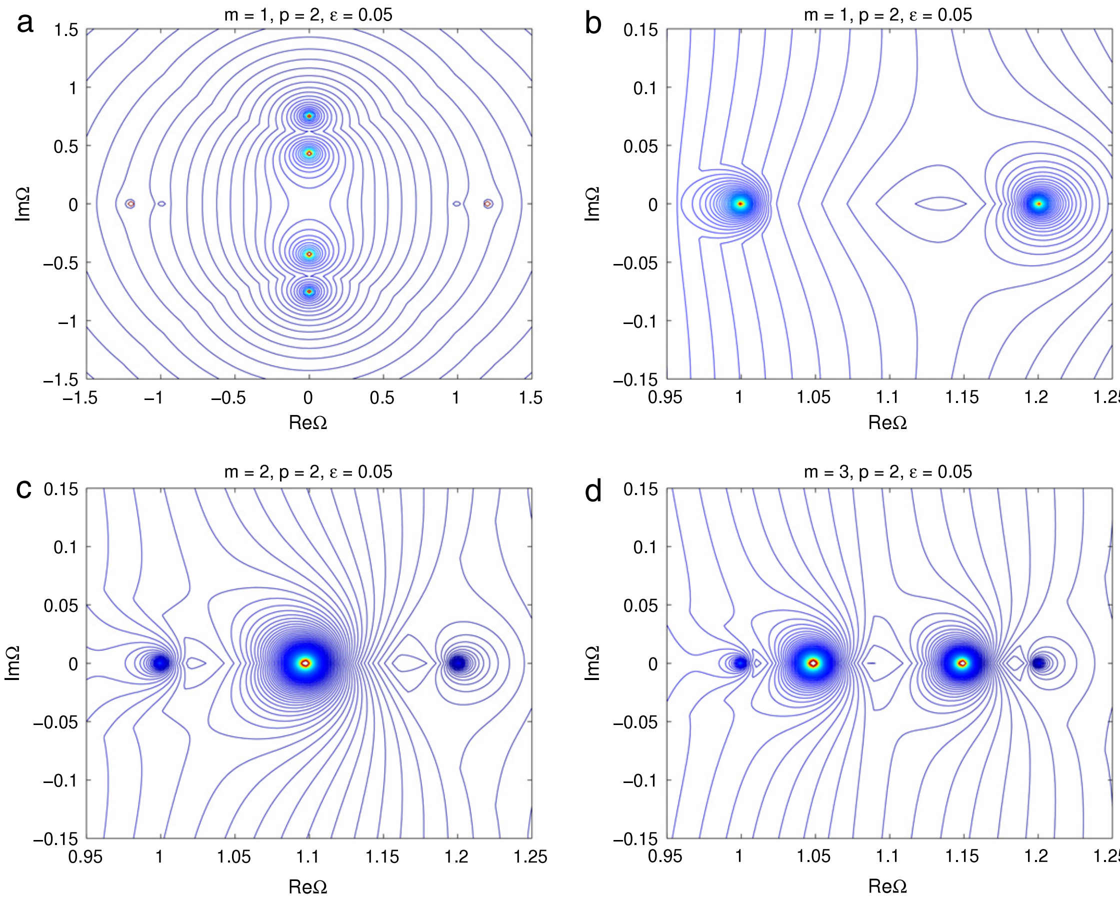


**Fig. 2.** Level sets for $\|A(\Omega,\epsilon)^{-1}\|_2$ in the $\Omega$-plane. The levels are equidistant on a logarithmic scale.

