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Three-dimensional gravity waves in a channel of variable depth

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Abstract

We consider existence of three-dimensional gravity waves traveling along a channel of variable depth. It is well known that the long-wave small-amplitude expansion for such waves results in the stationary Korteweg–de Vries equation, coefficients of which depend on the transverse topography of the channel. This equation has a single-humped solitary wave localized in the direction of the wave propagation. We show, however, that there exists an infinite set of resonant Fourier modes that travel at the same speed as the solitary wave does. This fact suggests that the solitary wave confined in a channel of variable depth is always surrounded by small-amplitude oscillatory disturbances in the far-field profile.

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1. Introduction

We consider a channel of variable depth, where the coordinate x is chosen in the direction of the wave propagation and the transverse coordinate y is chosen in the direction of the channel topography. The vertical coordinate z changes between the rigid bottom at $z = -H(y)$ and the free surface $z = \eta(x, y)$, where $H(y) \geqslant 0$ is given and $\eta(x, y)$ is unknown.

The stationary propagation of the homogeneous, incompressible and inviscid fluid in the direction of x with the constant velocity c is prescribed by the Euler equations. The Euler equations in the reference frame moving with the speed c take the form

$$\begin{cases} (c+u)u_x + vu_y + wu_z = -p_x, \\ (c+u)v_x + vv_y + wv_z = -p_y, \\ (c+u)w_x + vw_y + ww_z = -g - p_z, \\ u_x + v_y + w_z = 0, \end{cases} \tag{1.1}$$

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where (u, v, w) is the velocity field in the direction of (x, y, z) and p is a normalized pressure. The system of equations is defined in the domain

$$\Omega(\eta) = \{(x, y) \in \mathbb{R}^2, \quad -H(y) < z < \eta(x, y)\}, \tag{1.2}$$

subject to the boundary conditions at $z = -H(y)$ and $z = \eta(x, y)$ and the decay conditions at infinity. To be precise, the bottom boundary condition is

$$w + H'(y)v = 0 \text{ at } z = -H(y), \tag{1.3}$$

while the free surface boundary conditions are given by the kinematic condition

$$w = (c + u)\eta_x + v\eta_y \text{ at } z = \eta(x, y) \tag{1.4}$$

and the dynamic condition

$$p = p_0 \text{ at } z = \eta(x, y), \tag{1.5}$$

where p_0 is the constant atmosphere pressure. The air pressure can be normalized to $p_0 = 0$. We are looking for solutions close to solitary waves that have a sufficient decay to the equilibrium state $(u, v, w) = (0, 0, 0)$ and $\eta = 0$ as $|x|, |y| \to \infty$.

The fluid motion is vorticity free if the velocity vector can be represented by the scalar velocity potential $\varphi(x, y, z)$, such that $(u, v, w) = \nabla\varphi$. It is well known that the Euler equations in the domain $\Omega(\eta)$ can be closed at the three-dimensional Laplace equation for velocity potential

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0, \quad \forall (x, y, z) \in \Omega(\eta), \tag{1.6}$$

subject to the boundary conditions

$$\begin{cases} \partial_n \varphi|_{z=-H(y)} = 0, \\ (\partial_z - \eta_x \partial_x - \eta_y \partial_y)\varphi|_{z=\eta(x,y)} = c\eta_x, \end{cases} \tag{1.7}$$

and the Bernoulli condition

$$g\eta + c\varphi_x|_{z=\eta(x,y)} + \frac{1}{2}\left(\varphi_x^2 + \varphi_y^2 + \varphi_z^2\right)|_{z=\eta(x,y)} = 0, \tag{1.8}$$

where $\partial_n \varphi|_{z=-H(y)} = -\frac{(\partial_z + H'(y)\partial_y)\varphi}{\sqrt{1+(H'(y))^2}}\Big|_{z=-H(y)}$ is the outward normal derivative to the boundary $z = -H(y)$. If $H(y) = h$ on $y \in \mathbb{R}$, the following theorem was proved in [1]:

Theorem 1. *Suppose that* $\eta \in C^1(\mathbb{R}^2)$ *and* $\varphi \in C^1(\Omega(\eta))$ *solve system (1.6)–(1.8), such that* $\eta(x, y) \geqslant 0$ *on* $(x, y) \in \mathbb{R}^2$ *and* $\varphi(x, y, z) \to 0$ *as* $\sqrt{x^2 + y^2} \to \infty$. *Then, in fact,* $\eta = 0$ *and* $\varphi = 0$.

