# Multi-pulse solutions to the focusing NLS equation.

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# Non-Linear Schrodinger Equation

$$E\Psi = -\frac{d^2}{dx^2}\Psi + V\left(x\right)\Psi + \sigma\left|\Psi\right|^2\Psi$$

Energy

Periodic Potential

The non-linear term

The Defocusing NLS $\sigma = 1$

The Focusing NLS $\sigma = -1$

# For our purposes…

- Periodic Potential
  - The potential that we will use is

$$V(x) = \sin^2(x/2)$$

- Non-linear term
  - We will be examining the focusing NLS

$$\sigma = -1$$

  - All solutions are real

$$E\Psi = -\frac{d^2}{dx^2}\Psi + \sin^2(x/2)\,\Psi - \Psi^3$$

# Spectral Renormalization Method

$$E\Psi(x) = -\partial_x^2\Psi(x) + V(x)\Psi(x) - \Psi(x)^3$$

Define the linear operator L

$$L = -\partial xx + V(x)$$

$$(L - E)\Psi = \Psi^3$$

Consider the Iterations Scheme

$$u_{n+1} = (L - E)^{-1} u_n^3$$

# Will This Scheme Converge

$$u_{n+1} = (L-E)^{-1} u_n^3$$

- Lets say $u_n$ is $a_n \Psi$

$$a_{n+1} = a_n^3$$

- So we need to define a renormalization factor

$$M_n = \frac{\langle u_n, (L-E)u_n \rangle}{\langle u_n, u_n^3 \rangle} \qquad 1 = \frac{\langle \Psi, (L-E)\Psi \rangle}{\langle \Psi, \Psi^3 \rangle}$$

Renormalized Iteration Scheme

$$u_{n+1} = M_n^{3/2} (L-E)^{-1} u_n^3$$

# Numerical Methods

- There are three methods that we will be examining all based on the Spectral Renormalization method
  - Finite-Difference

$$\widehat{u_{n+1}} = M_n^{3/2} \left(L - IE\right)^{-1} \widehat{u_n^3}$$

  - Discrete Fourier

$$\widehat{u_{n+1}} = M_n^{3/2} \frac{\widehat{V\left(x\right) u_n} + \widehat{u_n^3}}{k^2 - E}$$

  - Bloch-Fourier

$$\tilde{\phi}_n = \int \phi\left(x\right) \bar{l}_m \left(x;k\right) dx \qquad \phi\left(x\right) = \int \sum_{m\in\mathbb{N}} \tilde{\phi}_n \left(k\right) l_m \left(x;k\right) dk$$

# How do we measure error?

- Compare with exact solution.