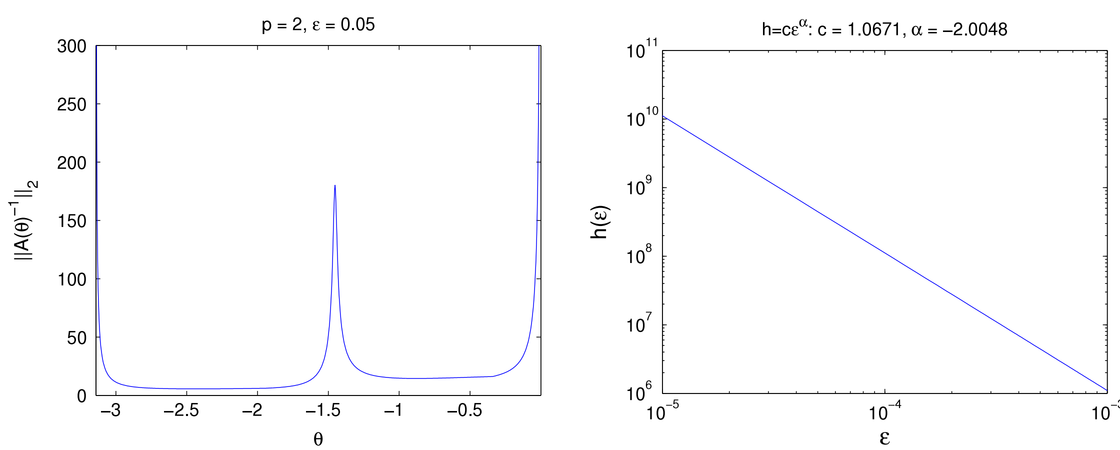


**Fig. 3.** Left: Norm $\|A(\Omega,\epsilon)^{-1}\|_2$ versus $\theta \in (-\pi, 0)$ for $m = 2$. Right: The value of local maxima of $\|A(\Omega,\epsilon)^{-1}\|_2$ in the neighborhood of $\theta = -\pi/2$ as a function of $\epsilon$.

## Appendix. Perturbative arguments for the cubic dNLS equation

We recall the coefficient matrices $A_\pm(\epsilon)$ from the proof of Lemma 2. In the case $p = 1$ (the cubic dNLS equation), these matrices are rewritten in the form

$$A_\pm(\epsilon) = \begin{bmatrix} -2M_\pm & -M_\pm \\ -N(\kappa_\pm) & 2\epsilon\sinh(\kappa_\pm)I - 2N(\kappa_\pm) \end{bmatrix},$$

where $\kappa_\pm > 0$ are uniquely defined by

$$2\epsilon(\cosh(\kappa_+) - 1) = 2, \qquad 2\epsilon(\cosh(\kappa_-) - 1) = 2 + 4\epsilon.$$

We recall that $\mathrm{Null}(A_\pm(\epsilon))$ and $\mathrm{Null}(M_\pm)$ are $(N-1)$ dimensional for any $\epsilon \in [0, \epsilon_0)$. It is clear from the explicit form of $A_\pm^*(\epsilon)$ that

$$u \in \mathrm{Null}(A_\pm^*(\epsilon)) \Leftrightarrow u = \begin{bmatrix} w \\ 0 \end{bmatrix}, \quad w \in \mathrm{Null}(M_\pm).$$