A more general case of a bounded continuously differentiable boundary $z = -H(y)$ can be incorporated in the proof of Lemma 3.2 in [1] and thus in the proof of Theorem 1, if the wave propagates in the x-direction (as in our case). The main reason for non-existence of a three-dimensional surface solitary wave is that the neighborhoods of infinity on $(x, y) \in \mathbb{R}^2$ are connected in $\Omega(\eta)$, such that the solution $\varphi(x, y, z)$ of the Laplace equation (1.6) must have the same (zero) constant as $\sqrt{x^2 + y^2} \to \infty$ (see Remark 2.2 in [1]). This result rules out a hope to find a three-dimensional gravity solitary wave in a channel with variable bottom, unless the y-axis is compactified by the assumption that $H(y) = 0$ on $|y| > L$ for some $0 < L < \infty$. If, instead of $\Omega(\eta)$, we consider a compactified space

$$\Omega_c(\eta) = \{x \in \mathbb{R}, -L < y < L, -H(y) < z < \eta(x, y)\}, \tag{1.9}$$

the arguments of Theorem 1 are not applicable as the neighborhoods of infinity on $x \in \mathbb{R}$ are disconnected in $\Omega_c(\eta)$. Therefore, we shall assume from now that $H(y)$ is a continuously differentiable positive function with a compact support on the interval $[-L, L]$.

Two-dimensional (y-independent) gravity and gravity–capillary solitary waves were proved to exist by using the spatial dynamics formulation and the central manifold reductions [2]. These methods were extended to the three-dimensional gravity–capillary solitary waves in [5], where y-periodic and x-localized waves were constructed in the channel of uniform depth $H(y) = h$ on $|y| < L$. A more general class of traveling gravity–capillary waves which are periodic in one coordinate and localized in the other coordinate, where both coordinates are oblique to the direction of the wave propagation, was constructed in [4]. Spatial dynamics for three-dimensional gravity waves in the channel of uniform depth was considered in [3] where well-posedness of the linearized evolution equations in x- and y-variables was proved.

In our paper, we shall consider traveling gravity waves in a channel of variable depth, modelled by the function $H(y)$ on the interval $[-L, L]$. The gravity waves are confined in the channel and satisfy the Neumann boundary conditions at the bottom of the channel. Two situations differ in details: (i) if $\lim_{y\to\pm L} H(y) = \lim_{y\to\pm L} H'(y) = 0$ and (ii) if either $H(y)$ or $H'(y)$ does not vanish in the limit $y \to \pm L$. In the former case, the boundary conditions (1.7) are continuous at the intersection of $z = 0$ and $z = -H(y)$ at $y = \pm L$ such that the surface elevation $\eta(x, y)$ is zero at $y = \pm L$ for any $x \in \mathbb{R}$. In the latter case, the surface elevation $\eta(x, y)$ is non-zero as $y \to \pm L$ and the walls at $y = \pm L$ must be added above $z = 0$, together with the Neumann boundary conditions at $y = \pm L$ to ensure that the fluid is contained in the domain $\Omega_c(\eta)$.

A crucial difference between our work and the previous works [2,4,5] is the fact that we consider gravity waves with *zero* surface tension. In terms of the spatial dynamics formulation, this difference shows up as follows. The dynamical system for gravity–capillary waves with non-zero surface tension is given by the semi-linear equations with a finite-dimensional center manifold. On the contrary, the dynamical system for gravity waves is given by the quasi-linear differential equations with an *infinite-dimensional* center manifold. In this case, the double zero eigenvalue for a bifurcating solitary wave coexists with an infinite set of simple purely imaginary eigenvalues. We shall prove that this situation is generic for any topography given by a smooth function $H(y)$ on $[-L, L]$.

Although the system of linearized equations for spatial dynamics is well posed [3], existence of the center manifold in the system of quasi-linear equations does not follow immediately from the theory of center manifold reductions. If this manifold can be constructed, one can apply the recent results of [6,7] to prove existence of local bounded solutions of the nonlinear system which represent a solitary wave surrounded by small-amplitude oscillatory disturbances in the far-field profile. When the system of dynamical equations is formulated as the Hamiltonian system and the Hamiltonian function is sign-definite at the eigenmodes of the linearized system corresponding to purely imaginary eigenvalues, the local solution can be extended to a global bounded solution [6]. In accordance with this theory, we will show that the Hamiltonian function is indeed sign-definite for the eigenmodes of the linearized system associated with gravity waves in a channel of arbitrary topography. However, we do not attempt here to prove existence of the center manifold in the nonlinear problem.

The article is organized as follows. In Section 2, we study eigenvalues of the boundary-value problem (1.6)–(1.8) linearized around the zero solution. In Section 3, we show that the Hamiltonian function of the linearized system is sign-definite for eigenmodes corresponding to purely imaginary eigenvalues. Section 4 concludes the article with discussions.