$$e_{actual}^{(n)} = \sup_{x \in \mathbb{R}} |u_n - \Psi|$$

- Compare $M_n$ with unity

$$e_M^{(n)} = |M_n - 1|$$

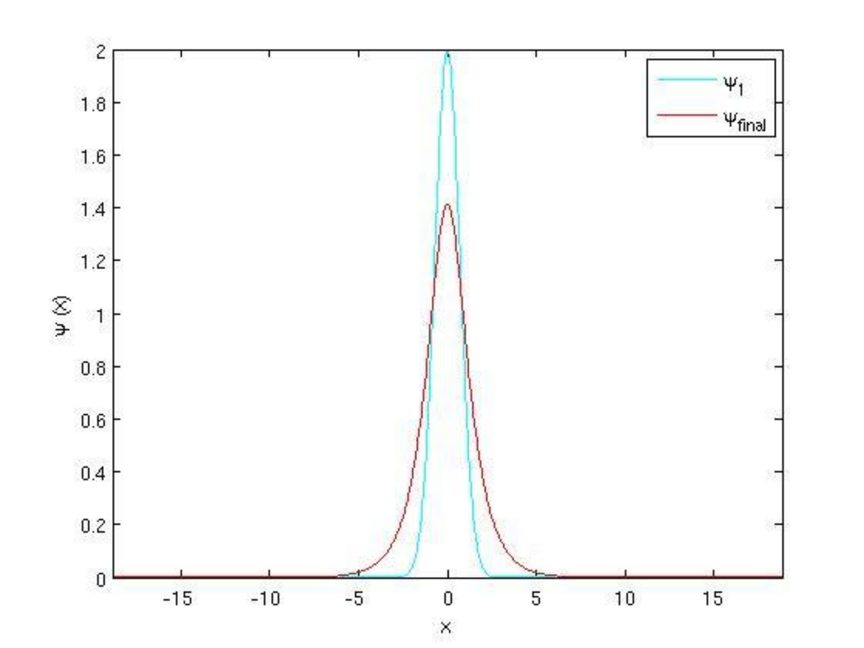
- Compare two successive iterations

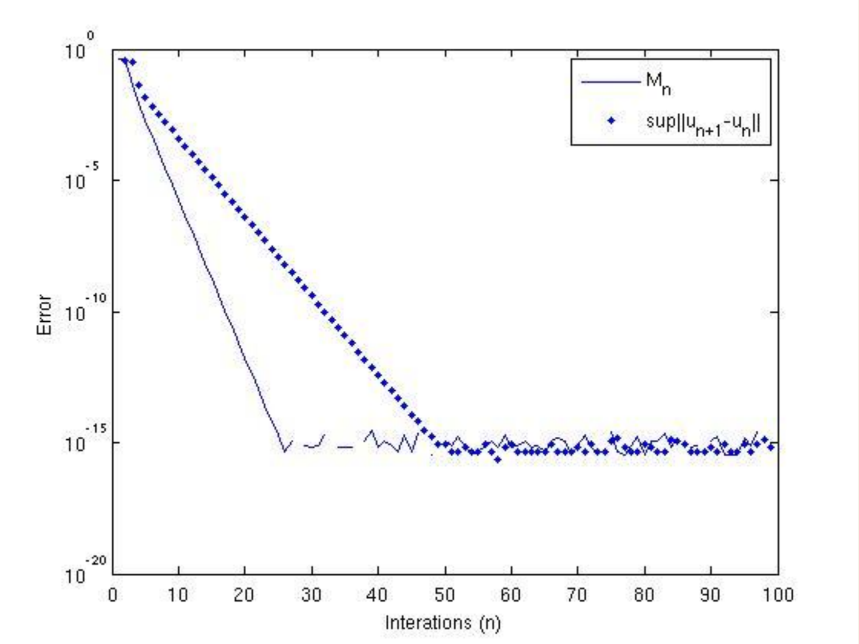
$$e_u^{(n)} = \sup_{x \in \mathbb{R}} |u_n - u_{n-1}|$$

# One-Pulse Solution with no potential

- Exact Solution for $V\left(x\right)=0$

$$\Psi=\sqrt{2\left|E\right|}sech^2\left(\sqrt{\left|E\right|}x\right)$$



# One-Pulse Solutions with Potential

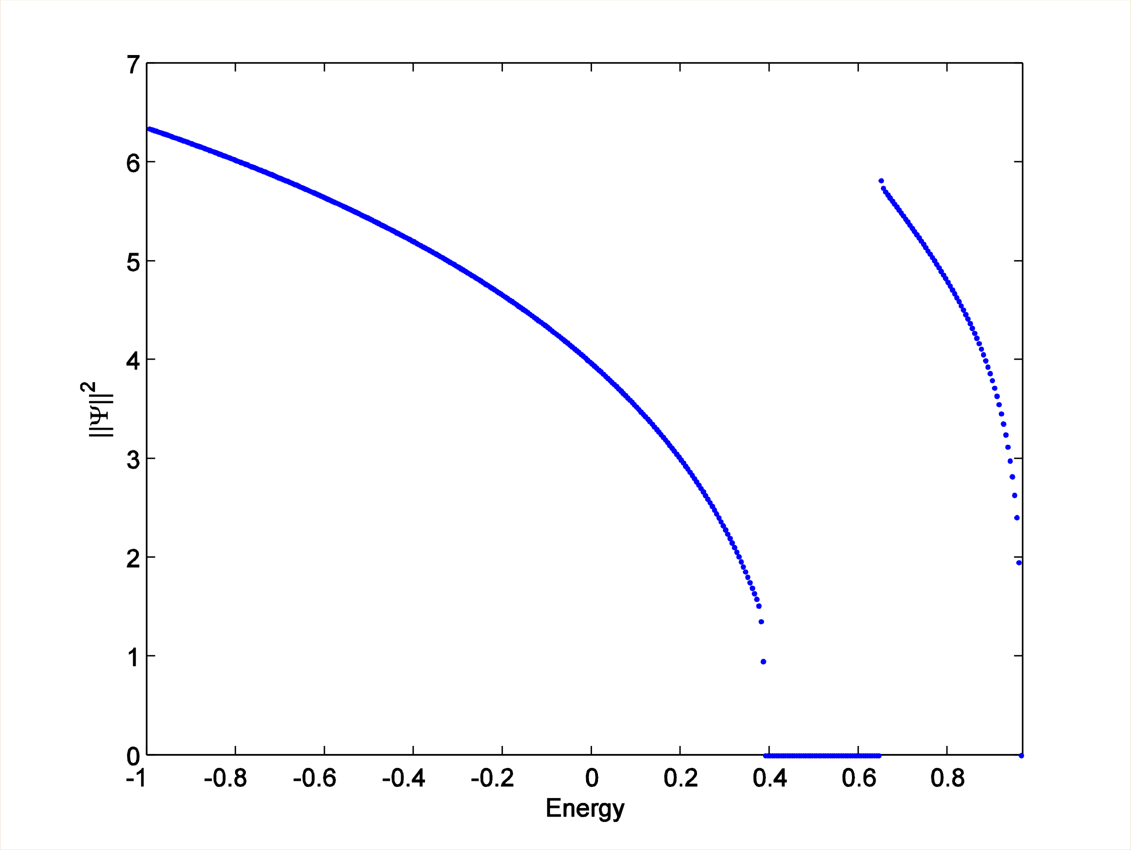
- Solutions can only exist where there is an extremum