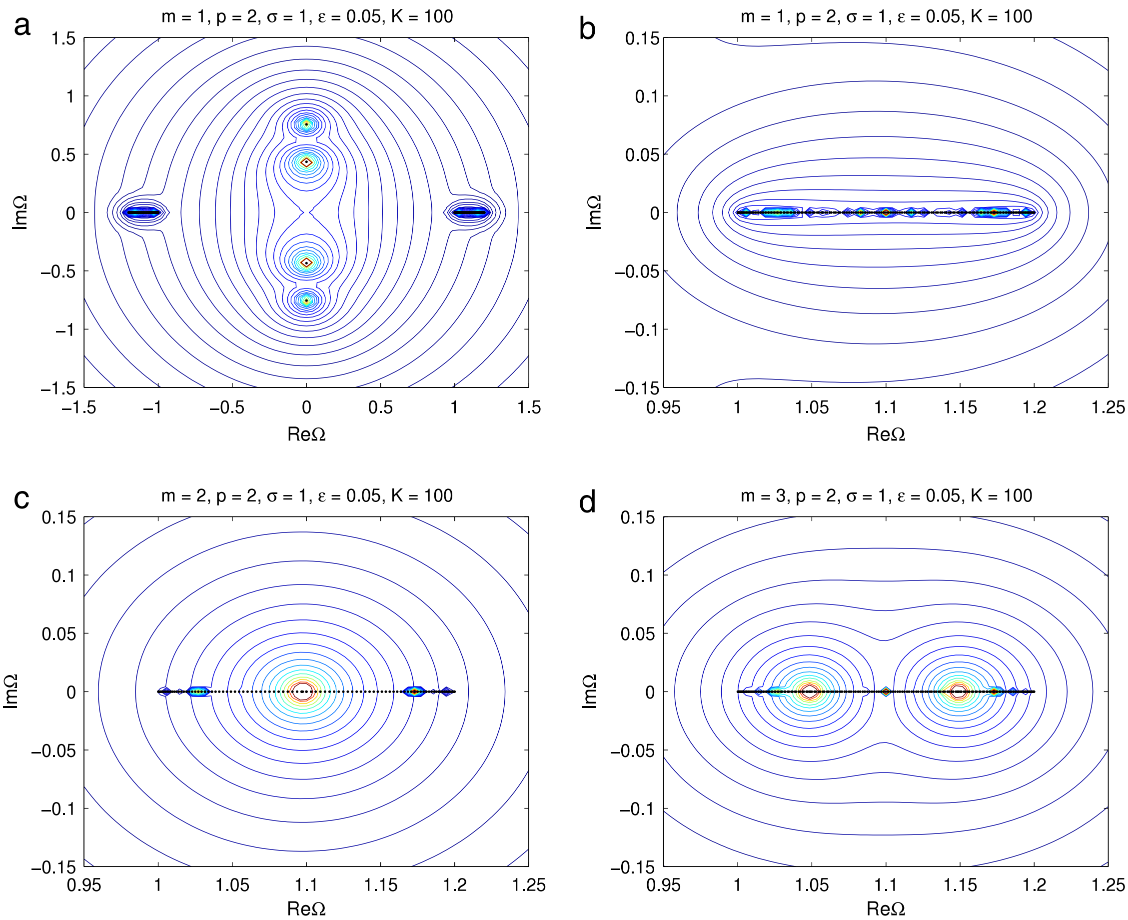


**Fig. 4.** The level sets of $\|(\tilde{L} - \Omega\tilde{I}_2)^{-1}\|_2$ in the $\Omega$-plane. The black dots represent eigenvalues of the matrix representation of operator $\tilde{L}$. The levels are equidistant on a logarithmic scale.

At $\epsilon = 0$, we also recall that $\mathrm{Null}(A_\pm(0))^2$ is $(2N-2)$ dimensional because of $(N-1)$ eigenvectors and $(N-1)$ generalized eigenvectors,

$$A_\pm(0)\begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad A_\pm(0)\begin{bmatrix} -w \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ w \end{bmatrix}, \quad w \in \mathrm{Null}(M_\pm).$$

We would like to show that $\mathrm{Null}(A_\pm(\epsilon))^2 = \mathrm{Null}(A_\pm(\epsilon))$ is $(N-1)$ dimensional for any $\epsilon \in (0, \epsilon_0)$. In other words, we would like to show that no solution $\tilde{u} \in \mathbb{C}^{2N}$ of the inhomogeneous equation $A_\pm(\epsilon)\tilde{u} = u \in \mathrm{Null}(A_\pm(\epsilon))$ exists for $\epsilon \in (0, \epsilon_0)$. This task is achieved by the perturbation theory. We will only consider the case $A_+(\epsilon)$, which corresponds to $\theta = 0$. The case $A_-(\epsilon)$ which corresponds to $\theta = -\pi$ can be considered similarly.

We shall only consider the case of the simply connected set $U_+ \cup U_-$ with $m_1 = m_2 = \cdots = m_{N-1} = 1$. The general case holds without any changes.

Thanks to the asymptotic expansions

$$\mathrm{e}^{-\kappa_+} = \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2), \qquad 2\epsilon\sinh(\kappa_+) = 2 + 2\epsilon + \mathcal{O}(\epsilon^2), \quad \text{as } \epsilon \to 0,$$

we obtain the asymptotic expansion

$$A_+(\epsilon) = \begin{bmatrix} -2M_+ & -M_+ \\ -I & O \end{bmatrix} + \epsilon \begin{bmatrix} O & O \\ -\frac{1}{2}J & 2I - J \end{bmatrix} + \mathcal{O}(\epsilon^2),$$

where $I$ and $O$ are identity and zero matrices in $\mathbb{R}^N$ and $J$ is the three-diagonal matrix in $\mathbb{R}^N$

$$J = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{55}$$

Note that $(2I - J)$ is a strictly positive matrix because it appears in the finite-difference approximation of the differential operator $-\partial_x^2$ subject to the Dirichlet boundary conditions.