2. Linear theory

Linearizing the boundary conditions (1.7) and (1.8) at the zero solution, we find that the Laplace equation (1.6) in the domain $\Omega_c(0)$ is supplemented by the boundary conditions

$$\begin{cases} \partial_n \varphi|_{z=-H(y)} = 0, \\ g\varphi_z|_{z=0} + c^2 \varphi_{xx}|_{z=0} = 0, \end{cases} \tag{2.1}$$

while the free surface is found from the relation $\eta = -\frac{c}{g}\varphi_x|_{z=0}$. The variables x and (y, z) are separated by using the Fourier transform in $x \in \mathbb{R}$ with parameter $k \in \mathbb{R}$. As a result, the boundary-value problem is rewritten in the form of the two-dimensional modified Helmholz equation

$$\varphi_{zz} + \varphi_{yy} - \mu\varphi = 0, \quad (y, z) \in \Omega_0, \tag{2.2}$$

subject to the boundary conditions

$$\begin{cases} \partial_n \varphi|_{z=-H(y)} = 0, \\ \partial_z \varphi|_{z=0} = \lambda \varphi|_{z=0}. \end{cases} \tag{2.3}$$

Here, $\mu = k^2 \in \mathbb{R}_+$ and $\lambda = \frac{c^2k^2}{g}$ are parameters of the problem and Ω_0 is the cross-section of the domain $\Omega_c(0)$ at any $x \in \mathbb{R}$ given by

$$\Omega_0 = \{-L < y < L, \quad -H(y) < z < 0\}. \tag{2.4}$$

Solutions of the boundary-value problem (2.2)–(2.3) are obtained in the following theorem.

Theorem 2. *Assume that $H(y)$ is a continuous function on the interval $[-L, L]$ such that for some $H_0, L_0 > 0$*

$$\lim_{\delta\to\infty} H\left(\frac{y}{\delta}\right) = H_0, \quad \forall y \in [-L_0, L_0] \subset \mathbb{R}.$$

The boundary-value problem (2.2)–(2.3) with $\mu = 0$ has an infinite set of eigenvalues $\{\lambda_m(0)\}_{m\in\mathbb{N}}$ with $\lambda_m(0) > 0$, in addition to the eigenvalue $\lambda_0(0) = 0$. Each eigenvalue is uniquely continued as a monotonically increasing function $\lambda_m(\mu)$ for $\mu \geqslant 0$ and $\lim_{\mu\to\infty} \frac{\lambda_m(\mu)}{\sqrt{\mu}} = 1$.

Proof. When $\lambda = 0$, the Neumann boundary-value problem (2.2)–(2.3) in a bounded domain Ω_0 has an infinite set of eigenvalues $\{\mu_m(0)\}_{m\geqslant 0}$, such that

$$\cdots \leqslant \mu_3(0) \leqslant \mu_2(0) \leqslant \mu_1(0) < \mu_0(0) = 0.$$

When $\lambda \in \mathbb{R}_+$, the Robin boundary-value problem (2.2)–(2.3) has an infinite set of eigenvalues $\{\mu_m(\lambda)\}_{m\geqslant 0}$, which are continuously differentiable functions of λ. We will prove that each eigenvalue $\mu_m(\lambda)$ is a strictly increasing function of λ. To do so, we consider a derivative problem

$$\mu_m'(\lambda)\varphi_m + \mu_m(\lambda)\partial_\lambda\varphi_m = \left(\partial_y^2 + \partial_z^2\right)\partial_\lambda\varphi_m, \quad (y,z) \in \Omega_0, \tag{2.5}$$

subject to the boundary conditions

$$\begin{cases} \partial_n\partial_\lambda\varphi_m|_{z=-H(y)} = 0, \\ \partial_z\partial_\lambda\varphi_m|_{z=0} = \lambda\partial_\lambda\varphi_m|_{z=0} + \varphi_m|_{z=0}. \end{cases} \tag{2.6}$$

By applying the second Green identity to the solutions of (2.5), we find that

$$\begin{aligned} \mu_m'(\lambda)\int_{\Omega_0}\varphi_m^2\,\mathrm{d}y\mathrm{d}z + \mu_m(\lambda)\int_{\Omega_0}\varphi_m\partial_\lambda\varphi_m\,\mathrm{d}y\mathrm{d}z &= \int_{\Omega_0}\varphi_m\left(\partial_y^2+\partial_z^2\right)\partial_\lambda\varphi_m\,\mathrm{d}y\mathrm{d}z \\ &= \int_{\Omega_0}\partial_\lambda\varphi_m\left(\partial_y^2+\partial_z^2\right)\varphi_m\,\mathrm{d}y\mathrm{d}z + \int_{\partial\Omega_0}(\varphi_m\partial_n\partial_\lambda\varphi_m - \partial_\lambda\varphi_m\partial_n\varphi_m)\,\mathrm{d}s, \end{aligned} \tag{2.7}$$

where s is a parameter along the boundary $\partial\Omega_0$ and n is the outward normal to the domain Ω_0. Using (2.2) and the boundary conditions (2.3) and (2.6), we obtain that

$$\mu_m'(\lambda) = \frac{\int_{-L}^{L}\varphi_m^2(y,0)\,\mathrm{d}y}{\int_{\Omega_0}\varphi_m^2(y,z)\,\mathrm{d}y\,\mathrm{d}z} > 0, \tag{2.8}$$

for any $\varphi_m \neq 0$. Since $\mu_m(\lambda)$ is strictly monotonic, there exists exactly one simple root $\lambda_m \geqslant 0$ for each $m \geqslant 0$, where $\mu_m(\lambda_m) = 0$, such that $\lambda_0 = 0$ and $\lambda_m > 0$ for $m \in \mathbb{N}$. Therefore, there exists a set of monotonically increasing functions $\{\lambda_m(\mu)\}_{m\in\mathbb{N}}$ on $\mu \geqslant 0$, such that $\lambda_m(0) = \lambda_m$.