$$V'(x) = 0, \qquad V''(x) \neq 0$$

- The solutions exist in $V(x) = \sin^2(x/2)$ at
  - $x = 0$ stable
  - $x = \pi$ unstable

# One-Pulse Solutions

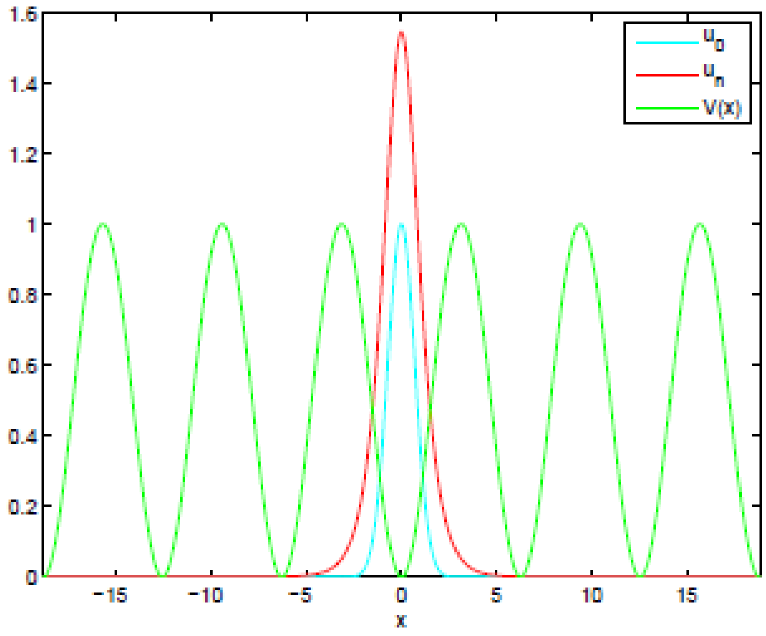
- Can only exist in Band Gaps.