Perturbative computations show that if $u \in \text{Null}(A_+(\epsilon))$, then $u$ is represented asymptotically as

$$u = \begin{bmatrix} \epsilon(2I - J)v \\ v \end{bmatrix} + \mathcal{O}(\epsilon^2),$$

where $v + 2\epsilon(2I - J)v + \mathcal{O}(\epsilon^2) = w \in \text{Null}(M_+)$.

Now, there exists a solution $\tilde{u} \in \mathbb{C}^{2N}$ of the inhomogeneous equation $A_+(\epsilon)\tilde{u} = u \in \text{Null}(A_+(\epsilon))$ if and only if $u \perp \text{Null}(A_+^*(\epsilon))$. For small $\epsilon \in (0, \epsilon_0)$, this condition implies that

$$\epsilon(2I - J)v + \mathcal{O}(\epsilon^2) = \epsilon(2I - J)w + \mathcal{O}(\epsilon^2) \perp w \in \text{Null}(M_+),$$

which is not possible since $(2I - J)$ is a strictly positive matrix.

## References


[1] P.G. Kevrekidis, The Discrete Nonlinear Schrödinger Equation: Mathematical Analysis, Numerical Computations and Physical Perspectives, in: Springer Tracts in Modern Physics, vol. 232, Springer, New York, 2009.
[2] G.L. Alfimov, V.A. Brazhnyi, V.V. Konotop, On classification of intrinsic localized modes for the discrete nonlinear Schrödinger equation, Physica D 194 (2004) 127–150.
[3] J.C. Eilbeck, P.S. Lomdahl, A.C. Scott, Soliton structure in crystalline acetanilide, Phys. Rev. B 30 (1984) 4703–4712.
[4] S. Aubry, G. Abramovici, Chaotic trajectories in the standard map. The concept of anti-integrability, Physica D 43 (1990) 199–219.
[5] R.S. MacKay, S. Aubry, Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators, Nonlinearity 7 (1994) 1623–1643.
[6] S. Aubry, Anti-integrability in dynamical and variational problems, Physica D 86 (1995) 284–296.
[7] S. Aubry, Breathers in nonlinear lattices: existence, linear stability and quantization, Physica D 103 (1997) 201–250.
[8] D. Bambusi, Exponential stability of breathers in Hamiltonian networks of weakly coupled oscillators, Nonlinearity 9 (1996) 433–457.
[9] D.E. Pelinovsky, P.G. Kevrekidis, D.J. Frantzeskakis, Stability of discrete solitons in nonlinear Schrödinger lattices, Physica D 212 (2005) 1–19.
[10] S. Cuccagna, On instability of excited states of the nonlinear Schrödinger equation, Physica D 238 (2009) 38–54.
[11] D.E. Pelinovsky, P.G. Kevrekidis, Stability of discrete dark solitons in nonlinear Schrödinger lattices, J. Phys. A: Math. Gen. 41 (2008) 185206. (10 pp).
[12] S. Cuccagna, M. Tarulli, On asymptotic stability of standing waves of discrete Schrödinger equation in $\mathbb{Z}$, SIAM J. Math. Anal. 41 (2009) 861–885.
[13] P.G. Kevrekidis, D.E. Pelinovsky, A. Stefanov, Asymptotic stability of small bound states in the discrete nonlinear Schrödinger equation in one dimension, SIAM J. Math. Anal. 41 (2009) 2010–2030.
[14] S. Cuccagna, Orbitally but not asymptotically stable ground states for the discrete NLS, Discrete Contin. Dyn. Syst. 26 (2010) 105–134.
[15] M. Johansson, S. Aubry, Growth and decay of discrete nonlinear Schrödinger breathers interacting with internal modes or standing-wave phonons, Phys. Rev. E 61 (2000) 5864–5879.
[16] P. Panayotaros, D. Pelinovsky, Periodic oscillations of discrete NLS solitons in the presence of diffraction management, Nonlinearity 21 (2008) 1265–1279.
[17] M. Chugunova, D. Pelinovsky, Count of unstable eigenvalues in the generalized eigenvalue problem, J. Math. Phys. 51 (2010) 052901 (19 pages).
[18] A. Komech, E. Kopylova, M. Kunze, Dispersive estimates for 1D discrete Schrödinger and Klein–Gordon equations, Appl. Anal. 85 (2006) 1487–1508.
[19] D.E. Pelinovsky, A. Stefanov, On the spectral theory and dispersive estimates for a discrete Schrödinger equation in one dimension, J. Math. Phys. 49 (2008) 113501. (17 pp).