To consider the asymptotic behavior of the functions $\lambda_m(\mu)$ as $\mu \to \infty$, we rescale the coordinates

$$\tilde{y} = \sqrt{\mu}y, \quad \tilde{z} = \sqrt{\mu}z, \quad \tilde{\Omega}_0 = \left\{(\tilde{y},\tilde{z}) : -\widetilde{L} < \tilde{y} < \widetilde{L}, -\widetilde{H}(\tilde{y}) < \tilde{z} < 0\right\}, \quad \widetilde{L} = \sqrt{\mu}L,$$

where $\widetilde{H}(\tilde{y}) = \sqrt{\mu}H\left(\frac{\tilde{y}}{\sqrt{\mu}}\right)$. The rescaled boundary-value problem is rewritten in the form

$$\varphi = \left(\partial_{\tilde{y}}^2 + \partial_{\tilde{z}}^2\right)\varphi, \quad (\tilde{y},\tilde{z}) \in \widetilde{\Omega}_0, \tag{2.9}$$

subject to the boundary conditions

$$\begin{cases} \partial_{\tilde{n}}\varphi|_{\tilde{z}=-\tilde{H}(\tilde{y})} = 0, \\ \partial_{\tilde{z}}\varphi|_{\tilde{z}=0} = \frac{\lambda}{\sqrt{\mu}}\varphi|_{\tilde{z}=0}. \end{cases} \tag{2.10}$$

Let $\widetilde{H}(\tilde{y}) = \widetilde{H}_0$ be constant and construct the explicit set of eigenvalues of the boundary-value problem (2.9)–(2.10) in the form

$$\frac{\lambda}{\sqrt{\mu}} = \sqrt{1 + \left(\frac{\pi m}{2\widetilde{L}}\right)^2} \tanh \widetilde{H}_0 \sqrt{1 + \left(\frac{\pi m}{2\widetilde{L}}\right)^2}, \quad m \geqslant 0.$$

Since $\widetilde{L} = \sqrt{\mu}L$ and $\widetilde{H}_0 \approx \sqrt{\mu}H_0$ for sufficiently large μ, we obtain that $\lim_{\mu\to\infty}\frac{\lambda}{\sqrt{\mu}} = 1$ for any fixed value of m. □

Corollary 3. *The Laplace equation (1.6) with the boundary values (2.1) has an infinite set of Fourier modes satisfying the dispersion relation $c = c_m(k)$ for any integer $m \geqslant 0$, such that $\lim_{k\to 0} c_0(k) < \infty$ and $\lim_{k\to 0} c_m(k) = \infty, \forall m \in \mathbb{N}$, while $\lim_{k\to\infty} c_m(k) = 0, \forall m \geqslant 0$.*

Proof. Let us consider a monotonically increasing curve $\lambda = \lambda_m(\mu)$ for $\mu \geqslant 0$ and $m \geqslant 0$. Since $\lambda = \frac{c^2k^2}{g}$ and $\mu = k^2$, we obtain that

$$\lim_{k\to 0} c_0^2(k) = \lim_{\mu\to 0} \frac{g\lambda_0(\mu)}{\mu} = \lim_{\lambda\to 0} \frac{g\lambda}{\mu_0(\lambda)} = \frac{g}{\mu_0'(0)}$$

and

$$\lim_{k\to 0} k^2 c_m^2(k) = \lim_{\mu\to 0} g\lambda_m(\mu) = g\lambda_m(0) > 0, \quad \forall m \in \mathbb{N},$$

such that $c_0(0) < \infty$ and $c_m(k) \to \infty$ as $k \to 0$ for $m \in \mathbb{N}$. Similarly, we obtain that

$$\lim_{k\to\infty} kc_m^2(k) = \lim_{\mu\to\infty} \frac{g\lambda_m(\mu)}{\sqrt{\mu}} = g, \quad \forall m \geqslant 0,$$

such that $c_m(k) \to 0$ as $k \to \infty$ for any integer $m \geqslant 0$. □

Remark 4. If $\lim_{y\to\pm L} H(y) = \lim_{y\to\pm L} H'(y) = 0$, then the boundary conditions (2.3) imply that

$$\lim_{y\to\pm L} \partial_z\varphi|_{z=0} = \lim_{y\to\pm L} \partial_z\varphi|_{z=-H(y)} = 0, \quad \lim_{y\to\pm L} \varphi|_{z=0} = 0.$$

Therefore, $\lim_{y\to\pm L}\eta = 0$. If either $H(y)$ or $H'(y)$ is non-zero at $y = \pm L$, then $\lim_{y\to\pm L}\eta \neq 0$ in general.