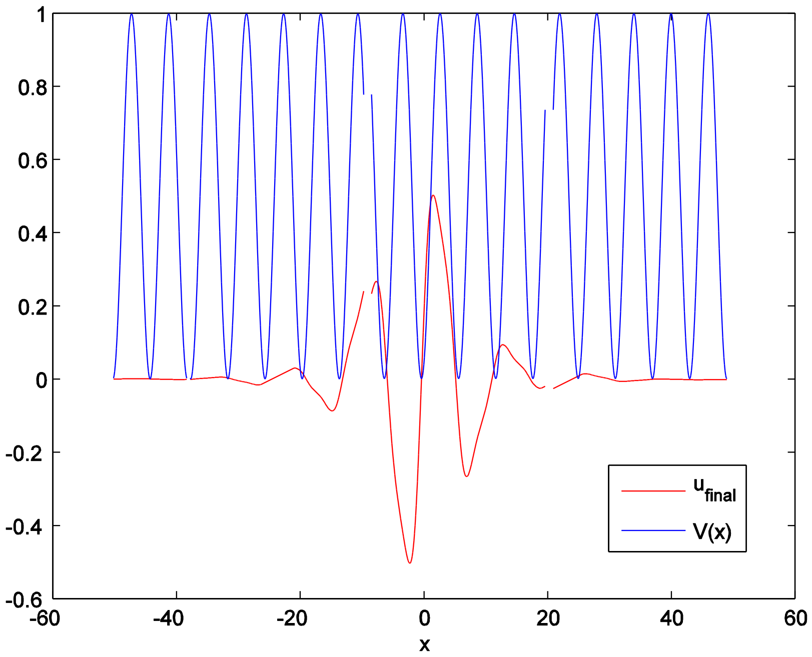


# One-Pulse Solutions

Semi-Infinite Gap

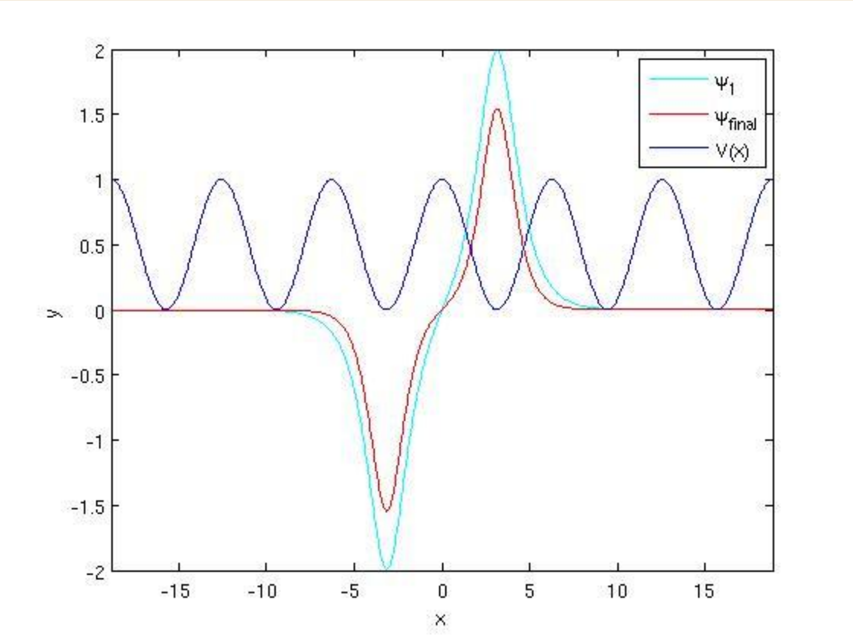
Band-Gap

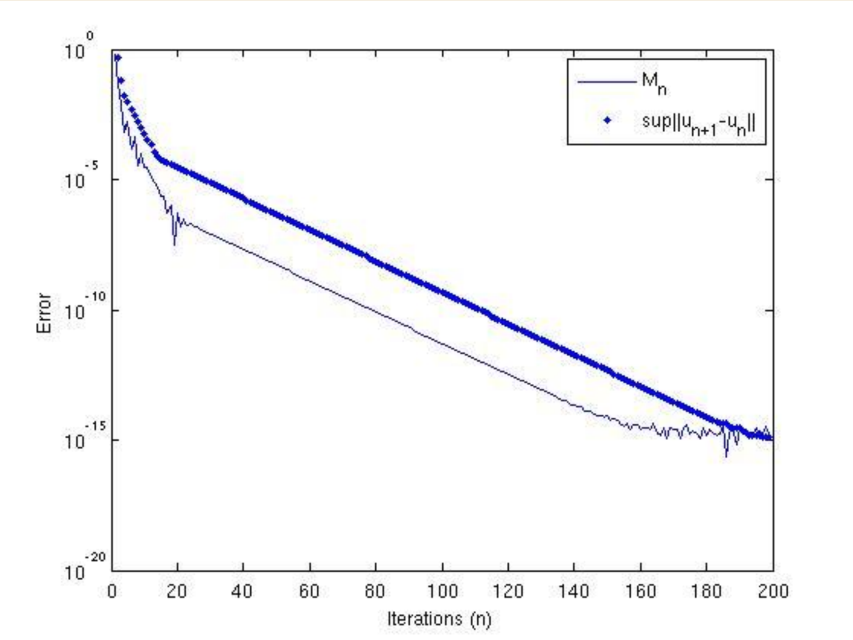


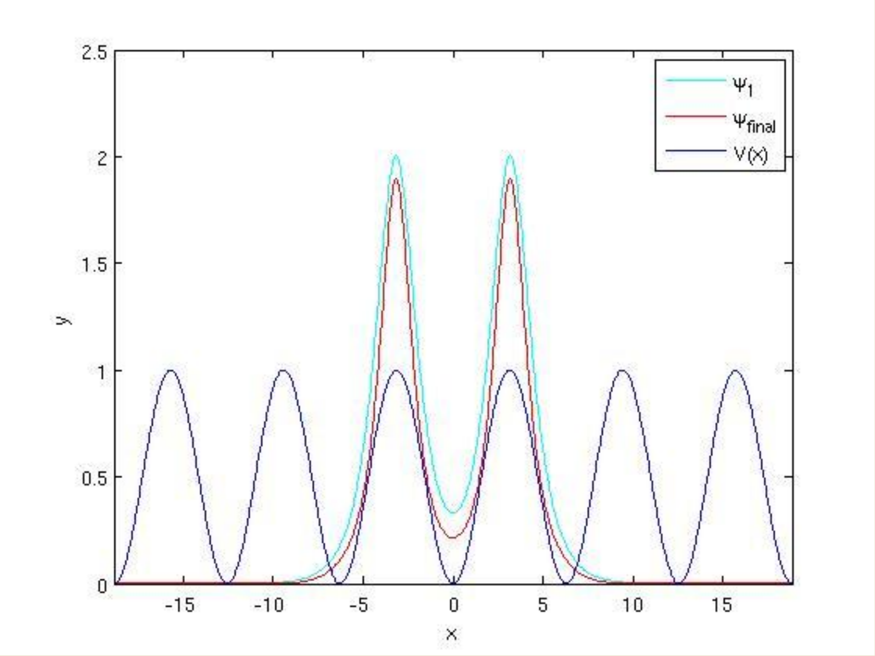