Example 5. Let the bottom be flat such that $H(y) = h$ on $y \in [-L, L]$. Since $\lim_{y\to\pm L} H(y) \neq 0$, we add the Neumann boundary conditions $\varphi_y|_{y=\pm L} = 0$ at the walls $y = \pm L$. The boundary-value problem (2.2)–(2.3) becomes separable in (y, z) with an explicit set of eigenfunctions

$$\varphi(y, z) = \cos k_m(y + L) \cosh \sqrt{k^2 + k_m^2}(z + h), \quad k_m = \frac{\pi m}{2L}, \quad \forall m \geqslant 0. \tag{2.11}$$

The dispersion relation $c = c_m(k)$ is found in the explicit form

$$c^2 = \frac{g\sqrt{k^2 + k_m^2}}{k^2} \tanh h\sqrt{k^2 + k_m^2}, \quad \forall k \in \mathbb{R}. \tag{2.12}$$

Fig. 1 shows the curves $c = c_m(k)$ for $m \geqslant 0$ versus $k \geqslant 0$. The curves $c_m(k)$ with $m \in \mathbb{N}$ diverge to a positive infinity as $k \to 0$, while the curve $c_0(k)$ converges to $\sqrt{gh}$. On the other hand, all curves converge to zero as $k \to \infty$. These behavior coincides with the predictions of Theorem 2. If a solitary wave bifurcates from the point $c = \sqrt{gh}$ to $c > \sqrt{gh}$, the solitary wave is in resonance with an infinite set of Fourier modes of the same speed c for all $m \in \mathbb{N}$.

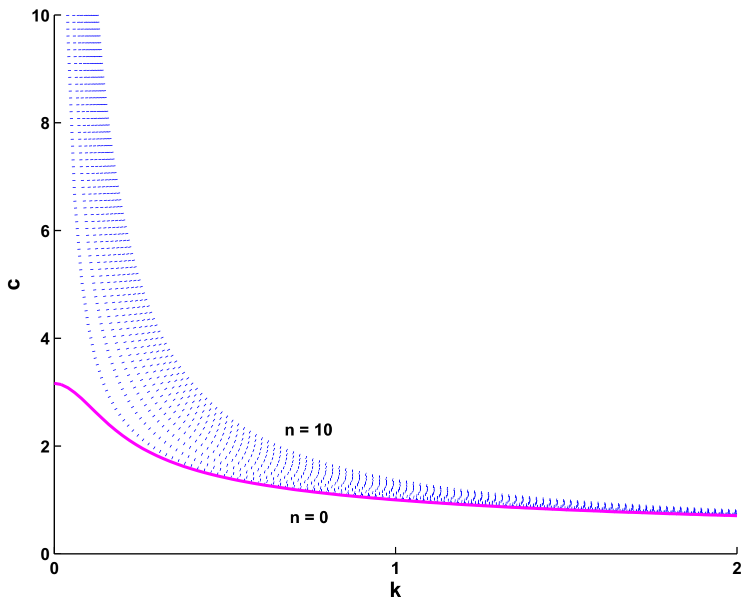

Fig. 1. Branches $c = c_m(k)$ of the dispersion relation (2.12) in the rectangular channel with $L = h = 10$ and $0 \leqslant m \leqslant 10$.

Proposition 6. *The dispersion relation* $c = c_0(k)$ *and the corresponding eigenmode are approximated near* $k = 0$ *by the asymptotic expansions*:

$$c_0^2(k) = c_0^2 + k^2 c_2 + \mathrm{O}(k^4), \quad \varphi(y,z) = 1 + k^2 \varphi_2(y,z) + \mathrm{O}(k^4), \tag{2.13}$$

where

$$c_0^2 = \frac{g}{2L} \int_{-L}^{L} H(y)\, dy, \quad c_2 = \frac{g}{2L} \int_{-L}^{L} \int_{-H(y)}^{0} \varphi_2(y,z)\, dz\, dy - \frac{c_0^2}{2L} \int_{-L}^{L} \varphi_2(y,0)\, dy \tag{2.14}$$

and $\varphi_2(y,z)$ *solves the inhomogeneous problem*

$$\left(\partial_z^2 + \partial_y^2\right) \varphi_2 = 1, \quad (y,z) \in \Omega_0, \tag{2.15}$$

subject to the boundary conditions

$$\begin{cases} \partial_n \varphi_2|_{z=-H(y)} = 0, \\ \partial_z \varphi_2|_{z=0} = \frac{c_0^2}{g}. \end{cases} \tag{2.16}$$

Proof. By Theorem 2 and direct substitutions, the constant solutions for the leading-order term of $\varphi(y,z)$ are unique solutions of the Laplace equation, so that we can normalize $\varphi_0 = 1$. The correction term $\varphi_2(y,z)$ is uniquely defined from the boundary-value problem (2.15)–(2.16). Applying the Green formula to this problem, we obtain the solvability condition resulting in the first equation (2.14) for c_0^2. More generally, applying the first Green identity to system (2.2)–(2.3), we obtain that

$$k^2 \int_{\Omega_0} \varphi \, dy \, dz = \int_{\Omega_0} (\varphi_{zz} + \varphi_{yy}) dy \, dz = \int_{\partial\Omega_0} \partial_n \varphi \, ds = \int_{z=-H(y)} \partial_n \varphi \, ds + \int_{-L}^{L} \partial_z \varphi|_{z=0} \, dy = \frac{c^2 k^2}{g} \int_{-L}^{L} \varphi(y, 0) \, dy. \tag{2.17}$$

Now the first equation (2.14) follows from the Green formula (2.17) after the asymptotic expansions (2.13). Extending the asymptotic expansion to the next order, we obtain the correction term c_2 in the form of the second equation (2.14). □

Example 7. Let $H(y) = h$ and find the correction terms $\varphi_2(y,z)$ and c_2 in the explicit form

$$\varphi_2 = \frac{1}{2}(h+z)^2, \quad c_2 = -\frac{1}{3} g h^3.$$

These corrections are the same as in the standard linear theory of two-dimensional surface water waves.

3. Hamiltonian for linearized system

We cast the nonlinear system (1.6)–(1.8) as a Hamiltonian system for spatial dynamical evolution, following the approach of [5] in the limit of no surface tension. We will show that the quadratic part of the Hamiltonian function at the eigenmodes of the linearized system is sign-definite, such that the energy of the Fourier modes with $k = k_m$, $m \in \mathbb{N}$ for the same value of $c > c_0(0)$ is of the same sign. This fact implies that a global bounded solution of the nonlinear spatial dynamical system (1.6)–(1.8), resembling a solitary wave surrounding by oscillatory disturbances in the far-field profile, can be constructed, similarly to the approach in [6,7], if existence of the center manifold can be proven for the nonlinear system (1.6)–(1.8). However, the existence of a center manifold and the construction of global solutions in the nonlinear system (1.6)–(1.8) is beyond the scopes of the present manuscript.

The Lagrangian function for the nonlinear system (1.6)–(1.8) is written in the form

$$\mathscr{L} = \frac{1}{2} \int_{\mathbb{R}} \int_{-L}^{L} \left[\int_{-H(y)}^{\eta(x,y)} \left(2c\varphi_x + \varphi_x^2 + \varphi_y^2 + \varphi_z^2 \right) dz + g\eta^2 \right] dy \, dx. \tag{3.1}$$

By using the new variables

$$\mu = \frac{z + H(y)}{\eta(x,y) + H(y)}, \quad \varphi = \phi(x, y, \mu(x,y,z)), \tag{3.2}$$

we transform the Lagrangian function into new form

$$\mathscr{L} = \frac{1}{2} \int_{\mathbb{R}} \int_{-L}^{L} \left[\int_0^1 \left(2c(\eta + H)\phi_x - 2c\mu\eta_x\phi_\mu + (\eta + H)\left(\phi_x - \frac{\mu\eta_x}{\eta + H}\phi_\mu\right)^2 \right. \right.$$
$$\left. \left. + (\eta + H)\left(\phi_y + \frac{H' - \mu(\eta_y + H')}{\eta + H}\phi_\mu\right)^2 + \frac{\phi_\mu^2}{\eta + H}\right) d\mu + g\eta^2 \right] dy \, dx. \tag{3.3}$$

We apply the Legendre transformation

$$\omega = \frac{\delta \mathscr{L}}{\delta \eta_x} = -\int_0^1 \mu\phi_\mu \left(c + \phi_x - \frac{\mu\eta_x}{\eta + H}\phi_\mu \right) d\mu, \tag{3.4}$$

$$\xi = \frac{\delta \mathscr{L}}{\delta \phi_x} = (\eta + H)(c + \phi_x) - \mu\eta_x\phi_\mu, \tag{3.5}$$

such that

$$\omega = -\int_0^1 \frac{\mu\phi_\mu\xi}{\eta + H} d\mu, \tag{3.6}$$

and obtain the Hamiltonian function in the form

$$\mathscr{H} = \int_{\mathbb{R}} \int_{-L}^{L} \left[\int_0^1 \xi \phi_x \, \mathrm{d}\mu + \omega \eta_x \right] \mathrm{d}y \, \mathrm{d}x - \mathscr{L}$$

$$= \frac{1}{2} \int_{\mathbb{R}} \int_{-L}^{L} \left[\int_0^1 \left(\frac{\xi^2}{\eta + H} - 2c\xi - (\eta + H) \left(\phi_y + \frac{H' - \mu(\eta_y + H')}{\eta + H} \phi_\mu \right)^2 - \frac{\phi_\mu^2}{\eta + H} \right) \mathrm{d}\mu \right.$$

$$\left. + c^2(\eta + H) - g\eta^2 \right] \mathrm{d}y \, \mathrm{d}x. \tag{3.7}$$