# Stable Two-Pulse Solution

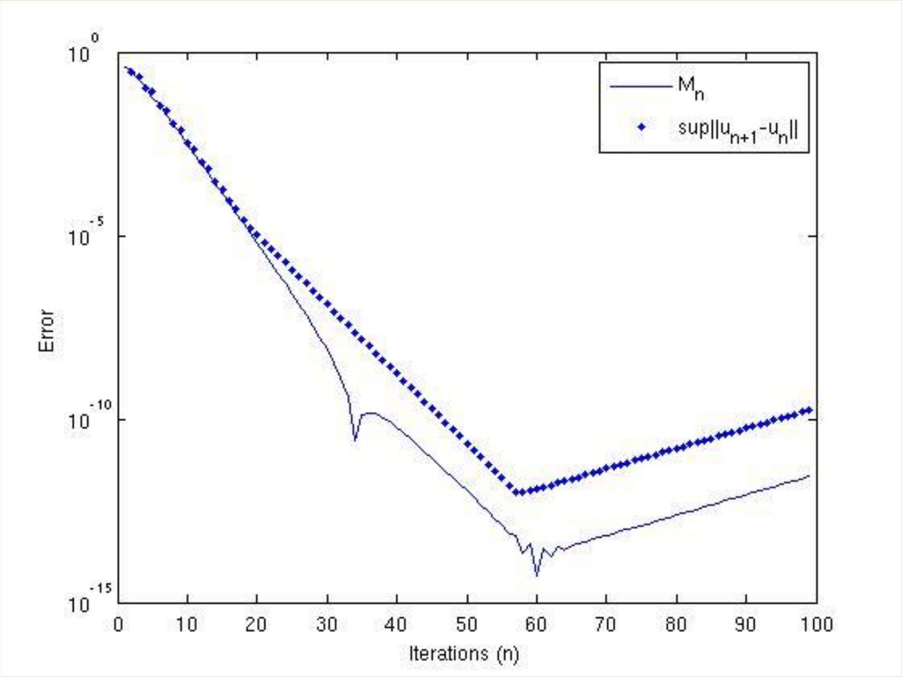
- We need to force Symmetry to ensure convergence



# Unstable Two-Pulse Solution



# Summary

- Computed One-Pulse Solutions
  - Semi-Infinite Gap
  - Band Gap
- Computed Two-Pulse Solutions
  - Semi-Infinite Gap