Although the Hamiltonian function $\mathscr{H}$ is not too complicated, the symplectic structure of nonlinear evolution equations for spatial dynamics is non-canonical since Eq. (3.6) gives a constraint on the canonical variables (ϕ, ξ) and (η, ω). Therefore, we shall avoid writing and analyzing equations of motion in the case of gravity waves. (These equations are analyzed in [5] for gravity–capillary waves.)

We note that $\eta = \omega = \phi = 0$ and $\xi = cH(y)$ gives a trivial solution of the Hamiltonian system, whose linearization recovers the linear system considered in Section 2. Therefore, using the substitution $\eta = \tilde{\eta}$, $\omega = \tilde{\omega}$, $\phi = \tilde{\phi}$ and $\xi = cH(y) + \tilde{\xi}$ and truncating equations (3.4) and (3.5) at the linear terms, we obtain the linearized constraints

$$\tilde{\xi} = c\tilde{\eta} + H(y)\tilde{\phi}_x, \quad \tilde{\omega} = -c \int_0^1 \mu \tilde{\phi}_\mu \, \mathrm{d}\mu. \tag{3.8}$$

Furthermore, truncating the Hamiltonian function $\mathscr{H}$ at the quadratic terms and dropping the tilde notations, we obtain the quadratic Hamiltonian function $\mathscr{H}_{\text{quad}}$ in the form

$$\mathscr{H}_{\text{quad}} = \frac{1}{2} \int_{\mathbb{R}} \int_{-L}^{L} \left[\int_0^1 \left(\frac{\xi^2}{H} - 2c\xi\eta - H \left(\phi_y + \frac{H'}{H}(1-\mu)\phi_\mu \right)^2 - \frac{\phi_\mu^2}{H} \right) \mathrm{d}\mu + \frac{c^2 - gH}{H} \eta^2 \right] \mathrm{d}y \, \mathrm{d}x$$

$$= \frac{1}{2} \int_{\mathbb{R}} \int_{-L}^{L} \left[\int_0^1 \left(H\phi_x^2 - H \left(\phi_y + \frac{H'}{H}(1-\mu)\phi_\mu \right)^2 - \frac{\phi_\mu^2}{H} \right) \mathrm{d}\mu - g\eta^2 \right] \mathrm{d}y \, \mathrm{d}x, \tag{3.9}$$

where we have substituted expression (3.8). Our main result of this section is described in the following theorem.

Theorem 8. *The Hamiltonian function $\mathscr{H}_{quad}$ is negative for solutions of the linear system (2.2)–(2.3).*

Proof. After the Fourier transform in x, the density of the Hamiltonian function $\mathscr{H}_{\text{quad}}$ becomes

$$\widehat{\mathscr{H}}_{\text{quad}} = \frac{1}{2} \int_{-L}^{L} \left[\int_0^1 \left(k^2 H|\phi|^2 - H \left| \phi_y + \frac{H'}{H}(1-\mu)\phi_\mu \right|^2 - \frac{|\phi_\mu|^2}{H} \right) \mathrm{d}\mu - g|\eta|^2 \right] \mathrm{d}y$$

$$= \frac{1}{2} \int_{-L}^{L} \left[\int_{-H(y)}^{0} \left(k^2|\varphi|^2 - |\varphi_y|^2 - |\varphi_z|^2 \right) \mathrm{d}z - g|\eta|^2 \right] \mathrm{d}y, \tag{3.10}$$

where the transformation (3.2) has been used. The last expression gives a Hamiltonian function for the linear system (2.2)–(2.3) with the constraint $\eta = -\frac{ick}{g} \varphi|_{z=0}$. Using the Helmholz equation (2.2) for the first term and applying the first Green identity, we obtain

$$\widehat{\mathscr{H}}_{\text{quad}} = \frac{1}{2} \int_{z=-H(y)} \varphi \partial_n \varphi \, \mathrm{d}s + \frac{1}{2} \int_{-L}^{L} \varphi \partial_z \varphi|_{z=0} \, \mathrm{d}y - \int_{-L}^{L} \left[\int_{-H(y)}^{0} \left(|\varphi_y|^2 + |\varphi_z|^2 \right) \mathrm{d}z + \frac{1}{2} g|\eta|^2 \right] \mathrm{d}y$$

$$= -\int_{-L}^{L} \left[\int_{-H(y)}^{0} \left(|\varphi_y|^2 + |\varphi_z|^2 \right) \mathrm{d}z \right] \mathrm{d}y, \tag{3.11}$$

where we have used the boundary conditions (2.3). Therefore, $\widehat{\mathscr{H}}_{\text{quad}} \leqslant 0$ and it reaches the zero value if and only if $\varphi \equiv 1$, which corresponds to the case $\lambda = \mu = 0$, that is $c^2 = c_0^2(0)$ for $k = 0$ and $m = 0$. □

Example 9. Let $H(y) = h$ on $y \in [-L, L]$ and consider the explicit solution (2.11). Then, direct computations show that

$$\widehat{\mathscr{H}}_{\text{quad}} = \frac{L}{2}\left[hk^2 - \frac{\sinh h\sqrt{k^2+k_m^2}\cosh h\sqrt{k^2+k_m^2}}{\sqrt{k^2+k_m^2}}(k^2+2k_m^2)\right] = -\frac{hLk^2}{2}\left[\frac{k^2+2k_m^2}{k^2}\frac{\sinh 2h\sqrt{k^2+k_m^2}}{2h\sqrt{k^2+k_m^2}} - 1\right],$$

which is non-positive. Moreover, if the bifurcation happens for $m = 0$, $k = 0$, for which $\widehat{\mathscr{H}}_{\text{quad}} = 0$, the other modes with $m \in \mathbb{N}$ and $k = k_m$ have strictly negative values of $\widehat{\mathscr{H}}_{\text{quad}}$.

Remark 10. Discussions in Section 6 of [3] imply that the results of Example 9 were known although no computations can be found in [3].

4. Discussions

It is well known that the formal long-wave small-amplitude expansions for solutions of the nonlinear system (1.6)–(1.8) in the form

$$\varphi = \frac{\varepsilon}{c}\left[\varphi_0(X) + \varepsilon^2\varphi_2(X,y,z) + \mathrm{O}(\varepsilon^4)\right], \quad \eta = \varepsilon^2\left[\eta_0(X) + \varepsilon^2\eta_2(X,y) + \mathrm{O}(\varepsilon^4)\right],$$

and

$$c^2 = c_0^2 + \varepsilon^2(\Delta c) + \mathrm{O}(\varepsilon^4),$$

where $X = \varepsilon x$ and ε is a formal small parameter, result in the fourth-order equation for $\varphi_0(X)$

$$c_2\varphi_0^{(\text{iv})} + b_2\varphi_0'\varphi_0'' + (\Delta c)\varphi_0'' = 0, \tag{4.1}$$

where the coefficient c_2 is determined by the second equation (2.14), parameter (Δc) is arbitrary, and the coefficient b_2 is found in the form

$$b_2 = 2 + \frac{1}{2L}\int_{-L}^{L}\partial_z^2\varphi_2(y,0)\,\mathrm{d}y = 3 - \frac{1}{2L}\int_{-L}^{L}\varphi_2''(y,0)\,\mathrm{d}y = 3.$$

The fourth-order ODE (4.1) is rewritten as the third-order ODE

$$c_2U''' + 3UU' + (\Delta c)U' = 0, \tag{4.2}$$

where $U = \varphi_0'(X)$. The free surface is approximated at the leading order by $\eta_0 = -\frac{1}{g}U(X)$. The third-order ODE (4.2) is usually referred to as the stationary Korteweg–de Vries equation. When the quadratic term is neglected in the ODE (4.2) and $\phi_0(X) \sim \mathrm{e}^{\mathrm{i}kX}$, the linear theory is recovered with $(\Delta c) = c_2k^2$. On the other hand, the nonlinear ODE (4.2) has a solitary wave solution in the form

$$U(X) = -(\Delta c)\operatorname{sech}^2(\kappa x), \quad \kappa = \sqrt{\frac{-(\Delta c)}{4c_2}}, \tag{4.3}$$

under the condition that $\operatorname{sign}((\Delta c)c_2) = -1$. The latter condition indicates that the speed of the solitary wave $c^2 = c_0^2 + \epsilon^2(\Delta c) + \mathrm{O}(\epsilon^4) > c_0^2$ is different from the speed of the linear waves $c_0^2(k) = c_0^2 + c_2k^2 + \mathrm{O}(k^4) \leqslant 0$, where the expansion of $c_0(k)$ for small k is defined by Proposition 6. Therefore, the solitary wave (4.3) bifurcates from the mode $m = 0$ at the point $k = 0$ and $c = c_0(0)$. However, according to Corollary 3, there exist infinitely many Fourier modes with $m \in \mathbb{N}$, which have the same $c = c_0(0)$ for $k = k_m$. The formal long-wave small-amplitude expansions neglect existence of these Fourier modes. It is expected that these Fourier modes lead to oscillatory disturbances far from the localization of a solitary wave. Construction of such local solutions to the full nonlinear system (1.6)–(1.8) remains an open problem up to the date.

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